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On Soft Normed Quasilinear Spaces

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Abstract — In this study, we investigate some properties of soft quasi-sequences and present new results. We then study the completeness of soft normed quasilinear space and present an analog of convergence and boundness results of soft quasi sequences in soft normed quasilinear spaces. Moreover, we define regular and singular subspaces of soft quasilinear spaces and draw several conclusions related to these notions. Afterward, we provide examples of these results in soft normed quasilinear spaces generalizing well-known results in soft linear normed spaces. Additionally, we offer new results concerning soft quasi subspaces of soft normed quasilinear spaces. Finally, we discuss the need for further research.

Keywords *Soft set, soft quasilinear space, soft normed quasilinear space, soft quasi vector, convergence*

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1. Introduction

Quasilinear spaces and normed quasilinear spaces were introduced by Aseev [1]. Then, several results on normed quasilinear spaces were obtained by defining proper quasilinear spaces in [2–4]. Later on, the quasilinear functions with bounded interval values were studied, and the Hahn-Banach extension theorem was analyzed in [5, 6], respectively. Then, quasilinear inner product spaces, generalizations of inner product spaces, were defined to develop quasilinear functional analysis in [7–10]. In addition, Yılmaz et al. [11] demonstrated that Hilbert quasilinear spaces are a special class of fuzzy number sequences. In [12, 13], Levent and Yılmaz included some quasilinear applications, such as signal processing.

Molodtsov [14] introduced soft sets in 1999. His next presentation covered a variety of applications of this theory in economics, engineering, and medicine. Following that, Maji et al. [15] presented several operations on soft sets. After, Das and Samanta introduced soft elements [16] and soft real numbers [17]. Additionally, they worked on soft linear operators, soft linear spaces, soft inner product spaces, and some of their features in [18–21]. Afterward, they introduced soft normed spaces in a novel perspective, along with soft inner product spaces and soft Hilbert spaces in [22, 23], respectively.

Bozkurt [24] introduced soft quasilinear spaces and soft normed quasilinear spaces, being more generalized than the previous notions of soft linear spaces and quasilinear spaces. Afterward, Bozkurt and Gönci defined soft inner product quasilinear spaces and soft Hilbert quasilinear spaces in [25]. Moreover, they worked on some properties of soft inner product quasilinear spaces.

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In this study, we give some properties of soft quasi-sequences and several new theorems related to their convergence in soft normed quasilinear spaces. Moreover, we study the completeness of soft normed quasilinear spaces. Besides, we define the regular and singular subspaces of a soft quasilinear space and find several conclusions that are related to these notions. In addition, we provide a few examples of soft normed quasilinear spaces. Finally, we discuss the need for further research.

2. Preliminaries

The objective of this section is to introduce some concepts in soft set theory and some basic notions, such as soft quasilinear spaces and soft normed quasilinear spaces, concerning soft set theory. Let Q be a universe, P be a set of parameters, $P(Q)$ be the power set of Q, and B be a non-empty subset of P.

Definition 2.1. [14] A pair (G, P) is called a soft set over Q, where G is a mapping defined by $G : P \to P(Q)$.

Definition 2.2. [19] A soft set (G, P) over Q is said to be an absolute soft set represented by $\widetilde{Q}$, if $G(\gamma) = Q$, for every $\gamma \in P$. A soft set (G, P) over Q is said to be a null soft set represented by Φ, if $G(\gamma) = \emptyset$, for every $\gamma \in P$.

Definition 2.3. [17] Let Q be a non-empty set and P be a non-empty parameter set. Then, a function $q : P \to Q$ is said to be a soft element of Q. A soft element q of Q is said to belong to a soft set G of Q, which is denoted by $q \in Q$, if $q(\gamma) \in G(\gamma)$, $\gamma \in P$. Thus, for a soft set G of Q with respect to the index set P, we get $G(\gamma) = \{q(\gamma), \gamma \in P\}$. A soft set (G, P) for which $G(\gamma)$ is a singleton set, for all $\gamma \in P$, can be determined with a soft element by simply determining the singleton set with the element that it contains, for all $\gamma \in P$.

The set of all the soft sets (G, P) over Q will be described by $S\left(\widetilde{Q}\right)$ for which $G(\gamma) \neq \emptyset$, for all $\gamma \in P$ and the collection of all the soft elements of (G, P) over Q will be denoted by $SE\left(\widetilde{Q}\right)$.

Definition 2.4. [24] Let Q be a quasilinear space, P be a parameter set, and G be a soft set over (Q, P). Then, G is said to be a soft quasilinear space of Q if $Q(\gamma)$ is a quasilinear subspace of Q, for every $\gamma \in P$.

Remark 2.5. [24] Soft quasi vectors in a soft quasilinear space are represented by $\widetilde{q}$, $\widetilde{w}$, and $\widetilde{z}$, and $\widetilde{a}$, $\widetilde{b}$, and $\widetilde{c}$ are used to specify soft real numbers.

Definition 2.6. If a soft quasi element $\widetilde{q}$ has an inverse, i.e., $\widetilde{q} - \widetilde{q} = \widetilde{\theta}$ such that $\widetilde{q}(\gamma) - \widetilde{q}(\gamma) = \widetilde{\theta}(\gamma)$, for every $\gamma \in P$, then it is called regular. If a soft quasi element $\widetilde{q}$ has no inverse, then it is called singular.

Definition 2.7. [24] Let $\widetilde{Q}$ be the absolute soft quasilinear space, i.e., $\widetilde{Q}(\gamma) = Q$, for every $\gamma \in P$ and $\mathbb{R}(P)$ denote all soft real numbers. Then, a mapping $\|.\| : SE\left(\widetilde{Q}\right) \longrightarrow \mathbb{R}(P)$ is said to be the soft norm on the soft quasilinear space $\widetilde{Q}$ if $\|.\|$ satisfies the following conditions:

i. $\|\widetilde{q}\| \geq 0$ if $\widetilde{q} \neq \widetilde{\theta}$, for every $\widetilde{q} \in \widetilde{Q}$.

ii. $\|\widetilde{q} + \widetilde{w}\| \leq \|\widetilde{q}\| + \|\widetilde{w}\|$, for every $\widetilde{q}, \widetilde{w} \in \widetilde{Q}$.

iii. $\|\widetilde{\alpha} \cdot \widetilde{q}\| = |\widetilde{\alpha}| \cdot \|\widetilde{q}\|$, for every $\widetilde{q} \in \widetilde{Q}$ and for every soft scalar $\widetilde{\alpha}$.

iv. If $\widetilde{q} \widetilde{\preceq} \widetilde{w}$, then $\|\widetilde{q}\| \leq \|\widetilde{w}\|$, for every $\widetilde{q}, \widetilde{w} \in \widetilde{Q}$.

v. If, for any $\varepsilon > 0$, there exists a quasi vector $\widetilde{q}_\varepsilon \in \widetilde{Q}$ such that $\widetilde{q} \widetilde{\preceq} \widetilde{w} + \widetilde{q}_\varepsilon$ and $\|\widetilde{q}_\varepsilon\| \leq \varepsilon$, then $\widetilde{q} \widetilde{\preceq} \widetilde{w}$, for any soft quasi vectors $\widetilde{q}, \widetilde{w} \in \widetilde{Q}$.

A soft quasilinear space $\widetilde{Q}$ with a soft norm $\|.\|$ on $\widetilde{Q}$ is called a soft normed quasilinear space and is indicated by $\left(\widetilde{Q}, \|.\|\right)$ or $\left(\widetilde{Q}, \|.\|, P\right)$.

Lemma 2.8. [26] Let $\left(\widetilde{Q}, \|.\|, P\right)$ be a soft normed quasilinear space and a soft quasi norm $\|.\|$ satisfy the condition:

$$\{\|\widetilde{q}\|(\gamma) : \widetilde{q}(\gamma) = q, \text{for } q \in Q \text{ and } \gamma \in P\} \text{ is a singleton set.} \tag{1}$$

Then, for every $\gamma \in P$, $\|.\|_\gamma : Q \to \mathbb{R}^+$ defined by $\|q\|_\gamma = \|\widetilde{q}\|(\gamma)$, for every $q \in Q$ and $\widetilde{q} \in \widetilde{Q}$ such that $\widetilde{q}(\gamma) = q$, is a quasi norm on Q.

Let $\widetilde{Q}$ be a soft normed quasilinear space. Then, soft Hausdorff or soft norm metric on $\widetilde{Q}$ is defined by

$$h_Q(\widetilde{q}, \widetilde{w}) = \inf\left\{\widetilde{r} \geq \widetilde{0} : \widetilde{q} \widetilde{\preceq} \widetilde{w} + \widetilde{q}_1^r,\ \widetilde{w} \widetilde{\preceq} \widetilde{q} + \widetilde{q}_2^r,\ \text{and } \|\widetilde{q}_i^r\| \leq \widetilde{r}\right\}$$

3. Some New Results Related to Soft Quasi Sequences

Throughout this section, let Q and W be two quasilinear spaces over field $\mathbb{R}$, P be a non-empty parameter set, and $\widetilde{Q}$ and $\widetilde{W}$ be two absolute soft quasilinear spaces, i.e., $\widetilde{Q}(\gamma) = Q$ and $\widetilde{W}(\gamma) = W$, for every $\gamma \in P$, respectively. $SE(\widetilde{Q})$ denotes the set of all the soft quasi vectors of $\widetilde{Q}$. The notations $\widetilde{q}$ and $\widetilde{w}$ demonstrate also soft quasi vectors of $\widetilde{Q}$. Further, we will take $\widetilde{\alpha}(\gamma) = \alpha$, for every soft scalar $\widetilde{\alpha}$ and $\gamma \in P$.

Definition 3.1. Let $\left(\widetilde{Q}, \|.\|, P\right)$ be a soft normed quasilinear space and $\{\widetilde{q}_n\}$ be a sequence of soft quasi vectors in $\widetilde{Q}$. If $h(\widetilde{q}_n, \widetilde{q}) \to \widetilde{0}$ as $n \to \infty$, then $\{\widetilde{q}_n\}$ is referred to as a convergent soft quasi sequence and converges to soft quasi vector $\widetilde{q} \in \widetilde{Q}$. In other words, for every $\widetilde{\varepsilon} > \widetilde{0}$, there exists $N \in \mathbb{N}$ such that the following condition applies for $n > N$,

$$\widetilde{q}_n \widetilde{\preceq} \widetilde{q} + \widetilde{q}_{1n}^{\varepsilon},\ \widetilde{q} \widetilde{\preceq} \widetilde{q}_n + \widetilde{q}_{2n}^{\varepsilon},\ \text{and } \|\widetilde{q}_{in}^{\varepsilon}\| \leq \widetilde{\varepsilon} \tag{2}$$

Example 3.2. Let $\widetilde{Q}$ be a soft normed linear space. In this case, $\widetilde{Q}$ is a soft normed quasilinear space. The partial ordering relation that gives $\widetilde{Q}$ a soft quasilinear space structure is equality. Moreover, if $\widetilde{Q}$ is a soft normed quasilinear space and every soft quasi vector $\widetilde{q}$ in $\widetilde{Q}$ has an inverse, then $\widetilde{Q}$ is called a soft normed linear space and partial order relation on $\widetilde{Q}$ turns into equality relation. Besides, $h_{\widetilde{Q}}(\widetilde{q}, \widetilde{w}) = \|\widetilde{q} - \widetilde{w}\|_{\widetilde{Q}}$.

Theorem 3.3. In a quasilinear soft normed space, the limit of a sequence is unique if it exists.

Proof.

Let $\left(\widetilde{Q}, \|.\|, P\right)$ be a soft normed quasilinear space. Suppose that $\{\widetilde{q}_n\}$ is a sequence of soft quasi vectors in $\widetilde{Q}$ such that $h(\widetilde{q}_n, \widetilde{q}) \to \widetilde{0}$ and $h(\widetilde{q}_n, \widetilde{w}) \to \widetilde{0}$ as $n \to \infty$ where $\widetilde{q} \neq \widetilde{w}$. If $h(\widetilde{q}_n, \widetilde{q}) \to \widetilde{0}$ as $n \to \infty$, then for every $\widetilde{\varepsilon} > \widetilde{0}$ there exists $N \in \mathbb{N}$ such that the following conditions are satisfied for $n > N$,

$$\widetilde{q}_n \widetilde{\preceq} \widetilde{q} + \widetilde{q}_{1n}^{\varepsilon},\ \widetilde{q} \widetilde{\preceq} \widetilde{q}_n + \widetilde{q}_{2n}^{\varepsilon},\ \text{and } \|\widetilde{q}_{in}^{\varepsilon}\| \leq \frac{\widetilde{\varepsilon}}{2}$$

Further, if $h(\widetilde{q}_n, \widetilde{w}) \to \widetilde{0}$ as $n \to \infty$, then, for every $\widetilde{\varepsilon} > \widetilde{0}$, there exists $M \in \mathbb{N}$ such that the following conditions are satisfied, for $n > M$,

$$\widetilde{q}_n \widetilde{\preceq} \widetilde{w} + \widetilde{w}_{1n}^{\varepsilon},\ \widetilde{w} \widetilde{\preceq} \widetilde{q}_n + \widetilde{w}_{2n}^{\varepsilon},\ \text{and } \|\widetilde{w}_{in}^{\varepsilon}\| \leq \frac{\widetilde{\varepsilon}}{2}$$

If we get $K = \max\{N, M\}$, then we get $\widetilde{q}_n \widetilde{\preceq} \widetilde{q} + \widetilde{q}_{1n}^{\varepsilon}$, $\widetilde{q} \widetilde{\preceq} \widetilde{q}_n + \widetilde{q}_{2n}^{\varepsilon}$, and $\|\widetilde{q}_{in}^{\varepsilon}\| \leq \frac{\widetilde{\varepsilon}}{2}$ and $\widetilde{q}_n \widetilde{\preceq} \widetilde{w} + \widetilde{w}_{1n}^{\varepsilon}$, $\widetilde{w} \widetilde{\preceq} \widetilde{q}_n + \widetilde{w}_{2n}^{\varepsilon}$, and $\|\widetilde{w}_{in}^{\varepsilon}\| \leq \frac{\widetilde{\varepsilon}}{2}$, for every $n > K$. Since $\widetilde{Q}$ is a soft normed quasilinear space, we obtain

$$\widetilde{q}\widetilde{\preceq}\widetilde{q}_n+\widetilde{q}^{\varepsilon}_{2n}\widetilde{\preceq}\widetilde{w}+\widetilde{w}^{\varepsilon}_{1n}+\widetilde{q}^{\varepsilon}_{2n}$$

and

$$\widetilde{w}\widetilde{\preceq}\widetilde{q}_n+\widetilde{w}^{\varepsilon}_{2n}\widetilde{\preceq}\widetilde{q}+\widetilde{q}^{\varepsilon}_{1n}+\widetilde{w}^{\varepsilon}_{2n}$$

for every $n>K$. Thus, we find

$$\|\widetilde{q}^{\varepsilon}_{in}+\widetilde{w}^{\varepsilon}_{in}\|\leq\widetilde{\varepsilon}$$

since $\|\widetilde{q}^{\varepsilon}_{in}\|\leq\frac{\widetilde{\varepsilon}}{2}$ and $\|\widetilde{w}^{\varepsilon}_{in}\|\leq\frac{\widetilde{\varepsilon}}{2}$, for every $\widetilde{\varepsilon}>\widetilde{0}$. This gives $\widetilde{q}\widetilde{\preceq}\widetilde{w}+\widetilde{w}^{\varepsilon}_{1n}+\widetilde{q}^{\varepsilon}_{2n}$, $\widetilde{w}\widetilde{\preceq}\widetilde{q}+\widetilde{q}^{\varepsilon}_{1n}+\widetilde{w}^{\varepsilon}_{2n}$, and $\|\widetilde{q}^{\varepsilon}_{in}+\widetilde{w}^{\varepsilon}_{in}\|\leq\widetilde{\varepsilon}$, for every $\widetilde{\varepsilon}>\widetilde{0}$. Since $\widetilde{Q}$ is a soft normed quasilinear space, we get $\widetilde{q}\widetilde{\preceq}\widetilde{w}$ and $\widetilde{w}\widetilde{\preceq}\widetilde{q}$ from Definition 2.7. Thus, we obtain $\widetilde{q}=\widetilde{w}$. This is a contradiction. In this way, the proof is complete. □

Definition 3.4. Let $\left(\widetilde{Q},\|.\|,P\right)$ be a soft normed quasilinear space and $\{\widetilde{q}_n\}$ be a sequence of soft quasi vectors in $\widetilde{Q}$. If $\{h(\widetilde{q}_n,\widetilde{q}_m):m,n\in\mathbb{N}\}$ is a bounded set, i.e., there exists $\widetilde{N}\widetilde{\succeq}\widetilde{0}$ such that $h(\widetilde{q}_n,\widetilde{q}_m)\widetilde{\preceq}\widetilde{N}$, for every $n,m\in\mathbb{N}$, then $\{\widetilde{q}_n\}$ is called a bounded soft quasi sequence in $\widetilde{Q}$.

Theorem 3.5. In a soft normed quasilinear space, every convergent sequence is bounded.

Proof.
Assume that $\{\widetilde{q}_n\}$ is a convergent sequence converging to $\widetilde{q}$ in $\widetilde{Q}$. Then, for every $\widetilde{\varepsilon}>\widetilde{0}$, there exists an $N\in\mathbb{N}$ such that, for all $n>N$, there are soft quasi vectors $\widetilde{q}^{\varepsilon}_{1n},\widetilde{q}^{\varepsilon}_{2n}\in\widetilde{Q}$ satisfying the conditions

$$\widetilde{q}_n\widetilde{\preceq}\widetilde{q}+\widetilde{q}^{\varepsilon}_{1n},\quad \widetilde{q}\widetilde{\preceq}\widetilde{q}_n+\widetilde{q}^{\varepsilon}_{2n},\quad \text{and}\quad \|\widetilde{q}^{\varepsilon}_{in}\|\leq\widetilde{\varepsilon}$$

This means $h(\widetilde{q}_n,\widetilde{q})\to\widetilde{0}$ as $n\to\infty$. For an arbitrary soft quasi element $\widetilde{q}_0\in\widetilde{Q}$, we can write

$$h(\widetilde{q}_n,\widetilde{q}_0)\widetilde{\preceq}h(\widetilde{q}_n,\widetilde{q})+h(\widetilde{q},\widetilde{q}_0)$$

from properties of Hausdorff metric. Let

$$\widetilde{M}=\max\left\{h(\widetilde{q}_1,\widetilde{q}),h(\widetilde{q}_2,\widetilde{q}),\cdots,\widetilde{1}+h(\widetilde{q},\widetilde{q}_0)\right\}$$

Then, we find $h(\widetilde{q}_n,\widetilde{q}_0)\widetilde{\preceq}\widetilde{M}$, if we take $h(\widetilde{q}_n,\widetilde{q})\widetilde{\preceq}\widetilde{1}$, for every $\widetilde{\varepsilon}>\widetilde{0}$. This gives $\widetilde{q}_n\in\widetilde{S}_{\widetilde{M}}(\widetilde{q}_0)$. □

Every bounded soft quasi-sequence is not necessarily convergent. For example, we take a bounded soft quasi sequence in soft normed quasilinear space $\widetilde{\Omega}_C(\mathbb{R})$ such that $\widetilde{q}_n(\gamma)=\{(-1)^n\}\in\Omega_C(\mathbb{R})$, for $\gamma\in P$. Clearly, this sequence is bounded in $\widetilde{\Omega}_C(\mathbb{R})$. However, $\widetilde{q}_n$ is not convergent in $\widetilde{\Omega}_C(\mathbb{R})$.

Definition 3.6. Let $\left(\widetilde{Q},\|.\|,P\right)$ be a soft normed quasilinear space and $\{\widetilde{q}_n\}$ be a sequence of soft quasi vectors in $\widetilde{Q}$. If, for every $\widetilde{\varepsilon}>\widetilde{0}$, there exists an $M\in\mathbb{N}$ such that $h(\widetilde{q}_n,\widetilde{q}_m)\preceq\widetilde{\varepsilon}$, for every $n,m>M$, then $\{\widetilde{q}_n\}$ is called a soft quasi-Cauchy sequence in $\widetilde{Q}$.

Theorem 3.7. In a soft normed quasilinear space, every convergent sequence is a Cauchy sequence.

Proof.
Assume that $\{\widetilde{q}_n\}$ is convergent to $\widetilde{q}$ in $\widetilde{Q}$. Then, for every $\widetilde{\varepsilon}>\widetilde{0}$, there exists an $N\in\mathbb{N}$ such that, for $n>N$, there are soft quasi vectors $\widetilde{q}^{\varepsilon}_{1n},\widetilde{q}^{\varepsilon}_{2n}\in\widetilde{Q}$ satisfying the conditions

$$\widetilde{q}_n\widetilde{\preceq}\widetilde{q}+\widetilde{q}^{\varepsilon}_{1n},\quad \widetilde{q}\widetilde{\preceq}\widetilde{q}_n+\widetilde{q}^{\varepsilon}_{2n},\quad \text{and}\quad \|\widetilde{q}^{\varepsilon}_{in}\|\leq\widetilde{\varepsilon}$$

Then, we clearly get $\widetilde{w}^{\varepsilon}_{1n},\widetilde{w}^{\varepsilon}_{2n}\in\widetilde{Q}$ such that $\widetilde{q}_m\widetilde{\preceq}\widetilde{q}+\widetilde{w}^{\varepsilon}_{1n}$, $\widetilde{q}\widetilde{\preceq}\widetilde{q}_m+\widetilde{w}^{\varepsilon}_{2n}$, and $\|\widetilde{w}^{\varepsilon}_{in}\|\leq\widetilde{\varepsilon}$, for all $m>N$. Similar to the proof of Theorem 3.5, if we get $K=\max\{N,M\}$, then

$$\widetilde{q}_n\widetilde{\preceq}\widetilde{q}+\widetilde{q}^{\varepsilon}_{1n}\widetilde{\preceq}\widetilde{q}_m+\widetilde{w}^{\varepsilon}_{2n}+\widetilde{q}^{\varepsilon}_{1n}$$

and

$$\widetilde{q}_m \widetilde{\preceq} \widetilde{q} + \widetilde{w}^{\varepsilon}_{1n} \widetilde{\preceq} \widetilde{q}_n + \widetilde{q}^{\varepsilon}_{2n} + \widetilde{w}^{\varepsilon}_{1n}$$

for every $n, m > K$. Moreover, if we take $\widetilde{2} \cdot \widetilde{\varepsilon} = \widetilde{\varepsilon}'$, then we have $\|\widetilde{q}^{\varepsilon}_{in} + \widetilde{w}^{\varepsilon}_{in}\| \leq \widetilde{\varepsilon}'$ since $\|\widetilde{q}^{\varepsilon}_{in}\| \leq \widetilde{\varepsilon}$ and $\|\widetilde{w}^{\varepsilon}_{in}\| \leq \widetilde{\varepsilon}$. This gives that every $\widetilde{\varepsilon}' > \widetilde{0}$, there exists a $K \in \mathbb{N}$ such that $h(\widetilde{q}_n, \widetilde{q}_m) \preceq \widetilde{\varepsilon}$, for every $n, m > K$. $\square$

The converse of Theorem 3.7 is not always correct.

Theorem 3.8. If there is a convergent soft quasi subsequence of a soft quasi-Cauchy sequence in a soft normed quasilinear space, this soft quasi-Cauchy sequence converges to the soft quasi vector at which the soft quasi-subsequence converges.

PROOF.
Let $\{\widetilde{q}_n\}$ be a soft quasi-Cauchy sequence in $\widetilde{Q}$. Then, for every $\widetilde{\epsilon} > \widetilde{0}$, there exists an $N \in \mathbb{N}$ such that, for $n, m > N$,

$$\widetilde{q}_n \widetilde{\preceq} \widetilde{q}_m + \widetilde{q}^{\epsilon}_{1n}, \quad \widetilde{q}_m \widetilde{\preceq} \widetilde{q}_n + \widetilde{q}^{\epsilon}_{2n}, \quad \text{and} \quad \|\widetilde{q}^{\epsilon}_{in}\| \leq \frac{\widetilde{\epsilon}}{2}$$

We define a convergent soft quasi subsequence of $\{\widetilde{q}_n\}$ with $\{\widetilde{q}_{n_k}\}$ and $\widetilde{q}_{n_k} \to \widetilde{q}$ as $n \to \infty$. Since $\{\widetilde{q}_n\}$ is a soft quasi-Cauchy sequence and $\{\widetilde{q}_{n_k}\}$ is a soft quasi subsequence of $\{\widetilde{q}_n\}$, then

$$\widetilde{q}_{n_m} \widetilde{\preceq} \widetilde{q}_n + \widetilde{k}^{\epsilon}_{1n}, \quad \widetilde{q}_n \widetilde{\preceq} \widetilde{q}_{n_m} + \widetilde{k}^{\epsilon}_{2n}, \quad \text{and} \quad \|\widetilde{k}^{\epsilon}_{in}\| \leq \frac{\widetilde{\epsilon}}{2}$$

Moreover, since $\widetilde{q}_{n_k} \to \widetilde{q}$ as $n \to \infty$, then, for $n, m > N$ and for every $\widetilde{\epsilon} > \widetilde{0}$, there exists an $N \in \mathbb{N}$ such that, for all $n > N$,

$$\widetilde{q}_{n_m} \widetilde{\preceq} \widetilde{q} + \widetilde{l}^{\epsilon}_{1n}, \quad \widetilde{q} \widetilde{\preceq} \widetilde{q}_{n_m} + \widetilde{l}^{\epsilon}_{2n}, \quad \text{and} \quad \|\widetilde{l}^{\epsilon}_{in}\| \leq \frac{\widetilde{\epsilon}}{2}$$

Then, from the above two inequalities,

$$\widetilde{q} \widetilde{\preceq} \widetilde{q}_n + \widetilde{l}^{\epsilon}_{2n} + \widetilde{k}^{\epsilon}_{1n} \quad \text{and} \quad \widetilde{q}_n \widetilde{\preceq} \widetilde{q} + \widetilde{l}^{\epsilon}_{1n} + \widetilde{k}^{\epsilon}_{2n}$$

for all $n > N$. Further, we find $\left\|\widetilde{l}^{\epsilon}_{1n} + \widetilde{k}^{\epsilon}_{2n}\right\| \leq \widetilde{\epsilon}$. This gives $\widetilde{q}_n \to \widetilde{q}$ as $n \to \infty$. $\square$

Theorem 3.9. In a soft normed quasilinear space, every Cauchy sequence is a bounded soft quasi-sequence.

PROOF.
Let $\{\widetilde{q}_n\}$ be a soft quasi-Cauchy sequence in $\widetilde{Q}$. Then, there exists $\widetilde{N} \widetilde{\succeq} \widetilde{0}$ such that $h(\widetilde{q}_k, \widetilde{q}_l) \widetilde{\preceq} \widetilde{1}$, for every $k, l > N$. If we take $\widetilde{K}(\gamma) = \max\limits_{1 \leq k,l \leq m} \{h(\widetilde{q}_k, \widetilde{q}_l)(\gamma)\}$, for all $\gamma \in P$, then

$$\begin{aligned} h(\widetilde{q}_k, \widetilde{q}_l)(\gamma) &\preceq h(\widetilde{q}_k, \widetilde{q}_m)(\gamma) + h(\widetilde{q}_m, \widetilde{q}_l)(\gamma) \\ &\preceq \widetilde{K}(\gamma) + \widetilde{1}(\gamma) \\ &= \left(\widetilde{K} + \widetilde{1}\right)(\gamma) \end{aligned}$$

for $1 \leq k \leq m$ and $l \geq m$. Thus, we find $h(\widetilde{q}_k, \widetilde{q}_l) \leq \left(\widetilde{K} + \widetilde{1}\right)$, for every $k, l \in \mathbb{N}$. This gives that $\{\widetilde{q}_n\}$ is a bounded soft quasi sequence in $\widetilde{Q}$. $\square$

Definition 3.10. Let $\left(\widetilde{Q}, \|.\|, P\right)$ be a soft normed quasilinear space and (S, P) be a soft quasi subset in $\widetilde{Q}$ such that $S(\gamma) \neq \emptyset$, for every $\gamma \in P$. If there exists a soft real number $\widetilde{m}$ such that $\|\widetilde{q}\| \leq \widetilde{m}$, for every $\widetilde{q} \in \widetilde{S}$, then the soft quasi subset (S, P) is referred to as bounded in $\widetilde{Q}$.

Example 3.11. Let $B = \{\widetilde{q} : \widetilde{q}(\gamma) \subseteq [0,1], \gamma \in P\}$, a soft quasi subset of soft quasilinear space $\widetilde{\Omega}_C(\mathbb{R})$ with $\|\widetilde{q}\| = \sup \|\widetilde{q}(\gamma)\|_{\Omega_C(\mathbb{R})}$. Then, the soft quasi subset B is bounded since

$$\|\widetilde{q}\| = \sup \|\widetilde{q}(\gamma)\|_{\Omega_C(\mathbb{R})} \leq \sup \|[0,1]\|_{\Omega_C(\mathbb{R})} = 1$$

Definition 3.12. Let $\left(\widetilde{Q}, \|.\|, P\right)$ be a soft normed quasilinear space. If every soft quasi-Cauchy sequences in $\widetilde{Q}$ converges to a soft quasi element in $\widetilde{Q}$, then $\widetilde{Q}$ is called a complete soft normed quasilinear space. Generally, a soft quasilinear Banach space can be described as a complete soft normed quasilinear space.

Theorem 3.13. Let $\left(\widetilde{Q}, \|.\|, P\right)$ be a soft normed quasilinear space. Then, the following statements are valid:

i. If $\widetilde{q}_n \longrightarrow \widetilde{q}$ and $\widetilde{w}_n \longrightarrow \widetilde{w}$, then $\widetilde{q}_n + \widetilde{w}_n \longrightarrow \widetilde{q} + \widetilde{w}$, i.e., according to the Hausdorff metric, the algebraic sum is continuous.

ii. If $\widetilde{q}_n \longrightarrow \widetilde{q}$ and $\widetilde{\gamma}_n \longrightarrow \widetilde{\gamma}$, then $\widetilde{\gamma}_n \cdot \widetilde{q}_n \longrightarrow \widetilde{\gamma} \cdot \widetilde{q}$, i.e., according to the Hausdorff metric, multiplication by soft real numbers is continuous. The sequence $\widetilde{\gamma}_n$ consists of soft scalars.

PROOF.
Let $\left(\widetilde{Q}, \|.\|, P\right)$ be a soft normed quasilinear space.

i. Suppose that $\widetilde{q}_n \longrightarrow \widetilde{q}$ and $\widetilde{w}_n \longrightarrow \widetilde{w}$. Then, for every $\widetilde{\varepsilon} > \widetilde{0}$, there exists an $N \in \mathbb{N}$ such that, for all $n > N$,

$$\widetilde{q}_n \widetilde{\preceq} \widetilde{q} + \widetilde{q}_{1n}^{\varepsilon}, \quad \widetilde{q} \widetilde{\preceq} \widetilde{q}_n + \widetilde{q}_{2n}^{\varepsilon}, \quad \text{and} \quad \|\widetilde{q}_{in}^{\varepsilon}\| \leq \widetilde{\varepsilon}$$

and

$$\widetilde{w}_n \widetilde{\preceq} \widetilde{w} + \widetilde{w}_{1n}^{\varepsilon}, \quad \widetilde{w} \widetilde{\preceq} \widetilde{w}_n + \widetilde{w}_{2n}^{\varepsilon}, \quad \text{and} \quad \|\widetilde{w}_{in}^{\varepsilon}\| \leq \widetilde{\varepsilon}$$

Therefore,

$$\widetilde{q}_n + \widetilde{w}_n \widetilde{\preceq} \widetilde{q} + \widetilde{w} + \widetilde{q}_{1n}^{\varepsilon} + \widetilde{w}_{1n}^{\varepsilon}, \quad \widetilde{q} + \ \widetilde{w} \widetilde{\preceq} \widetilde{q}_n + \widetilde{w}_n + \widetilde{q}_{2n}^{\varepsilon} + \widetilde{w}_{2n}^{\varepsilon}, \quad \text{and} \quad \|\widetilde{q}_{in}^{\varepsilon} + \widetilde{w}_{in}^{\varepsilon}\| \leq \widetilde{\varepsilon}'$$

such that $\widetilde{\varepsilon}' = \widetilde{2}\widetilde{\varepsilon}$. Thus, $\widetilde{q}_n + \widetilde{w}_n \longrightarrow \widetilde{q} + \widetilde{w}$.

Similarly, it can be demonstrated that soft real number multiplication is continuous. □

Theorem 3.14. Let $\left(\widetilde{Q}, \|.\|, P\right)$ be a soft normed quasilinear space. The soft quasi-norm is continuous according to the Hausdorff metric.

PROOF.
Suppose that $\widetilde{q}_n \longrightarrow \widetilde{q}$. Then, for every $\widetilde{\varepsilon} > \widetilde{0}$, there exists an $N \in \mathbb{N}$ such that, for all $n > N$,

$$\widetilde{q}_n \widetilde{\preceq} \widetilde{q} + \widetilde{q}_{1n}^{\varepsilon}, \quad \widetilde{q} \widetilde{\preceq} \widetilde{q}_n + \widetilde{q}_{2n}^{\varepsilon}, \quad \text{and} \quad \|\widetilde{q}_{in}^{\varepsilon}\| \leq \widetilde{\varepsilon}$$

Since $\widetilde{Q}$ is a soft normed quasilinear space with $\|.\|$,

$$\|\widetilde{q}_n\| \leq \|\widetilde{q}\| + \|\widetilde{q}_{1n}^{\varepsilon}\| \quad \text{and} \quad \|\widetilde{q}\| \widetilde{\leq} \|\widetilde{q}_n\| + \|\widetilde{q}_{2n}^{\varepsilon}\|$$

This gives $\|\widetilde{q}_n\| \to \|\widetilde{q}\|$ as $n \to \infty$ because $\|\widetilde{q}_{in}^{\varepsilon}\| \leq \widetilde{\varepsilon}$. □

Theorem 3.15. Let $\left(\widetilde{Q}, \|.\|, P\right)$ be a soft normed quasilinear space and $\{\widetilde{q}_n\}$ and $\{\widetilde{w}_n\}$ be two soft quasi-Cauchy sequences in $\widetilde{Q}$. Then, $\{\widetilde{q}_n + \widetilde{w}_n\}$ is soft quasi-Cauchy sequence in $\widetilde{Q}$.

PROOF.
Let $\{\widetilde{q}_n\}$ and $\{\widetilde{w}_n\}$ be two soft quasi-Cauchy sequences in $\widetilde{Q}$. Then, for every $\widetilde{\varepsilon} > \widetilde{0}$, there exist

$N, M \in \mathbb{N}$ such that, for all $n, m > N$ and $n, m > M$,

$$\widetilde{q}_n \widetilde{\preceq} \widetilde{q}_m + \widetilde{q}_{1n}^{\varepsilon}, \quad \widetilde{q}_m \widetilde{\preceq} \widetilde{q}_n + \widetilde{q}_{2n}^{\varepsilon}, \quad \text{and} \quad \|\widetilde{q}_{in}^{\varepsilon}\| \leq \frac{\widetilde{\varepsilon}}{2}$$

and

$$\widetilde{w}_n \widetilde{\preceq} \widetilde{w}_m + \widetilde{w}_{1n}^{\varepsilon}, \quad \widetilde{w}_m \widetilde{\preceq} \widetilde{w}_n + \widetilde{w}_{2n}^{\varepsilon}, \quad \text{and} \quad \|\widetilde{w}_{in}^{\varepsilon}\| \leq \frac{\widetilde{\varepsilon}}{2}$$

If we take $K = \max\{N, M\}$, then, for every $\widetilde{\varepsilon} > \widetilde{0}$, there exists a $K \in \mathbb{N}$ such that, for all $n, m > K$,

$$\widetilde{q}_n + \widetilde{w}_n \widetilde{\preceq} \widetilde{q}_m + \widetilde{w}_m + \widetilde{q}_{1n}^{\varepsilon} + \widetilde{w}_{1n}^{\varepsilon} \quad \text{and} \quad \widetilde{q}_m + \ \widetilde{w}_m \widetilde{\preceq} \widetilde{q}_n + \widetilde{w}_n + \widetilde{q}_{2n}^{\varepsilon} + \widetilde{w}_{2n}^{\varepsilon}$$

Moreover, $\|\widetilde{q}_{in}^{\varepsilon} + \widetilde{w}_{in}^{\varepsilon}\| \leq \widetilde{\varepsilon}$ since $\|\widetilde{q}_{in}^{\varepsilon}\| \leq \frac{\widetilde{\varepsilon}}{2}$ and $\|\widetilde{w}_{in}^{\varepsilon}\| \leq \frac{\widetilde{\varepsilon}}{2}$. This gives $\{\widetilde{q}_n + \widetilde{w}_n\}$ is a soft quasi-Cauchy sequence in $\widetilde{Q}$. $\square$

Theorem 3.16. Let $\left(\widetilde{Q}, \|.\|, P\right)$ be a soft normed quasilinear space. The following statements are provided:

i. Assume that $\widetilde{q}_n \to \widetilde{q}$ and $\widetilde{w}_n \to \widetilde{w}$. If $\widetilde{q}_n \widetilde{\preceq}\ \widetilde{w}_n$, for every $n \in \mathbb{N}$, then $\widetilde{q} \widetilde{\preceq} \widetilde{w}$.

ii. Assume that $\widetilde{q}_n \to \widetilde{q}$ and $\widetilde{w}_n \to \widetilde{q}$. If $\widetilde{q}_n \widetilde{\preceq} \widetilde{m}_n \widetilde{\preceq}\ \widetilde{w}_n$, for every $n \in \mathbb{N}$, then $\widetilde{m}_n \to \widetilde{q}$.

iii. If $\widetilde{q}_n +\ \widetilde{w}_n \to \widetilde{q}$ and $\widetilde{w}_n \to \widetilde{\theta}$, then $\widetilde{q}_n \to \widetilde{q}$.

Proof.
Let $\left(\widetilde{Q}, \|.\|, P\right)$ be a soft normed quasilinear space.

i. Let $\widetilde{q}_n \to \widetilde{q}$, $\widetilde{w}_n \to \widetilde{w}$, and $\widetilde{q}_n \widetilde{\preceq}\ \widetilde{w}_n$, for every $n \in \mathbb{N}$. Then, for every $\widetilde{\varepsilon} > \widetilde{0}$, there exists an $N \in \mathbb{N}$ such that, for all $n > N$,

$$\widetilde{q}_n \widetilde{\preceq} \widetilde{q} + \widetilde{q}_{1n}^{\varepsilon}, \quad \widetilde{q} \widetilde{\preceq} \widetilde{q}_n + \widetilde{q}_{2n}^{\varepsilon}, \quad \text{and} \quad \|\widetilde{q}_{in}^{\varepsilon}\| \leq \frac{\widetilde{\varepsilon}}{2}$$

and

$$\widetilde{w}_n \widetilde{\preceq} \widetilde{w} + \widetilde{w}_{1n}^{\varepsilon}, \quad \widetilde{w} \widetilde{\preceq} \widetilde{w}_n + \widetilde{w}_{2n}^{\varepsilon}, \quad \text{and} \quad \|\widetilde{w}_{in}^{\varepsilon}\| \leq \frac{\widetilde{\varepsilon}}{2}$$

Moreover, since $\widetilde{q}_n \widetilde{\preceq}\ \widetilde{w}_n$, for every $n \in \mathbb{N}$,

$$\widetilde{q} \widetilde{\preceq} \widetilde{q}_n + \widetilde{q}_{2n}^{\varepsilon} \widetilde{\preceq} \widetilde{w}_n + \widetilde{q}_{2n}^{\varepsilon} \widetilde{\preceq} \widetilde{w} + \widetilde{q}_{2n}^{\varepsilon} + \widetilde{w}_{1n}^{\varepsilon}$$

Further,

$$\|\widetilde{q}_{2n}^{\varepsilon} + \widetilde{w}_{1n}^{\varepsilon}\| \leq \widetilde{\varepsilon}$$

This gives $\widetilde{q} \widetilde{\preceq} \widetilde{w}$ because $\widetilde{Q}$ is a soft normed quasilinear space.

ii. Let $\widetilde{q}_n \to \widetilde{q}$, $\widetilde{w}_n \to \widetilde{q}$, and $\widetilde{q}_n \widetilde{\preceq} \widetilde{m}_n \widetilde{\preceq}\ \widetilde{w}_n$, for every $n \in \mathbb{N}$. Then, for every $\widetilde{\varepsilon} > \widetilde{0}$, there exists an $N \in \mathbb{N}$ such that, for all $n > N$,

$$\widetilde{q}_n \widetilde{\preceq} \widetilde{q} + \widetilde{q}_{1n}^{\varepsilon}, \quad \widetilde{q} \widetilde{\preceq} \widetilde{q}_n + \widetilde{q}_{2n}^{\varepsilon}, \quad \text{and} \quad \|\widetilde{q}_{in}^{\varepsilon}\| \leq \widetilde{\varepsilon}$$

and

$$\widetilde{w}_n \widetilde{\preceq} \widetilde{w} + \widetilde{w}_{1n}^{\varepsilon}, \quad \widetilde{w} \widetilde{\preceq} \widetilde{w}_n + \widetilde{w}_{2n}^{\varepsilon}, \quad \text{and} \quad \|\widetilde{w}_{in}^{\varepsilon}\| \leq \widetilde{\varepsilon}$$

Moreover, since $\widetilde{q}_n \widetilde{\preceq}\ \widetilde{m}_n$ for every $n \in \mathbb{N}$,

$$\widetilde{q} \widetilde{\preceq} \widetilde{m}_n + \widetilde{q}_{2n}^{\varepsilon}$$

Further, as $\widetilde{m}_n \widetilde{\preceq}\ \widetilde{w}_n$ for every $n \in \mathbb{N}$,

$$\widetilde{m}_n \widetilde{\preceq} \widetilde{w} + \widetilde{w}_{1n}^{\varepsilon}$$

Besides, because $\|\widetilde{q}_{2n}^{\varepsilon}\| \leq \widetilde{\varepsilon}$ and $\|\widetilde{w}_{1n}^{\varepsilon}\| \leq \widetilde{\varepsilon}$, for every $n \in \mathbb{N}$, $\widetilde{m}_n \to \widetilde{q}$ as $n \to \infty$.

iii. Let $\widetilde{q}_n + \widetilde{w}_n \to \widetilde{q}$ and $\widetilde{w}_n \to \widetilde{\theta}$. Then, for every $\widetilde{\varepsilon} > \widetilde{0}$, there exists an $N \in \mathbb{N}$ such that, for all $n > N$,

$$\widetilde{q}_n + \widetilde{w}_n \widetilde{\preceq} \widetilde{q} + \widetilde{q}^{\varepsilon}_{1n}, \quad \widetilde{q} \widetilde{\preceq} \widetilde{q}_n + \widetilde{w}_n + \widetilde{q}^{\varepsilon}_{2n}, \quad \text{and} \quad \|\widetilde{q}^{\varepsilon}_{in}\| \leq \frac{\widetilde{\varepsilon}}{2}$$

and

$$\widetilde{w}_n \widetilde{\preceq} \widetilde{\theta} + \widetilde{w}^{\varepsilon}_{1n}, \quad \widetilde{\theta} \widetilde{\preceq} \widetilde{w}_n + \widetilde{w}^{\varepsilon}_{2n}, \quad \text{and} \quad \|\widetilde{w}^{\varepsilon}_{in}\| \leq \frac{\widetilde{\varepsilon}}{2}$$

From these above relations,

$$\widetilde{q}_n + \widetilde{\theta} \widetilde{\preceq} \widetilde{q}_n + \widetilde{w}_n + \widetilde{w}^{\varepsilon}_{2n} \widetilde{\preceq} \widetilde{q} + \widetilde{q}^{\varepsilon}_{1n} + \widetilde{w}^{\varepsilon}_{2n}$$

and

$$\widetilde{q} \widetilde{\preceq} \widetilde{q}_n + \widetilde{w}_n + \widetilde{q}^{\varepsilon}_{2n} \widetilde{\preceq} \widetilde{q}_n + \widetilde{\theta}_n + \widetilde{q}^{\varepsilon}_{2n} + \widetilde{w}^{\varepsilon}_{1n}$$

Besides, $\|\widetilde{q}^{\varepsilon}_{1n} + \widetilde{w}^{\varepsilon}_{2n}\| \leq \widetilde{\varepsilon}$ and $\|\widetilde{q}^{\varepsilon}_{2n} + \widetilde{w}^{\varepsilon}_{1n}\| \leq \widetilde{\varepsilon}$ since $\|\widetilde{q}^{\varepsilon}_{in}\| \leq \frac{\widetilde{\varepsilon}}{2}$ and $\|\widetilde{w}^{\varepsilon}_{in}\| \leq \frac{\widetilde{\varepsilon}}{2}$. This gives $\widetilde{q}_n \to \widetilde{q}$ as $n \to \infty$. □

4. Some New Results Concerning to Soft Quasi Subspaces of Soft Normed Quasilinear Space

In this section, we provide some results on soft quasilinear subspaces of soft normed quasilinear spaces. Further, we define the regular and singular soft quasi vectors of a soft quasilinear space and exemplify them.

Lemma 4.1. Let $\widetilde{W}$ and $\widetilde{Z}$ be soft closed subspaces of a soft normed quasilinear space $\widetilde{Q}$ satisfying Condition 1 and $\widetilde{W}$ be a closed proper subset of the subspace $\widetilde{Z}$. Then, for $\widetilde{\varepsilon} \widetilde{\geq} \widetilde{0}$, there exists $\widetilde{z} \in \widetilde{Q} \backslash \widetilde{W}$ with $\|\widetilde{z}\| \widetilde{\geq} \widetilde{1}$ such that, for all $\widetilde{w} \in \widetilde{W}$, the inequality $\|\widetilde{z} - \widetilde{w}\| \widetilde{\geq} \widetilde{1} - \widetilde{\varepsilon}$ is satisfied.

Proof.
Suppose that $\widetilde{\varepsilon} \widetilde{\geq} \widetilde{0}$ and $\widetilde{\varepsilon}(\gamma) = \varepsilon_\gamma > 0$, for every parameter $\gamma \in P$. Since $\widetilde{Q}$ satisfies Condition 1, $\widetilde{Z}(\gamma) = Z_\gamma$ is a closed subspace of the normed quasilinear space Q such that $\widetilde{Q}(\gamma) = Q$. From Reisz's Lemma for normed quasilinear space provided in [4], there exists $\widetilde{z}(\gamma) \in Q \backslash Z_\gamma$ with $\|\widetilde{z}(\gamma)\|_\gamma \geq 1$ such that for all $\widetilde{w}(\gamma) \in W_\gamma$ the inequality

$$\|\widetilde{z}(\gamma) - \widetilde{w}(\gamma)\|_\gamma \geq 1 - \varepsilon_\gamma$$

is satisfied. This gives that, for $\widetilde{\varepsilon} \widetilde{\geq} \widetilde{0}$, there exists $\widetilde{z} \in \widetilde{Q} \backslash \widetilde{W}$ with $\|\widetilde{z}\| \widetilde{\geq} \widetilde{1}$ such that, for all $\widetilde{w} \in \widetilde{W}$, the inequality $\|\widetilde{z} - \widetilde{w}\| \widetilde{\geq} \widetilde{1} - \widetilde{\varepsilon}$ is satisfied. □

Lemma 4.2. Let $\widetilde{Q}$ be a soft quasilinear space. Then, in the soft quasilinear space $\widetilde{Q}$, the soft element $\widetilde{\theta}$ is minimal in $\widetilde{Q}$, i.e., $\widetilde{q} = \widetilde{\theta}$ if $\widetilde{q} \widetilde{\preceq} \widetilde{\theta}$.

Proof.
Assume that $\widetilde{q}$ is a soft quasi vector in $\widetilde{Q}$ and $\widetilde{q} \widetilde{\preceq} \widetilde{\theta}$. Since $(\widetilde{-1})\widetilde{q} \widetilde{\preceq} (\widetilde{-1})\widetilde{q}$, for soft scalar $\widetilde{-1}$, and $\widetilde{Q}$ is a soft quasilinear space, then

$$\widetilde{q} + (\widetilde{-1})\widetilde{q} \widetilde{\preceq} \widetilde{\theta} + (\widetilde{-1})\widetilde{q} = (\widetilde{-1})\widetilde{q}$$

Further,

$$\widetilde{\theta} = \left(\widetilde{1} + (\widetilde{-1})\right)\widetilde{q} = \widetilde{q} + (\widetilde{-1})\widetilde{q} \widetilde{\preceq} \widetilde{\theta} + (\widetilde{-1})\widetilde{q} = (\widetilde{-1})\widetilde{q}$$

from properties of soft quasilinear space. Thus, $(\widetilde{-1})\widetilde{\theta} \widetilde{\preceq} (\widetilde{-1})\left((\widetilde{-1})\widetilde{q}\right) = \widetilde{q}$. Moreover, $(\widetilde{-1})\widetilde{\theta} = \widetilde{\theta}$. Therefore, $\widetilde{\theta} \widetilde{\preceq} \widetilde{q}$. This gives $\widetilde{q} = \widetilde{\theta}$. Consequently, the soft element $\widetilde{\theta}$ is minimal in soft quasilinear space $\widetilde{Q}$. □

Definition 4.3. Let $\widetilde{Q}$ be a soft quasilinear space. A soft quasi vector $\widetilde{q}^{'} \in \widetilde{Q}$ is named an inverse of a soft quasi vector $\widetilde{q} \in \widetilde{Q}$ if $\widetilde{q} + \widetilde{q}^{'} = \widetilde{\theta}$. The inverse of a soft quasi vector is unique if there exists.

Lemma 4.4. If any soft quasi vector in the soft quasilinear space $\widetilde{Q}$ has an inverse soft quasi vector in $\widetilde{Q}$, then the partial order relation in $\widetilde{Q}$ is achieved through equality. As a result, the distributive property is valid. Therefore, $\widetilde{Q}$ is a soft linear space.

Proof.
The proof is similar to the quasilinear spaces if take as $\widetilde{q}(\gamma) = q$ and $\widetilde{q}^{'}(\gamma) = q^{'}$, for all parameter γ. □

Definition 4.5. In a soft quasilinear space $\widetilde{Q}$, a soft quasi vector with an inverse is called a regular soft quasi vector, and a soft quasi vector without an inverse is called a singular soft quasi vector. The set of all the regular and singular soft quasi vectors of $\widetilde{Q}$ is denoted by $\widetilde{Q}_r$ and $\widetilde{Q}_s$, respectively.

Here, the subspace of all the regular soft quasi vectors of the soft quasilinear space $\widetilde{Q}$ is called the soft regular subspace of $\widetilde{Q}$. Similarly, The subspace of all the singular soft quasi vectors of the soft quasilinear space $\widetilde{Q}$ is called the soft singular subspace of $\widetilde{Q}$.

Definition 4.6. Let $\widetilde{Q}$ be a soft quasilinear space and $\widetilde{W} \subseteq \widetilde{Q}$. If $\widetilde{W}$ is a soft quasilinear space with the same operations in $\widetilde{Q}$ and the same partial order relation in $\widetilde{Q}$, then $\widetilde{W}$ is called a soft sub-quasilinear space of $\widetilde{Q}$.

Theorem 4.7. Let $\widetilde{Q}$ be a soft quasilinear space and $\widetilde{W} \subseteq \widetilde{Q}$. Then, $\widetilde{W}$ is a soft sub-quasilinear space of $\widetilde{Q}$ if and only if $\widetilde{\alpha}\widetilde{w}_1 + \widetilde{\beta}\widetilde{w}_2 \in \widetilde{W}$, for every soft quasi vector $\widetilde{w}_1, \widetilde{w}_2 \in \widetilde{W}$ and soft scalars $\widetilde{\alpha}, \widetilde{\beta}$.

Proof.
The theorem can be proved in a similar way to that of soft linear spaces. □

Example 4.8. Consider the absolute soft quasi set generated by $\Omega_C(\mathbb{R})$ and defined by $\widetilde{\Omega}_C(\mathbb{R})$, i.e., $\widetilde{\Omega}_C(\mathbb{R})\ (\gamma) = \Omega_C(\mathbb{R})$, for every $\gamma \in P$. Let

$$\widetilde{W} = \{\widetilde{w} : \widetilde{w}(\gamma) = [a, b],\ a, b \in \mathbb{R},\ a < b,\ \text{and}\ \gamma \in P\} \cup \left\{\widetilde{0}\right\}$$

Clearly, $\widetilde{W}$ consists of all the soft quasi vectors in which image is a singular element of $\Omega_C(\mathbb{R})$ under the parameter γ. Since

$$\left(\widetilde{\alpha}\widetilde{w}_1 + \widetilde{\beta}\widetilde{w}_2\right)(\gamma) = \widetilde{\alpha}(\gamma)\,\widetilde{w}_1(\gamma) + \widetilde{\beta}(\gamma)\,\widetilde{w}_2(\gamma) = \alpha\widetilde{w}_1(\gamma) + \beta\widetilde{w}_2(\gamma) \in \Omega_C(\mathbb{R})$$

for every soft quasi vectors $\widetilde{w}_1, \widetilde{w}_2 \in \widetilde{W}$ and soft scalars $\widetilde{\alpha}, \widetilde{\beta}$, then $\widetilde{W}$ is a soft subquasilinear space of $\widetilde{\Omega}_C(\mathbb{R})$. Moreover, by Definition 4.5, we get $\widetilde{W}$ is a soft singular subspace of $\widetilde{\Omega}_C(\mathbb{R})$. For another soft quasi set

$$\widetilde{M} = \{\widetilde{m} : \widetilde{m}(\gamma) = \{m\} \in \mathbb{R}, \forall\gamma \in P\}$$

$\widetilde{M}$ is a soft subspace of $\widetilde{\Omega}_C(\mathbb{R})$ since

$$\left(\widetilde{\alpha}\widetilde{m}_1 + \widetilde{\beta}\widetilde{m}_2\right)(\gamma) = \widetilde{\alpha}(\gamma)\,\widetilde{m}_1(\gamma) + \widetilde{\beta}(\gamma)\,\widetilde{m}_2(\gamma) = \alpha m_1 + \beta m_2 \in \mathbb{R}$$

for every soft quasi vectors $\widetilde{m}_1, \widetilde{m}_2 \in \widetilde{M}$ and soft scalars $\widetilde{\alpha}, \widetilde{\beta}$. Further, every soft quasi vector $\widetilde{m} \in \widetilde{M}$ has an inverse because $\widetilde{\Omega}_C(\mathbb{R})$ is an absolute soft quasilinear space. Thus, $\widetilde{M}$ is a soft regular subspace of $\widetilde{\Omega}_C(\mathbb{R})$.

Theorem 4.9. Every regular soft quasi vector in a soft quasilinear space $\widetilde{Q}$ is minimal.

PROOF.
Let $\widetilde{q} \in \widetilde{Q}_r$ be an arbitrary soft quasi vector and $\widetilde{w} \widetilde{\preceq} \widetilde{q}$, for any $\widetilde{w} \in \widetilde{Q}$. Then,

$$\widetilde{w} + \widetilde{q}^{'} \widetilde{\preceq} \widetilde{q} + \widetilde{q}^{'} = \widetilde{\theta}$$

since $\widetilde{q}$ is a soft quasi-regular vector in $\widetilde{Q}$. From Lemma 4.2, $\widetilde{w} + \widetilde{q}^{'} = \widetilde{\theta}$. Thus, $\widetilde{w} = \widetilde{q}$ because the inverse of a soft quasi vector is unique if there exists. Hence, an arbitrary soft quasi vector $\widetilde{q}$ in $\widetilde{Q}$ is minimal. □

Theorem 4.10. Let $\widetilde{Q}$ be a soft normed quasilinear space. Then, the soft quasi set $\widetilde{Q}_r$ is a closed subspace of $\widetilde{Q}$.

PROOF.
Let $\widetilde{q}, \widetilde{w} \in \widetilde{Q}_r$ and soft scalars $\widetilde{\alpha}, \widetilde{\beta} \in \mathbb{R}(P)$. It is necessary to prove that $\widetilde{\alpha}\widetilde{q} + \widetilde{\beta}\widetilde{w} \in \widetilde{Q}_r$ to show that $\widetilde{Q}_r$ is a subspace of $\widetilde{Q}$. As $\widetilde{q}, \widetilde{w} \in \widetilde{Q}_r$, there exist $\widetilde{q}^{'}, \widetilde{w}^{'} \in \widetilde{Q}$ such that $\widetilde{q} + \widetilde{q}^{'} = \widetilde{\theta}$ and $\widetilde{w} + \widetilde{w}^{'} = \widetilde{\theta}$. Since $\widetilde{Q}$ is a soft normed quasilinear space,

$$\widetilde{\alpha}\widetilde{q} + \widetilde{\beta}\widetilde{w} + \widetilde{\alpha}\widetilde{q}^{'} + \widetilde{\beta}\widetilde{w}^{'} = \widetilde{\alpha}\left(\widetilde{q} + \widetilde{q}^{'}\right) + \widetilde{\beta}\left(\widetilde{w} + \widetilde{w}^{'}\right) = \widetilde{\theta}$$

This gives $\widetilde{\alpha}\widetilde{q} + \widetilde{\beta}\widetilde{w} \in \widetilde{Q}_r$.

The soft quasi sequence $\{\widetilde{q}_n\}$ in $\widetilde{Q}_r$ converges to $\widetilde{q} \in \widetilde{Q}$, i.e., $\widetilde{q}_n \to \widetilde{q} \in \widetilde{Q}$ as $n \to \infty$. Since $\widetilde{Q}$ is a soft normed quasilinear space, $-\widetilde{q}_n \to -\widetilde{q}$ as $n \to \infty$. Hence, $\widetilde{q}_n - \widetilde{q}_n \to \widetilde{q} - \widetilde{q}$. Since $\widetilde{q}_n \in \widetilde{Q}_r$, $\widetilde{q}_n - \widetilde{q}_n = \widetilde{\theta}$ and then $\widetilde{q} - \widetilde{q} = \widetilde{\theta}$. This gives $\widetilde{q} \in \widetilde{Q}_r$. Therefore, $\widetilde{Q}_r$ is a closed subspace of $\widetilde{Q}$. □

Theorem 4.11. Let $\widetilde{Q}$ be a soft quasilinear space and $\widetilde{q}, \widetilde{w} \in \widetilde{Q}$. If $\widetilde{q} + \widetilde{w} \in \widetilde{Q}_r$, then $\widetilde{q} \in \widetilde{Q}_r$ and $\widetilde{w} \in \widetilde{Q}_r$.

PROOF.
Assume that $\widetilde{q} + \widetilde{w} \in \widetilde{Q}_r$ and $\widetilde{q}$ are not soft quasi-regular vectors of $\widetilde{Q}$. Then, there exists a soft quasi vector $\widetilde{m} \in \widetilde{Q}_r$ such that $(\widetilde{q} + \widetilde{w}) + \widetilde{m} = \widetilde{\theta}$. Thus, $\widetilde{q} + (\widetilde{w} + \widetilde{m}) = \widetilde{\theta}$ because $\widetilde{Q}$ is a soft quasilinear space. This implies that the soft quasi vector $\widetilde{q}$ has an inverse soft quasi vector $\widetilde{w} + \widetilde{m}$. However, this contradicts the assumption $\widetilde{q} \notin \widetilde{Q}_r$. Because, if $\widetilde{q} \in \widetilde{Q}_r$, then $\widetilde{q}$ has an inverse soft quasi vector $\widetilde{q}^{'} \in \widetilde{Q}_r$ such that $\widetilde{q} + \widetilde{q}^{'} = \widetilde{\theta}$. As a result, the assumption is not correct and thus $\widetilde{q} \in \widetilde{Q}_r$. In a similar way, it can be observed that $\widetilde{w}$ is a soft quasi-regular vector of $\widetilde{Q}$. □

Theorem 4.12. Let $\widetilde{Q}$ be a soft quasilinear space. If $\widetilde{q} \in \widetilde{Q}_r$ and $\widetilde{w} \in \widetilde{Q}_s$, then $\widetilde{q} + \widetilde{w} \in \widetilde{Q}_r$.

PROOF.
The proof is similar to the proof of Teorem 4.11. □

As in quasilinear spaces, soft quasilinear spaces have a soft quasi-singular vector containing each soft quasi-regular vector.

5. Conclusion

This study provided some results on the convergence and boundedness of a soft quasi sequence in a soft quasilinear space. Further, it investigated some properties of regular and singular subspaces of a soft quasilinear space. In future works, some algebraic properties of soft quasilinear spaces, such as basis, dimensions, and properness, can be studied depending on the descriptions of soft quasilinear spaces. Moreover, whether the class of soft fuzzy sets has a soft quasilinear space structure can be investigated. Applying the soft quasi concept to them is worth studying.

Author Contributions

All the authors contributed equally to this work. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

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