GU J Sci 30(1): 293-302 (2017)

Gazi University

Journal of Science

http://dergipark.gov.tr/gujs

Some Fixed Point and Common Fixed Point Results Through Ω-Distance Under Nonlinear Contractions

K. ABODAYEH [2,*], W. SHATANAWI [1,2], A. BATAIHAH [3], A.H. ANSARI [4]

[1] *Department of Mathematics, Faculty of Science Hashemite University, Zarqa, Jordan*

[2] *Department of Mathematics and general courses Prince Sultan University Riyadh, Saudi Arabia*
[3] *Ministry of Education, Irbid, Jordan*
[4] *Department of Mathematics, Islamic Azad University, Karaj, Iran*

Article Info

Received: 26/04/2016
Accepted: 23/06/2016

Keywords

G-Metric Space
Ω-Distance
Fixed Point Theory

Abstract

Saadati et al [1] introduced the notion of Ω-distance mappings and studied some existing fixed point results. In this paper, we employ the notion of Ω-distance and some functions defined on [0,∞) to introduce some new nonlinear contractions of Suzuki types. We utilize our new notions to formulate and proved serval fixed and common fixed point results. Our results improve and extend many results in literature.

1. INTRODUCTION

In 2010, Saadati *et.al.* [1] employed the notion of G-metric spaces in sense of Mustafa and Sims [2] to introduce a new notion in mathematics called Ω-distance. Their paper is crucial to obtain new results in such a space. Recently, Samet et al in [3] and [4] connected the fixed and common fixed point theorems in G-metric spaces to standard metric space, thus their method plays a major role to reduce some fixed and common fixed point results to standard metric spaces. It is worth mentioning that the method of Samet et al in [3] and [4] is not working in fixed and common fixed point results involving Ω-distance. For More information about Ω-distance, we refer the reader to [5]-[9]. Also, for some works in G-metric space, we refer the reader to [10]-[19].

We begin with the definition of G-metric spaces.

Definition 1.1. [2]. Let X be a nonempty set, and let $G: X \times X \times X \to [0, \infty)$ be a function that satisfies the following conditions:

(G1) $G(x, y, z) = 0$ if $x = y = z$;

(G2) $G(x, x, y) > 0$ for all $x, y \in X$ with $x \neq y$;

(G3) $G(x, y, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$;

(G4) $G(x, y, z) = G(p\{x, y, z\})$, for any permutation of x, y, z;

(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$.

Then the function G is called a *generalized metric space*, or more specifically G-metric on X, and the pair (X, G) is called a G-metric space.

Definition 1.2. [2]. Let (X, G) be a G-metric space, and let (x_n) be a sequence of points of X. We say that (x_n) is G-convergent to x if for any $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that $G(x, x_n, x_m) < \epsilon$, for all $n, m \geq k$.

*Corresponding author, e-mail: swasfi@hu.edu.jo

Definition 1.3. [2]. Let (X, G) be a G-metric space. A sequence (x_n) in X is said to be *G-Cauchy* if for every $\epsilon > 0$, there exists $k \in$ N such that $G(x_n, x_m, x_l) < \epsilon$ for all $n, m, l \geq k$.

Definition 1.4. [14]. A G-metric space (X, G) is said to be *G-complete* or *complete G-metric* space if every G-Cauchy sequence in (X, G) is G-convergent in (X, G).

The notion of Ω-distance in the sense of Saadati et al. [1] is defined as follows:

Definition 1.5. [1]. Let (X, G) be a G-metric space. Then a function $\Omega: X \times X \times X \to [0, \infty)$ is called an Ω-*distance* on X if the following conditions are satisfied:

$\Omega(x, y, z) \leq \Omega(x, a, a) + \Omega(a, y, z)$ for all $x, y, z, a \in X$,

for any $x, y \in X$, the functions $\Omega(x, y, .), \Omega(x, ., y): X \to X$ are lower semi

continuous,

for each $\epsilon > 0$, there exists $\delta > 0$ such that if $\Omega(x, a, a) \leq \delta$ and $\Omega(a, y, z) \leq \delta$, then $G(x, y, z) \leq \epsilon$.

Definition 1.6. [1]. Let (X,G) be a G-metric space and Ω be an Ω-distance on X. Then we say that X is Ω-*bounded* if there exists $M \geq 0$ such that $\Omega(x, y, z) \leq M$ for all $x, y, x \in X$.

Saadati et al. [1] proved the following crucial lemma in the setting of Ω-distance.

Lemma 1.1. [1]. Let X be a metric space with metric G and Ω be an Ω-distance on X. Let (x_n),(y_n) be sequences in X, and (α_n),(β_n) be sequences in $[0, \infty)$ converging to zero. Then for all $x, y, z, a \in X$, we have the following:

If $\Omega(y, x_n, x_n) \leq \alpha_n$ and $\Omega(x_n, y, z) \leq \beta_n$ for $n \in$ N, then $G(y, y, z) < \epsilon$ and hence $y = z$;

If $\Omega(y_n, x_n, x_n) \leq \alpha_n$ and $\Omega(x_n, y_m, z) \leq \beta_n$ for all $m > n \in$ N, then $G(y_n, y_m, z) \to 0$ and hence $y_n \to z$;

If $\Omega(x_n, x_m, x_l) \leq \alpha_n$ then the sequence (x_n) is a G-Cauchy sequence, for all $m, n, l \in$ N with $n \leq m \leq l$,;

If $\Omega(x_n, a, a) \leq \alpha_n$ for any $n \in$ N, then (x_n) is a G-Cauchy sequence.

In 2008, Suzuki [20] introduced a nonlinear contraction of special form and proved many results as a generalization of the Banach contraction theorem.

Very recently, Abodayeh et al. [21] utilized the concept of altering distance function in the sense of Khan et al [22] to drive some contractive conditions of Suzuk's types and obtained some fixed point theorems in the setting of Ω-distance. In this paper, we utilize the notion of Ω-distance and functions on $[0, \infty)$ to introduce some nonlinear contractions and prove many fixed and common fixed point theorems in the setting of Ω-distance. Our new results improved the results of Abodayeh et al. [21].

2. MAIN RESULT

2.1. Fixed Point Results.

We start our work by introducing the definition of almost perfect function:

Definition 2.1. We call a non-decreasing function $\psi: [0, \infty) \to [0, \infty)$ an *almost perfect* function if ψ satisfies the following conditions:

$\psi(t) = 0$ iff $t = 0$, and

If (t_n) is a sequence in $[0, \infty)$ with $\psi(t_n) \to 0$, then $t_n \to 0$.

Remark: Every altering distance is an almost perfect function.

Now, we introduce the following two definitions:

Definition 2.2. Let Ω be an Ω-distance mapping on a G-metric space X. A mapping $T: X \to X$ is called an (Ω, ψ)-*Suzuki-contraction* if there exists $k \in [0,1)$ and an almost perfect function ψ such that if$x, y, z \in X$, $p, q \in$ N and $p \leq q$with $(1 - k)\Omega(x, T^p x, T^q x) \leq \Omega(x, y, z)$, then

$$\psi\Omega(Tx,Ty,Tz) \leq k\psi\Omega(x,y,z).$$

Definition 2.3. Let Ω be an Ω-distance mapping on a G-metric space X. A mapping $T: X \to X$ is called a generalized-(Ω,ψ)-Suzuki-contraction if there exists $k \in [0,1)$ and an almost perfect function ψ such that if $x,y,z \in X$, $p,q \in \mathrm{N}$, and $q \geq p$ with $(1-k)\Omega(x,T^p x,T^q x) \leq \Omega(x,y,z)$, then

$$\psi\Omega(Tx,Ty,Tz) \leq k \max\{\psi\Omega(x,Tx,Tx),\psi\Omega(y,Ty,Ty),\psi\Omega(z,Tz,Tz)\}.$$

Now, we introduce and prove our first results:

Theorem 2.1. Let Ω be an Ω-distance mapping on a complete G-metric space X such that X is Ω-bounded. Let $T: X \to X$ be an (Ω,ψ)-Suzuki-contraction mapping that satisfies the following condition:

(i) For all $u \in X$ if $Tu \neq u$, then $\inf\{\Omega(x,Tx,u): x \in X\} > 0$.

Then T has a fixed point in X. Moreover, for any fixed point $z \in X$ of T, we have $\Omega(z,z,z) = 0$

Proof. Given $x_0 \in X$, define a sequence (x_n) in X inductively by putting $x_{n+1} = Tx_n$, $n \in \mathrm{N}$.

Given $n,m,l \in \mathrm{N}$ with $n < m \leq l$. Let $m = n+s$ and $l = m+t$ with $s \in \mathrm{N}$ and $t \in \mathrm{N} \cup \{0\}$. We note that

$$\begin{aligned}&(1-k)\Omega(x_n,Tx_{m-1},Tx_{l-1}) = (1-k)\Omega(x_n,Tx_{n+s-1},Tx_{n+t-1})\\ &\leq \Omega(x_n,x_{n+s},x_{n+t})\end{aligned}$$

So, we have

$$\begin{aligned}\psi\Omega(Tx_n,Tx_m,Tx_l) &= \psi\Omega(Tx_n,Tx_{n+s},Tx_{n+t})\\ &\leq k\psi\Omega(x_n,x_{n+s},x_{n+t})\\ &= k\psi\Omega(Tx_{n-1},Tx_{n+s-1},Tx_{n+t-1})\\ &\leq k^2\psi\Omega(x_{n-1},x_{n+s-1},x_{n+t-1})\\ &= k^2\psi\Omega(Tx_{n-2},Tx_{n+s-2},Tx_{n+t-2})\\ &\vdots\\ &\leq k^n\psi\Omega(Tx_0,Tx_s,Tx_t).\end{aligned} \tag{2.1}$$

Since Ω is bounded, there exists $M > 0$ such that $\Omega(x,y,z) \leq M$ for all $x,y,z \in X$.

So, we get

$$\begin{aligned}\psi\Omega(Tx_n,Tx_m,Tx_l) &= \psi\Omega(Tx_n,Tx_{n+s},Tx_{n+s+t})\\ &\leq k^n\psi(M).\end{aligned} \tag{2.2}$$

Take $s = 1$ and $t = 0$ in (2.2), we get

$$\psi\Omega(Tx_n,Tx_{n+1},Tx_{n+1}) \leq k^n\psi(M). \tag{2.3}$$

By taking the limit as $n \to \infty$ in (2.2) and (2.3), we get

$$\lim_{n,m,l\to\infty} \psi\,\Omega(Tx_n,Tx_m,Tx_l) = 0$$

and

$$\lim_{n\to\infty} \psi\,\Omega(Tx_n,Tx_{n+1},Tx_{n+1}) = 0.$$

Using the properties of ψ, we get

$$\lim_{n,m,l\to\infty} \Omega\,(Tx_{n+1},Tx_m,Tx_l) = 0 \tag{2.4}$$

and

$$\lim_{n\to+\infty} \Omega\,(Tx_n,Tx_{n+1},Tx_{n+1}) = 0. \tag{2.5}$$

Using the definition of Ω-distance with helping from Inequalities (2.4) and (2.5), we conclude that (x_n) is a G-Cauchy sequence and hence (Tx_n) converges to an element $u \in X$. Given $\epsilon > 0$. Since (x_n) is a G-Cauchy sequence, there exists $n_0 \in$ N such that

$\Omega(x_n, x_m, x_l) < \epsilon$, for all $n, m, l \geq n_0$. Thus, $\liminf_{l\to\infty} \Omega(x_n, x_m, x_l) \leq \epsilon$, for all $n, m \geq n_0$.

Using the fact that Ω is a lower semi-continuous, we get

$$\Omega(x_n, x_m, u) \leq \liminf_{l\to\infty} \Omega(x_n, x_m, x_l) \leq \epsilon, \text{for all } n, m \geq N. \tag{2.6}$$

Now if we substitute $m = n + 1$ in (2.6), gives us $\Omega(x_n, x_{n+1}, u) \leq \epsilon$, for all $n \geq N$.

Assume that $Tu \neq u$. Then condition (i) implies that

$$0 < \inf\{\Omega(x, Tx, u) : x \in X\} \leq \inf\{\Omega(x_n, x_{n+1}, u) : n \geq N\} \leq \epsilon,$$

for all $\epsilon > 0$ which is a contradiction. Therefore, $Tu = u$. Let z be a fixed point of T. Then

$$(1-k)\Omega(z, T^p z, T^q z) = (1-k)\Omega(z, z, z) \leq \Omega(z, z, z)$$

And thus

$$\psi\Omega(z, z, z) = \psi\Omega(Tz, Tz, Tz) \leq k\psi\Omega(z, z, z).$$

Since $k < 1$, and ψ is a an almost perfect function, then $\Omega(z, z, z) = 0$. □

Theorem 2.2. Let Ω be an Ω-distance mapping on a complete G-metric space X such that X is Ω-bounded. Let $T: X \to X$ be a mapping such that T is an (Ω, ψ)-Suzuki-contraction mapping. If T is continuous, then T has a fixed point in X. Moreover, for any fixed point $z \in X$ of T, we have $\Omega(z, z, z) = 0$

Proof. Given $x_0 \in X$, define a sequence (x_n) in X inductively by putting $x_{n+1} = Tx_n$, $n \in$ N. Follow the same way as in the proof of Theorem 2.1, we prove that (x_n) is a G-Cauchy sequence in X. Since X is G-complete, there is $u \in X$ such that (x_n) converges to u. Since T is continuous, then $x_{n+1} = Tx_n \to Tu$. By uniqueness of limit, we conclude that $Tu = u$.

□

Theorem 2.3. Let Ω be an Ω-distance mapping on a complete G-metric space X such that X is Ω-bounded. Let $T: X \to X$ a generalized-(Ω, ψ)-Suzuki-contraction mapping that satisfies the following condition:

(ii): For all $u \in X$ if $Tu \neq u$, then $\inf\{\Omega(x, Tx, u) : x \in X\} > 0$.

Then T has a fixed point in X. Moreover, for any fixed point $z \in X$ of T, we have $\Omega(z, z, z) = 0$

Proof. Given $x_0 \in X$, define a sequence (x_n) in X inductively by putting $x_{n+1} = Tx_n$, $n \in$ N.

For any $n \in$ N, we have

$$\begin{aligned}(1-k)\Omega(x_n, Tx_n, Tx_n) &= (1-k)\Omega(x_n, x_{n+1}, x_{n+1}) \\ &\leq \Omega(x_n, x_{n+1}, x_{n+1}).\end{aligned}$$

Since T is a generalized-(Ω, ψ)-Suzuki-contraction, we get

$$\begin{aligned}\psi\Omega(Tx_n, Tx_{n+1}, Tx_{n+1}) &\leq k \max\{\psi\Omega(x_n, Tx_n, Tx_n), \psi\Omega(x_{n+1}, Tx_{n+1}, Tx_{n+1})\} \\ &= k \max\{\psi\Omega(x_n, Tx_n, Tx_n), \psi\Omega(Tx_n, Tx_{n+1}, Tx_{n+1})\}\end{aligned}$$

Since $k \in [0,1)$, we have

$$\begin{aligned}\psi\Omega(Tx_n, Tx_{n+1}, Tx_{n+1}) &\leq k\psi\Omega(x_n, Tx_n, Tx_n) \\ &= k\psi\Omega(Tx_{n-1}, Tx_n, Tx_n) \\ &\leq k^2\psi\Omega(x_{n-1}, Tx_{n-1}, Tx_{n-1}) \\ &\vdots \\ &\leq k^{n+1}\psi\Omega(x_0, Tx_0, Tx_0).\end{aligned} \tag{2.7}$$

Letting $n \to +\infty$ in (2.7) we get

$$\lim_{n\to+\infty} \psi\,\Omega(Tx_n, Tx_{n+1}, Tx_{n+1}) = 0.$$

Using the properties of ψ, we get

$$\lim_{n\to+\infty} \Omega\,(Tx_n, Tx_{n+1}, Tx_{n+1}) = 0. \tag{2.8}$$

Now, given $n, m, l \in$ I with $n < m \leq l$, we have

$$(1-k)\Omega(x_n, Tx_{m-1}, Tx_{l-1}) = (1-k)\Omega(x_n, x_m, x_l) \\ \leq \Omega(x_n, x_m, x_l).$$

Since T is a generalized-(Ω, ψ)-Suzuki-contraction, we get

$$\begin{aligned} \psi\Omega(Tx_n, Tx_m, Tx_l) &\leq k \max\{\psi\Omega(x_n, Tx_n, Tx_n), \psi\Omega(x_m, Tx_m, Tx_m), \psi\Omega(x_l, Tx_l, Tx_l)\} \\ &= k \max\{\psi\Omega(Tx_{n-1}, Tx_n, Tx_n), \psi\Omega(Tx_{m-1}, Tx_m, Tx_m), \psi\Omega(Tx_{l-1}, Tx_l, Tx_l)\}. \end{aligned}$$

Using (2.7), we get

$$\psi\Omega(Tx_n, Tx_{n+1}, Tx_{n+1}) \leq k \max\{k^n\psi\Omega(x_0, Tx_0, Tx_0), k^m\psi\Omega(x_0, Tx_0, Tx_0), k^l\psi\Omega(x_0, Tx_0, Tx_0)\}.$$

Since $n < m \leq l$, we get

$$\psi\Omega(Tx_n, Tx_{n+1}, Tx_{n+1}) \leq k^{n+1}\psi\Omega(x_0, Tx_0, Tx_0). \tag{2.9}$$

By taking the limit as $n \to \infty$ in (2.9), we get

$$\lim_{n,m,l\to\infty} \psi\,\Omega(Tx_n, Tx_m, Tx_l) = 0.$$

Taking into account, the properties of ψ, we get

$$\lim_{n,m,l\to\infty} \Omega\,(Tx_{n+1}, Tx_m, Tx_l) = 0 \tag{2.10}$$

Inequalities (2.8) and (2.10) imply that $(x_{n+1}) = (Tx_n)$ is a G-Cauchy sequence and hence (Tx_n) converges to an element $u \in X$. Given $\epsilon > 0$. Since (x_n) is a G-Cauchy sequence, there exists $n_0 \in$ N such that

$\Omega(x_n, x_m, x_l) < \epsilon$, for all $n, m, l \geq n_0$. Thus, $\lim_{l\to\infty} \inf\,\Omega\,(x_n, x_m, x_l) \leq \epsilon$, for all $n, m \geq n_0$.

Using the fact that Ω is a lower semi-continuous, we get

$$\Omega(x_n, x_m, u) \leq \lim_{l\to\infty} \inf\,\Omega\,(x_n, x_m, x_l) \leq \epsilon, \text{for all } n, m \geq N. \tag{2.11}$$

Putting $m = n + 1$ in (2.11), gives us $\Omega(x_n, x_{n+1}, u) \leq \epsilon$, for all $n \geq N$.

Assume that $Tu \neq u$. Then condition (ii) implies that

$$0 < \inf\{\Omega(x, Tx, u): x \in X\} \leq \inf\{\Omega(x_n, x_{n+1}, u): n \geq N\} \leq \epsilon,$$

for all $\epsilon > 0$ which is a contradiction. Therefore, $Tu = u$. Let $z = Tz$. Then

$(1-k)\Omega(z, T^pz, T^qz) = (1-k)\Omega(z, z, z) \leq \Omega(z, z, z)$.

We have

$$\psi\Omega(z, z, z) = \psi\Omega(Tz, Tz, Tz) \leq k\psi\Omega(z, z, z).$$

Since $k < 1$ and ϕ is almost perfect function, we have $\Omega(z, z, z) = 0$.

Theorem 2.4. Let Ω be an Ω-distance mapping on a complete G-metric space X such that X is Ω-bounded. Let $T: X \to X$ be a mapping such that T is a generalized-(Ω, ψ)-Suzuki-contraction mapping. If T is continuous, then T has a fixed point in X. Moreover, for any fixed point $z \in X$ of T, we have $\Omega(z, z, z) = 0$.

Proof. Given $x_0 \in X$, define a sequence (x_n) in X inductively by putting $x_{n+1} = Tx_n$, $n \in$ N. Following the same technique as in the proof of Theorem 2.3, one can easily show that (x_n) is a G-Cauchy sequence in X. Using the continuity of T, we may prove that T has a fixed point.

As an application of Theorems 2.3 and 2.4, we formulate the following results:

Corollary 2.1. Let Ω be an Ω-distance mapping on a complete G-metric space. Let $T: X \to X$ be a mapping such that there exist $k \in [0,1)$ with the property that if $x, y, z \in X$, $p, q \in$ N and $p \leq q$ such that $(1-k)\Omega(x, T^p x, T^q x) \leq \Omega(x, y, z)$, then

$$\Omega(Tx, Ty, Tz) \leq k \max\{\Omega(x, Tx, Tx), \Omega(y, Ty, Ty), \Omega(z, Tz, Tz)\}.$$

Also, suppose that T satisfies the following property:

(iii) For all $u \in X$ if $Tu \neq u$, then $\inf\{\Omega(x, Tx, u): x \in X\} > 0$.

Then T has a fixed point in X. Moreover, for any fixed point $z \in X$ of T, we have $\Omega(z, z, z) = 0$.

Proof. Defining $\psi: [0, +\infty) \to [0, +\infty)$ by $\psi(t) = t$. Then ψ is an almost perfect function. Note that T is a generalized-(Ω, ψ)-Suzuki contraction. Moreover, T satisfies all the hypotheses of Theorem 2.3.

□

Corollary 2.2. Let Ω be an Ω-distance mapping on a complete G-metric space. Let $T: X \to X$ be a mapping such that there exists $k \in [0,1)$ with the property that if $x, y, z \in X$, $p, q \in$ N and $p \leq q$ such that $(1-k)\Omega(x, T^p x, T^q x) \leq \Omega(x, y, z)$, then

$$\Omega(Tx, Ty, Tz) \leq k \max\{\Omega(x, Tx, Tx), \Omega(y, Ty, Ty), \Omega(z, Tz, Tz)\}.$$

If T is continuous, then T has a fixed point in X. Moreover, for any fixed point $z \in X$ of T, we have $\Omega(z, z, z) = 0$.

Proof. Defining $\psi: [0, +\infty) \to [0, +\infty)$ by $\psi(t) = t$. Then ψ is an almost perfect function. Note that T satisfies all the hypotheses of Theorem 2.4.

2.2. Common Fixed Point Results

In 2011, Luong and Thuan [23] introduced the class of Φ as follows:

Let $\phi: [0, \infty) \to [0, \infty)$ be a function satisfies the following conditions:

ϕ is continuous and nondecreasing.

$\phi(t) = 0$ iff $t = 0$,

$\phi(t + s) \leq \phi(t) + \phi(s)$.

The set of all functions ϕ is denoted by Φ.

Now, we utilize the class of Φ and Ω-distance to introduce the following contraction of Suzuki type:

Definition 2.4. Let Ω be an Ω-distance mapping on a G-metric space X. Let $S, T: X \to X$ be two mappings such that $TX \subseteq SX$. The pair (T, S) is called an (Ω, ϕ)-*Suzuki-contraction* if there exists $k \in [0,1)$ and $\phi \in$ Φ such that if $x, u, v, y, z \in X$ with $(1-k)\Omega(x, Tu, Tv) \leq \Omega(x, Sy, Sz)$, then

$$\phi\Omega(Tx, Ty, Tz) \leq k\phi\Omega(Sx, Sy, Sz).$$

Theorem 2.5. Let Ω be an Ω-distance mapping on a complete G-metric space X such that X is Ω-bounded. Let $T, S: X \to X$ be two mappings such that the pair (T, S) is an (Ω, ϕ)-Suzuki-contraction mapping that satisfies the following condition:

(iv) For all $u \in X$ if $Tu \neq u$ or $Su \neq u$, then $\inf\{\Omega(Sx, Tx, u): x \in X\} > 0$.

Then T and S have a common fixed point in X. Moreover, for any common fixed point $z \in X$ of T and S, we have $\Omega(z, z, z) = 0$.

Proof. Given $x_0 \in X$, define a sequence (x_n) in X inductively by setting $Sx_{n+1} = Tx_n$, $n \in$ N.

Since

$$\begin{aligned}&(1-k)\Omega(x_n,Tx_n,Tx_n)=(1-k)\Omega(x_n,Sx_{n+1},Sx_{n+1})\\&\leq \Omega(x_n,Sx_{n+1},Sx_{n+1})\end{aligned}$$

holds for every $n\in$ N, we have

$$\begin{aligned}\phi\Omega(Tx_n,Tx_{n+1},Tx_{n+1}) &\leq k\phi\Omega(Sx_n,Sx_{n+1},Sx_{n+1})\\ &= k\phi\Omega(Tx_{n-1},Tx_n,Tx_n)\\ &\leq k^2\phi\Omega(Sx_{n-1},Sx_n,Sx_n)\\ &= k^2\phi\Omega(Tx_{n-2},Tx_{n-1},Tx_{n-1})\\ &\vdots\\ &\leq k^n\phi\Omega(Tx_0,Tx_1,Tx_1)\end{aligned} \tag{2.12}$$

By taking the limit in (2.12) as $n\to+\infty$, we get

$$\lim_{n\to+\infty}\phi\,\Omega(Tx_n,Tx_{n+1},Tx_{n+1})=0.$$

Since ϕ is continuous, we conclude that

$$\lim_{n\to+\infty}\Omega\,(Tx_n,Tx_{n+1},Tx_{n+1})=0. \tag{2.13}$$

Given $s\in$ N, since

$$\begin{aligned}&(1-k)\Omega(x_n,Tx_n,Tx_{n+s})=(1-k)\Omega(x_n,Sx_{n+1},Sx_{n+s+1})\\&\leq \Omega(x_n,Sx_{n+1},Sx_{n+s+1})\end{aligned}$$

holds for every $n\in$ N, we have

$$\begin{aligned}\phi\Omega(Tx_n,Tx_{n+1},Tx_{n+s+1}) &\leq k\phi\Omega(Sx_n,Sx_{n+1},Sx_{n+s+1})\\ &= k\phi\Omega(Tx_{n-1},Tx_n,Tx_{n+s})\\ &\leq k^2\phi\Omega(Sx_{n-1},Sx_n,Sx_{n+s})\\ &= k^2\phi\Omega(Tx_{n-2},Tx_{n-1},Tx_{n+s-1})\\ &\vdots\\ &\leq k^n\phi\Omega(Tx_0,Tx_1,Tx_s)\end{aligned} \tag{2.14}$$

Given $n,m,l\in$ N with $l\geq m>n+1$. Then Part (a) of Definition 1.5, implies that

$$\begin{aligned}\Omega(Tx_{n+1},Tx_m,Tx_l)\leq \Omega(Tx_{n+1},Tx_{n+2},Tx_{n+2})+\Omega(Tx_{n+2},Tx_{n+3},Tx_{n+3})\\ +\cdots+\Omega(Tx_{m-1},Tx_m,Tx_l).\end{aligned}$$

Using the properties of ϕ, we get

$$\begin{aligned}\phi(\Omega(Tx_{n+1},Tx_m,Tx_l)) &\leq \phi[\Omega(Tx_{n+1},Tx_{n+2},Tx_{n+2})\\ &\quad +\Omega(Tx_{n+2},Tx_{n+3},Tx_{n+3})+\cdots+\Omega(Tx_{m-1},Tx_m,Tx_l)]\\ &\leq \phi(\Omega(Tx_n,Tx_{n+1},Tx_{n+1}))\\ &\quad +\phi(\Omega(Tx_{n+1},Tx_{n+2},Tx_{n+2}))+\cdots+\phi(\Omega(Tx_{m-1},Tx_m,Tx_l)).\end{aligned} \tag{2.15}$$

Using (2.12), (2.14) and (2.15), we get

$$\begin{aligned}\phi(\Omega(Tx_{n+1},Tx_m,Tx_l)) &\leq k^{n+1}\phi\Omega(Tx_0,Tx_1,Tx_1)+k^{n+2}\phi\Omega(Tx_0,Tx_1,Tx_s)\\ &\quad +\cdots+k^{m-1}\phi\Omega(Tx_0,Tx_1,Tx_s)+k^m\phi\Omega(Tx_0,Tx_1,Tx_s).\end{aligned} \tag{2.16}$$

Since Ω is bounded, there exists $M>0$ such that $\Omega(x,y,z)\leq M$ for all $x,y,z\in X$.

Now, using (2.16), we get

$$\phi(\Omega(Tx_{n+1},Tx_m,Tx_l))\leq k^{n+1}\phi(M)+k^{n+2}\phi(M)+\cdots+k^m\phi(M)$$

$$\leq k^{n+1}\phi(M)[1 + k + k^2 + k^3 + \cdots]$$
$$= \frac{k^{n+1}\phi(M)}{1-k} \tag{2.17}$$

By taking the limit as $n \to \infty$ in (2.17), we have $\lim_{n,m,l\to\infty} \phi\, \Omega(Tx_n, Tx_m, Tx_l) = 0.$

Since ϕ is continuous, we have

$$\lim_{n,m,l\to\infty} \Omega\,(Tx_{n+1}, Tx_m, Tx_l) = 0. \tag{2.18}$$

Inequalities (2.13) and (2.18) imply that (Tx_n) is a G-Cauchy sequence and hence (Tx_n) converges to an element $u \in X$. For all $\epsilon > 0$, since (Tx_n) is a G-Cauchy sequence, there exists $N \in$ N such that

$\Omega(Tx_n, Tx_m, Tx_l) < \epsilon$, for all $n, m, l \geq N$. Thus,

$$\lim_{l\to\infty} \inf\, \Omega\,(Tx_n, Tx_m, Tx_l) \leq \epsilon \quad \forall n, m \geq N.$$

The lower semi-continuity of Ω implies that

$$\Omega(Tx_n, Tx_m, u) \leq \lim_{l\to\infty} \inf\, \Omega\,(Tx_n, Tx_m, Tx_l) \leq \epsilon, \text{ for all } n, m \geq N. \tag{2.19}$$

Considering $m = n + 1$ in (2.19), gives us $\Omega(Tx_n, Tx_{n+1}, u) \leq \epsilon$, for all $n \geq N$.

Assume that $Tu \neq u$ or $Su \neq u$. Then condition (iv) implies that

$$0 < \inf\{\,\Omega(Sx, Tx, u): x \in X\} \leq \inf\{\,\Omega(Tx_n, Tx_{n+1}, u): n \geq N\} \leq \epsilon,$$

for all $\epsilon > 0$ which is a contradiction. Therefore, $Tu = u$ and $Su = u$. Let $z = Tz$ and $z = Sz$. Then

$$(1-k)\Omega(z, Tz, Tz) = (1-k)\Omega(z, Sz, Sz)$$
$$\leq \Omega(z, Sz, Sz).$$

So,

$$\phi\Omega(z, z, z) = \phi\Omega(Tz, Tz, Tz) \leq k\phi\Omega(Sz, Sz, Sz) = k\phi\Omega(z, z, z).$$

Since $k < 1$ and $\phi \in \Phi$, we have $\Omega(z, z, z) = 0$.

Theorem 2.6. Let Ω be an Ω-distance mapping on a complete G-metric space X such that X is Ω-bounded. Let $T, S: X \to X$ be two mappings such that the pair (T, S) is an (Ω, ϕ)-Suzuki-contraction mapping. If T and S are continuous and commute, then T and S have a common coincidence point.

Proof. Given $x_0 \in X$, define a sequence (x_n) in X inductively by setting $Sx_{n+1} = Tx_n$, $n \in$ N. Following the same technique as in the proof of Theorem 2.1, we prove that (Tx_n) is a Cauchy sequence in the complete G-metric space X. So there exists $u \in X$ such that $Tx_n \to u$ and $Sx_n = Tx_{n+1} \to u$. Since S and T are continuous and commute, we get

$$S(Tx_n) \to Su$$

and

$$S(Tx_n) = T(Sx_n) \to Tu.$$

By uniqueness of limit, we conclude that $Su = Tu$; that is, u is a coincidence point of T and S.

Now, as an application of Theorems 2.5 and 2.6, we have the following results:

Corollary 2.3. Let Ω be an Ω-distance mapping on a complete G-metric space X such that X is Ω-bounded. Let $T: X \to X$ be mapping such that there exist $k \in [0,1)$ and $\phi \in \Phi$ with the property that if $x, y, z, u, v \in X$ such that $(1-k)\Omega(x, Tu, Tv) \leq \Omega(x, y, z)$, then

$$\phi\Omega(Tx, Ty, Tz) \leq k\phi\Omega(x, y, z).$$

Assume T satisfies the following property:

For all $u \in X$ if $Tu \neq u$, then

$$\inf\{\,\Omega(x, Tx, u): x \in X\} > 0. \tag{2.20}$$

Then T has a fixed point in X. Moreover, for any fixed point $z \in X$ of T, we have $\Omega(z, z, z) = 0$.

Proof. Take S to be the identity mapping in Theorem 2.5. Then the pair (T, S) is (Ω, ϕ)-Suzuki-contraction. Moreover, S and T satisfy all the conditions of Theorem 2.5. So T has a fixed point.

Corollary 2.4. Let Ω be an Ω-distance mapping on a complete G-metric space X such that X is Ω-bounded. Let $T: X \to X$ be mapping such that there exist $k \in [0,1)$ and $\phi \in \Phi$ with the property that if $x, y, z, u, v \in X$ such that $(1-k)\Omega(x, Tu, Tv) \leq \Omega(x, y, z)$, then

$$\phi\Omega(Tx, Ty, Tz) \leq k\phi\Omega(x, y, z).$$

If T is continuous, then T has a fixed point in X. Moreover, for any fixed point $z \in X$ of T, we have $\Omega(z, z, z) = 0$.

Proof. Take S to be the identity mapping in Theorem 2.6. Then the pair (T, S) is (Ω, ϕ)-Suzuki-contraction. Moreover, S and T satisfy all the conditions of Theorem 2.6. So T and S have a coincidence point u; that is $Tu = Su = u$. So u is a fixed point of T.

Remarks:

Theorem 2.2 of [21] is a special case of Theorem 2.1.

Theorem 2.5 of [21] is a special case of Theorem 2.3.

3. CONCLUSION

We introduced a new type of functions called almost perfect function. We utilized our function and the notion of Omega-distance to introduce many fixed and common fixed point theorems. Our results generalized many interesting results found in the literature.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors

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