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PT-/non-PT-Symmetric and Non-Hermitian Generalized Woods-Saxon Potential: Feynman Path Integral Approach

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Abstract

In this paper, we examined the treatment path integral of the PT-/non-PT Symmetric and non-Hermitian Generalized Wood Saxon Potentials. Kernel, energy spectrums and corresponding normalized wave functions of these potentials states are obtained. We used parametric time and point transformation for constructed kernel via path integral.

1. INTRODUCTION

Symmetry plays an important role in determining the dynamics of a given physical system. It is well known in the standard quantum mechanics that Hermitian Hamiltonian is required to have the real eigenvalues. In the PT- symmetric quantum mechanics, Hamiltonian has real spectrum although it is not Hermitian [1,2], where P denotes parity, and T operator denotes time reversal operator. They act on the position and momentum operators as P: x→-x, p→-p and T: x→x, p→p, i→-i. First time Bender and Boetcher, later many authors have studied several Hamiltonians with complex or real spectra using different analytic and numeric methods [3-10]. One of the analytic methods providing exact solutions is the Feynman's path integral method. This method gives quantum mechanical amplitude for a point particle at a position x_a at time t_a to reach a position x_b at time t_b integrate over all possible paths connecting by the classical action. This also is the energy dependent Green's function for the Schrödinger equation. A Feynman Path integral formalism to derive kernel of several potentials have been improved in Ref.[11]. The solution of the path integral for the three-dimensional H-atom has been studied by Duru and Kleinert. The radial path integral for the Woods-Saxon potential and Hulthen potential for s-waves was exactly solved [12]. The purpose of this paper is to obtain the energy spectrum and the wave functions of the PT-/Non-PT-Symmetric and non-Hermitian Generalized Woods Saxon Potential using Feynman's Path integral method.

The interactions between nucleon with heavy nucleus are described using nuclear and Coulomb potentials. The Woods-Saxon Potential is one of the important nuclear potential which has many applications in physics [13]. Woods Saxon plus Coulomb potentials known as Modified Woods-Saxon Potential is used for a unified description of the entrance channel fusion barrier and the fission barrier of fusion-fission reactions based on the Skyrme energy-density functional approach [14,15]. We examined Generalized Woods Saxon Potential which is near the Modified Woods-Saxon Potential. Generalized Woods-Saxon Potential is given by [10,13]

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$$V_q(x) = -\frac{V_1}{1+qe^{\frac{(r-R_0)}{a}}} - \frac{V_2 e^{\frac{(r-R_0)}{a}}}{\left(1+qe^{\frac{(r-R_0)}{a}}\right)^2} \tag{1}$$

where V_1 and V_2 determine the potential depth, R_0 is the width of the potential, a is the surface thickness and q is a deformation parameter. Selecting some specific q values for Eq.(1) it reduces to the well known types. For example, q=0 it happens Generalized Morse potential. We obtained energy spectrum and corresponding wave functions of PT-/Non-PT-Symmetric and non-Hermitian Generalized Morse potential [20]. For q=1, V_2=0 it reduces to standard Woods-Saxon potential and for q=-1 and V_1=0 it happens Hulthen potential.

This paper is organized as follows: In section 2, we derive Kernel of PT and non-Hermitian Generalized Woods Saxon Potential. In Section 3, we obtain the energy eigenvalues and the corresponding wave functions. In section 4, we find kernel, energy spectrum and wave functions of Non- PT-symmetric and non-Hermitian Generalized Woods-Saxon Potential.

2. KERNEL FOR PT-SYMMETRIC AND NON-HERMITIAN GENERALIZED WOODS-SAXON POTENTIAL

Kernel express probability amplitude of particle moving to position x_b at time t_b from position x_a at time t_a. The kernel of a point particle moving in the V(x) potential in one dimension is

$$K(x_b,t_b;x_a,t_a) = \int \frac{DxDp}{2\pi} \exp\left\{\frac{i}{\hbar}\int dt\left[p\dot{x} - \frac{p^2}{2m} - V(x)\right]\right\}. \tag{2}$$

If V_1 and V_2 are real and $\alpha \to i\alpha$ (1) potential becomes

$$V_q(x) = -\frac{V_1}{1+qe^{i\alpha x}} - \frac{V_2 e^{i\alpha x}}{\left(1+qe^{i\alpha x}\right)^2} \tag{3}$$

It is called as PT-symmetric and also non-Hermitian, since the property $V^*(x) = V(x)$ is exist.

In the time sliced representation, kernel of potential (3) is given as

$$K(x_b,x_a;T) = \lim_{N\to\infty}\int\prod_{j=1}^{N} dx_j \prod_{j=1}^{N+1}\frac{dp_j}{2\pi\hbar}\exp\left\{\frac{i}{\hbar}\sum_{j=1}^{N+1}\left[p_j\Delta x_j - \left(\frac{p_j^2}{2m} + \frac{V_1}{1+qe^{i\alpha x}} + \frac{V_2 e^{i\alpha x}}{\left(1+qe^{i\alpha x}\right)^2}\right)\varepsilon\right]\right\} \tag{4}$$

where $\varepsilon = t_J - t_{j-1} = \dfrac{T}{N+1}, \Delta x_j = x_j - x_{j-1}, x_a = x(0), x_b = x(T) = x(t_{N+1})$.

To bring the path integral of (4) into a solvable form, its variables are changed as

$$\frac{1}{1+qe^{i\alpha x}} = \cos^2\theta \qquad p_x = -\frac{i\alpha}{2}\sin\theta\cos\theta\, p_\theta \qquad 0 \le \theta \le \pi \tag{5}$$

Because of this transformation the contribution to Jacobien performed kernel happens $\dfrac{\alpha}{2i}\sin\theta_b\cos\theta_b$. Thus (5) takes the form

$$K(x_b,t_b;x_a,t_a)=\frac{\alpha}{2i}\sin\theta_b\cos\theta_b\int D\theta Dp_\theta$$
$$\times\exp\left[i\int dt\left(p_\theta\dot{\theta}-\frac{\alpha^2\sin^2\theta\cos^2\theta}{4}\frac{p_\theta^2}{2\mu}+V_1\cos^2\theta+\frac{V_2}{q}\sin^2\theta\cos^2\theta\right)\right]. \tag{6}$$

To eliminate the ($\alpha^2\sin^2\theta\cos^2\theta$)/4) part in the kinetic energy term we define a new time parameter [12-15]

$$\frac{dt}{ds}=\frac{4}{\alpha^2\sin^2\theta\cos^2\theta} \qquad \text{or} \qquad t=\frac{4}{\alpha^2}\int\frac{ds'}{\sin^2\theta\cos^2\theta} \tag{7}$$

Using Fourier transformation of δ- function, (7) can be written as

$$1=\int dS\int\frac{dE}{2\pi}\frac{4}{\alpha^2\sin^2\theta\cos^2\theta}\exp\left[i\left(ET-\int ds\frac{4E}{\alpha^2\sin^2\theta\cos^2\theta}\right)\right] \tag{8}$$

Here $S=s_b-s_a$. Using Eq.(7) and Eq.(8), we perform Eq.(6) as following

$$K(x_b,t_b;x_a,t_a)=\frac{2}{i\alpha\sin\theta_b\cos\theta b}\int_{-\infty}^{\infty}\frac{dE}{2\pi}e^{iET}\int_0^{\infty}dS\int D\theta Dp_\theta e^{i\frac{4V_2}{q\alpha^2}S}$$
$$\times\exp\left[i\int_0^S ds\left(p_\theta\dot{\theta}-\left[\frac{p_\theta^2}{2\mu}-\frac{\frac{4\mu}{\alpha^2}(E+V_1)}{\sin^2\theta}-\frac{\frac{4\mu}{\alpha^2}E}{\cos^2\theta}\right]\right)\right] \tag{9}$$

To symmetries the factor in front of (6) we use

$$\frac{1}{\sin\theta_b\cos\theta_b}=\frac{2}{\sqrt{\sin 2\theta_a\sin 2\theta_b}}\exp\left(i\int_0^S dsi\frac{\cos 2\theta}{\sin 2\theta}\dot{\theta}\right) \tag{10}$$

Thus we can write (6) as

$$K(x_b,x_a;T)=\int_0^{\infty}dSe^{iS/2\mu}\int_{-\infty}^{\infty}\frac{dE}{2\pi}e^{iET}\frac{4q}{i\alpha\sqrt{\sin 2\theta_a\cos 2\theta_b}}K(\theta_b,\theta_a;S) \tag{11}$$

where

$$K(\theta_b,\theta_a;S)=\int D\theta Dp_\theta\exp\left\{i\int_0^S ds\left[p_\theta\dot{\theta}-\frac{p_\theta^2}{2\mu}-\frac{1}{2\mu}\left(\frac{K(K-1)}{\sin^2\theta}+\frac{\lambda(\lambda-1)}{\cos^2\theta}\right)-\frac{ip_\theta\cos 2\theta}{2\mu\sin 2\theta}\right]\right\} \tag{12}$$

and K and λ are

$$K=\frac{1}{2}\left[1+\sqrt{\frac{32}{\alpha^2}\mu(V_1+E)}\right]$$
$$\lambda=\left[1+\sqrt{\frac{32}{\alpha^2}\mu E}\right] \tag{13}$$

We have from 1 to N+1 in Eq. (4) time graded path integral formula. Instead of if we had to have taken from 0 to N, contribution to Jacobien would be $\frac{\alpha}{2}\sin\theta_a\cos\theta_a$.

$$\dot{\theta}_j \to \dot{\theta}_j \pm \frac{i\cos 2\theta_j}{2\mu \sin 2\theta_j} \quad \text{or} \qquad \frac{\theta_j - \theta_{j-1}}{\varepsilon} \to \frac{\theta_j - \theta_{j-1}}{\varepsilon} \pm \frac{ip_\theta \cos 2\theta_j}{2\mu \sin 2\theta_j} \tag{14}$$

In $\varepsilon \to 0$ limit it will give a vanishing contribution.

Eq. (9) is converted to the path integral for Pöschl-Teller potential which is known exact solution [8]. Kernel for Pöschl-Teller potential:

$$K(\theta_b, \theta_a; S) = \int D\theta Dp_\theta \exp\left\{ i\int_0^S ds \left[p_\theta \dot{\theta} - \frac{p_\theta^2}{2\mu} - \frac{1}{2\mu}\left(\frac{K(K-1)}{\sin^2\theta} + \frac{\lambda(\lambda-1)}{\cos^2\theta} \right) \right] \right\}. \tag{15}$$

Kernel can be also expressed in terms of wave functions as

$$K(\theta_b, \theta_a; S) = \sum_{n=0}^{\infty} \exp\left[-i(S/2\mu)(K+\lambda+2n)^2 \right] \psi_n(\theta_a)\psi^*_n(\theta_b) \tag{16}$$

where

$$\begin{aligned} \psi_n(\theta) &= \sqrt{2(K+\lambda+2n)} \sqrt{\frac{\Gamma(n+1)\Gamma(K+\lambda+n)}{\Gamma(\lambda+n+1/2)\Gamma(K+n+1/2)}} \\ &\times (\cos\theta)^\lambda (\sin\theta)^K P_n^{(K-1/2,\lambda-1/2)}(1-2\sin^2\theta) \end{aligned} \tag{17}$$

Eq. (11) can be expressed

$$K(x_b, x_a; T) = \int_{-\infty}^{\infty} \frac{dE}{2\pi} e^{iET} \int_0^\infty dS e^{i\left(4V_2/q\alpha^2\right)S} \frac{4q}{i\alpha\sqrt{\sin 2\theta_a \cos 2\theta_b}} K(\theta_b, \theta_a; S) \tag{18}$$

If we integrate over S to obtain the energy -dependent Green's function it can be written as

$$G(x_b, x_a; E) = \frac{8\mu q}{\alpha\sqrt{\sin 2\theta_a \cos 2\theta_b}} \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \frac{dE}{2\pi} \frac{e^{iET}}{(K+\lambda+2n)^2 - 1} \psi_n(\theta_a)\psi^*_n(\theta_b) \tag{19}$$

Using Green’s function in (19), kernel of physical system is represented in the form

$$K(x_b, x_a; E) = \sum_{n=0}^{\infty} e^{-iE_nT} \varphi_n(x_a)\varphi^*_n(x_b) = \sum_{n=0}^{\infty} \exp\left\{ \left[-\frac{1}{8\mu\alpha q(n+1)^2} \right] \right\} \varphi_n(u_a)\varphi^*_n(u_b) \tag{20}$$

3. ENERGY SPECTRUM AND WAVE FUNCTIONS FOR PT-SYMMETRIC AND NON-HERMITIAN GENERALIZED WOODS-SAXON POTENTIAL

If we integrate over dS and dE to Kernel in Eq.(18), we can get the energy eigenvalues as

$$E_n = \frac{1}{8\mu\alpha^2(n+1)^2} \left[2\mu\alpha^2 \frac{V_1}{q} - (n+1)^2 \right]^2 \tag{21}$$

and normalized wave functions in terms of Jacobi polynomials [22] are

$$\begin{aligned} \phi(x) &= \frac{1}{2\sqrt{2}\sqrt{n+1}} \sqrt{4(n+1)^2 - (\lambda_n - K_n)^2} \sqrt{\frac{\Gamma(n+1)\Gamma(K_n+\lambda_n+n)}{\Gamma(\lambda_n+n+1/2)\Gamma(K_n+n+1/2)}} \\ &\times \frac{\exp[(K_n - 1/2)\alpha x/2]}{(1+e^{-\alpha x})^{K+n-1/2}} P_n^{(K_n-1/2,\lambda-1/2)}\left(-\frac{1+e^{-i\alpha x}}{1-e^{-i\alpha x}} \right) \end{aligned} \tag{22}$$

where we got

$$K_n = \frac{1}{8\mu\alpha^2(n+1)^2}\left[(n+1)^2 - 2\mu\alpha^2\frac{V_1}{q}\right]$$
$$\lambda_n = \frac{1}{2} + \frac{1}{(n+1)}\left[(n+1)^2 + 2\mu\alpha^2\frac{V_1}{q}\right] \tag{-(23)}$$

Here we see that there are real spectra for the PT Symmetric and Non Hermitian potential case. We can see that these results coincide with the results in Ref. [10].

4. NON- PT-SYMMETRIC AND NON-HERMITIAN GENERALIZED WOODS-SAXON POTENTIAL

Non PT-symmetric and Non-Hermitian Generalized Woods-Saxon Potential is determined by taking (1/a)→(i/a), V_0→A+iB and q→iq as

$$V_q(x) = -\frac{(A+iB)}{1+qe^{i\alpha x}} - \frac{V_2 e^{i\alpha x}}{\left(1+qe^{i\alpha x}\right)^2} \tag{24}$$

Following the same steps above, we obtain the wave functions and energy spectrum for the Non PT-Symmetric and non-Hermitian Generalized Woods-Saxon Potential. Defining a suitable coordinate transformation, we express kernel as

$$K(x_b,t_b;x_a,t_a) = \frac{q}{2a}\sin\theta_b\cos\theta_b\int D\theta Dp_\theta$$
$$\times\exp\left[i\int dt\left(p_\theta\dot{\theta} + \frac{\alpha^2\sin^2\theta\cos^2\theta}{4}\frac{p_\theta^2}{2\mu} + (A+iB)\sin^2\theta + \frac{V_2}{q}\sin^2\theta\cos^2\theta\right)\right] \tag{25}$$

To get kernel form sec. (2) parametric time is defined

$$\frac{dt}{ds} = -\frac{4}{\alpha^2\sin^2\theta\cos^2\theta} \qquad \text{or} \qquad t = -\frac{4}{\alpha^2}\int\frac{ds'}{\sin^2\theta\cos^2\theta} \tag{26}$$

If we follow the steps in sec. (2) we obtain energy eigenvalues as

$$E_n = \frac{1}{8\mu\alpha^2(n+1)^2}\left[2\mu\alpha^2\frac{(A+iB)}{q} - (n+1)^2\right]^2 \tag{27}$$

And Normalized wave functions are

$$\phi(x) = \frac{1}{2\sqrt{2}\sqrt{n+1}}\sqrt{4(n+1)^2 - (\lambda_n - K_n)^2}\sqrt{\frac{\Gamma(n+1)\Gamma(K+\lambda+n)}{\Gamma(\lambda+n+1/2)\Gamma(K+n+1/2)}}$$
$$\times\frac{\exp[(K_n - 1/2)\alpha x/2]}{(1+e^{-\alpha x})^{K+n-1/2}}P_n^{(K-1/2,\lambda-1/2)}\left(-\frac{1+e^{-i\alpha x}}{1-e^{-i\alpha x}}\right) \tag{.(28)}$$

Kn and λn are the same in Eq. (22). It is seen that energy spectra is real only if Re(V_0)=0. This expression is identical to that given in Ref. [10]

5. CONCLUSION

In this study, we used path integral formalism to find kernel, energy spectrum and wave functions of PT-/non-PT Symmetric and non-Hermitian Generalized Woods-Saxon Potentials. We have seen that kernel for the PT-/non-PT Symmetric and non-Hermitian Generalized Woods-Saxon Potentials can be obtained exactly and using this kernel we get energy eigenvalues and the corresponding wave functions. We showed that PT-/non-PT Symmetric and non-Hermitian forms of potentials have real energy spectra by restricting the potential parameters. Our results are in agreement with the finding of other methods.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors

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