

Qualitative Behavior of the Difference Equation

$$x_{n+1} = \frac{\alpha x_{n-m} + \eta x_{n-k} + \sigma x_{n-l} + \delta x_n}{\beta + \gamma x_{n-k} x_{n-l} (x_{n-k} + x_{n-l})}$$

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Abstract

In this paper, we discuss some qualitative properties of the positive solutions to the following rational nonlinear difference equation $x_{n+1} = \frac{\alpha x_{n-m} + \eta x_{n-k} + \sigma x_{n-l} + \delta x_n}{\beta + \gamma x_{n-k} x_{n-l} (x_{n-k} + x_{n-l})}$, $n = 0, 1, 2, \dots$ where the parameters $\alpha, \beta, \gamma, \delta, \eta, \sigma \in (0, \infty)$, while m, k, l are positive integers, such that $m < k < l$. The initial conditions $x_{-m}, \dots, x_{-k}, \dots, x_{-l}, \dots, x_{-1}, \dots, x_0$ are arbitrary positive real numbers. We will give some numerical examples to illustrate our results.

1. Introduction

The study of solution of nonlinear rational recursive sequence of high order is quite challenging and rewarding. Every dynamical system $a_{n+1} = f(a_n)$ determines a difference equation and vice versa. An interesting class of nonlinear difference equations is the class of solvable difference equations, and one of the interesting problems is to find equations that belong to this class and to solve them in closed form or in explicit form [1–25]. Note that most of these equation often show increasingly complex behavior such as the existence of a bounded. The qualitative study of difference equations is a fertile research area and increasingly attracts many mathematicians. This topic draws its importance from the fact that many real life phenomena are modeled using difference equations. The applications of these difference equations can be found on the economy, biology and so on. It is known that nonlinear difference equations are capable of producing a complicated behavior regardless its order. The objective of this article is to investigate some qualitative behavior of the solutions of the nonlinear difference equation

$$x_{n+1} = \frac{\alpha x_{n-m} + \eta x_{n-k} + \sigma x_{n-l} + \delta x_n}{\beta + \gamma x_{n-k} x_{n-l} (x_{n-k} + x_{n-l})}, \quad n = 0, 1, 2, \dots \quad (1.1)$$

where the parameters $\alpha, \beta, \gamma, \delta, \eta, \sigma \in (0, \infty)$, while m, k, l are positive integers, such that $m < k < l$. The initial conditions $x_{-m}, \dots, x_{-k}, \dots, x_{-l}, \dots, x_{-1}, \dots, x_0$ are arbitrary positive real numbers equation (1.1) has been discussed in [26] when $m = 1, k = 2$ and $l = 4$, and in [27] when $\delta = 0$, where some global behavior of the more general nonlinear rational difference equation (1.1), we need the following well-known definitions and results [28–34].

Definition 1.1. A difference equation of order $(k + 1)$ is of the form

$$x_{n+1} = F(x_n, x_{n-1}, \dots, x_{-k}), \quad n = 0, 1, 2, \dots \quad (1.2)$$

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where F is a continuous function which maps some set J^{k+1} into J and J is a set of real numbers. An equilibrium point \tilde{x} of this equation is a point that satisfies the condition $\tilde{x} = F(\tilde{x}, \tilde{x}, \dots, \tilde{x})$. That is, the constant sequence $\{x_n\}_{n=-k}^{\infty}$ with $x_n = \tilde{x}$ for all $n \geq -k$ is a solution of that equation.

Definition 1.2. Let $\tilde{x} \in (0, \infty)$ be an equilibrium point of the difference equation (1.2). Then

- (i) An equilibrium point \tilde{x} of the difference equation (1.2) is called locally stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that, if $x_{-k}, \dots, x_{-1}, x_0 \in (0, \infty)$ with $|x_{-k} - \tilde{x}| + \dots + |x_{-1} - \tilde{x}| + |x_0 - \tilde{x}| < \delta$, then $|x_n - \tilde{x}| < \varepsilon$ for all $n \geq -k$.
- (ii) An equilibrium point \tilde{x} of the difference equation (1.2) is called locally asymptotically stable if it is locally stable and there exists $\gamma > 0$ such that, if $x_{-k}, \dots, x_{-1}, x_0 \in (0, \infty)$ with $|x_{-k} - \tilde{x}| + \dots + |x_{-1} - \tilde{x}| + |x_0 - \tilde{x}| < \gamma$, then

$$\lim_{n \rightarrow \infty} x_n = \tilde{x}.$$

- (iii) An equilibrium point \tilde{x} of the difference equation (1.2) is called a global attractor if for every $x_{-k}, \dots, x_{-1}, x_0 \in (0, \infty)$ we have

$$\lim_{n \rightarrow \infty} x_n = \tilde{x}.$$

- (iv) An equilibrium point \tilde{x} of the equation (1.2) is called globally asymptotically stable if it is locally stable and a global attractor.
- (v) An equilibrium point \tilde{x} of the difference equation (1.2) is called unstable if it is not locally stable.

Definition 1.3. A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$. A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with prime period p if p is the smallest positive integer having this property.

Definition 1.4. We say that a sequence $\{x_n\}_{n=-l}^{\infty}$ is bounded and persisting if there exists positive constants m and M such that

$$m \leq x_n \leq M \quad \text{for all } n \geq -k.$$

Definition 1.5. A positive semicycle of $\{x_n\}_{n=-k}^{\infty}$ consists of "a string" of terms x_l, x_{l+1}, \dots, x_m all greater than or equal to \tilde{x} , with $l \geq -k$ and $m \leq \infty$ such that

$$\text{either } l = -k \text{ or } l > -k \text{ and } x_{l-1} < \tilde{x},$$

and

$$\text{either } m = \infty \text{ or } m < \infty \text{ and } x_{m+1} < \tilde{x}.$$

A negative semicycle of $\{x_n\}_{n=-k}^{\infty}$ consists of "a string" of terms x_l, x_{l+1}, \dots, x_m all less than \tilde{x} , with $l \geq -k$ and $m \leq \infty$ such that

$$\text{either } l = -k \text{ or } l > -k \text{ and } x_{l-1} \geq \tilde{x},$$

and

$$\text{either } m = \infty \text{ or } m < \infty \text{ and } x_{m+1} \geq \tilde{x}.$$

Definition 1.6. The linearized equation of equation (1.2) about the equilibrium point \tilde{x} is the linear difference equation

$$y_{n+1} = \sum_{i=0}^k \frac{\partial F(\tilde{x}, \tilde{x}, \dots, \tilde{x})}{\partial x_{n-i}} y_{n-i}. \tag{1.3}$$

Now assume that the characteristic equation associated with equation (1.3) is

$$p(\lambda) = p_0 \lambda^k + p_1 \lambda^{k-1} + \dots + p_{k-1} \lambda + p_k = 0 \tag{1.4}$$

where

$$p_i = \partial F(\tilde{x}, \tilde{x}, \dots, \tilde{x}) / \partial x_{n-i}.$$

Theorem 1.7. Assume that $p_i \in \mathbb{R}$, $i = 1, 2, \dots$, and $k \in \{0, 1, 2, \dots\}$. Then

$$\sum_{i=1}^k |p_i| < 1$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+k} + p_1 x_{n+k-1} + \dots + p_k x_n = 0, \quad n = 0, 1, 2, \dots$$

Theorem 1.8 (The Linearized Stability Theorem). Suppose F is a continuously differentiable function defined on an open neighbourhood of the equilibrium \tilde{x} . Then the following statements are true.

- (i) If all roots of the characteristic equation (1.4) of the linearized equation (1.3) have an absolute value less than one, then the equilibrium point \tilde{x} is locally asymptotically stable.
- (ii) If at least one root of equation (1.4) has an absolute value greater than one, then the equilibrium point \tilde{x} is unstable.

2. Change of Variables

By using the change of variables $x_n = \left(\frac{\beta}{\gamma}\right)^{\frac{1}{3}} y_n$, the equation (1.1) reduces to the following difference equation

$$y_{n+1} = \frac{ry_{n-m} + ty_{n-k} + uy_{n-l} + sy_n}{1 + y_{n-k}y_{n-l}(y_{n-k} + y_{n-l})}, \quad n = 0, 1, 2, \dots \quad (2.1)$$

where $r = \frac{\alpha}{\beta} > 0$, $s = \frac{\delta}{\beta} > 0$, $u = \frac{\sigma}{\beta} > 0$, $t = \frac{\eta}{\beta} > 0$ and the initial conditions $y_{-l}, \dots, y_{-k}, \dots, y_{-m}, \dots, y_{-l}, y_0 \in (0, \infty)$. In the next section, we shall study the global behavior of equation (2.1).

3. The Dynamics of Equation (2.1)

The equilibrium points \tilde{y} of the equation (2.1) are the positive solutions of the equation

$$\tilde{y} = \frac{[r+s+t+u]\tilde{y}}{1 + 2\tilde{y}^3}. \quad (3.1)$$

Thus $\tilde{y}_1 = 0$ is always an equilibrium point of the equation (2.1). If $(r+s+t+u) > 1$, then the only positive equilibrium point \tilde{y}_2 of equation (2.1) is given by

$$\tilde{y}_2 = \left(\frac{[r+s+t+u]-1}{2}\right)^{\frac{1}{3}}. \quad (3.2)$$

Let us introduce a continuous function $F : (0, \infty)^4 \rightarrow (0, \infty)$ which is defined by

$$F(v_0, v_1, v_2, v_3) = \frac{rv_0 + sv_1 + tv_2 + uv_3}{1 + v_2^2v_3 + v_2v_3^2}. \quad (3.3)$$

Consequently, we get

$$\frac{\partial F(v_0, v_1, v_2, v_3)}{\partial v_0} = \frac{r}{1 + v_2^2v_3 + v_2v_3^2},$$

$$\frac{\partial F(v_0, v_1, v_2, v_3)}{\partial v_1} = \frac{s}{1 + v_2^2v_3 + v_2v_3^2},$$

$$\frac{\partial F(v_0, v_1, v_2, v_3)}{\partial v_2} = \frac{t(1 + v_2^2v_3 + v_2v_3^2) - (rv_0 + sv_1 + tv_2)(2v_2v_3 + v_3^2)}{(1 + v_2^2v_3 + v_2v_3^2)^2},$$

$$\frac{\partial F(v_0, v_1, v_2, v_3)}{\partial v_3} = \frac{u(1 + v_2^2v_3 + v_2v_3^2) - (rv_0 + sv_1 + tv_2 + uv_3)(v_2^2 + 2v_2v_3)}{(1 + v_2^2v_3 + v_2v_3^2)^2}.$$

At $\tilde{y}_1 = 0$, we have $\frac{\partial F(0,0,0,0)}{\partial v_0} = r$, $\frac{\partial F(0,0,0,0)}{\partial v_1} = s$, $\frac{\partial F(0,0,0,0)}{\partial v_2} = t$, $\frac{\partial F(0,0,0,0)}{\partial v_3} = u$, and the linearized equation of equation (2.1) about $\tilde{y}_1 = 0$, is the equation

$$z_{n+1} - \rho_0 z_n - \rho_1 z_{n-m} - \rho_2 z_{n-k} - \rho_3 z_{n-l} = 0 \quad (3.4)$$

where $\rho_0 = s$, $\rho_1 = r$, $\rho_2 = t$, $\rho_3 = u$. At $\tilde{y}_2 = \left(\frac{[r+s+t+u]-1}{2}\right)^{\frac{1}{3}}$, we have

$$\frac{\partial F(\tilde{y}_2, \tilde{y}_2, \tilde{y}_2, \tilde{y}_2)}{\partial v_0} = \frac{r}{1 + 2\tilde{y}_2^3} = \frac{r}{1 + ([r+s+t+u]-1)} = \frac{r}{[r+s+t+u]},$$

$$\frac{\partial F(\tilde{y}_2, \tilde{y}_2, \tilde{y}_2, \tilde{y}_2)}{\partial v_1} = \frac{s}{1 + 2\tilde{y}_2^3} = \frac{s}{1 + ([r+s+t+u]-1)} = \frac{s}{[r+s+t+u]},$$

$$\frac{\partial F(\tilde{y}_2, \tilde{y}_2, \tilde{y}_2, \tilde{y}_2)}{\partial v_2} = \frac{2t - 3([r+s+t+u]-1)}{2[r+s+t+u]},$$

$$\frac{\partial F(\tilde{y}_2, \tilde{y}_2, \tilde{y}_2, \tilde{y}_2)}{\partial v_3} = \frac{2u - 3([r+s+t+u] - 1)}{2[r+s+t+u]}.$$

And the linearized equation of equation (2.1) about $\tilde{y}_2 = \left(\frac{[r+s+t+u]-1}{2}\right)^{\frac{1}{3}}$ is the equation

$$z_{n+1} - \rho_0 z_n - \rho_1 z_{n-m} - \rho_2 z_{n-k} - \rho_3 z_{n-l} = 0, \tag{3.5}$$

where $\rho_0 = \frac{s}{[r+s+t+u]}$, $\rho_1 = \frac{r}{[r+s+t+u]}$, $\rho_2 = \frac{2t-3([r+s+t+u]-1)}{2[r+s+t+u]}$, $\rho_3 = \frac{2u-3([r+s+t+u]-1)}{2[r+s+t+u]}$.

Theorem 3.1. (i) If $[r+s+t+u] < 1$, then the equilibrium point $\tilde{y}_1 = 0$ is locally asymptotically stable.

(ii) If $[r+s+t+u] > 1$, then the equilibrium point $\tilde{y}_1 = 0$ is unstable.

(iii) If $[r+s+t+u] > 1$, $2t > 3([r+s+t+u] - 1)$, then the equilibrium point $\tilde{y}_2 = \left(\frac{[r+s+t+u]-1}{2}\right)^{\frac{1}{3}}$ is unstable.

Proof. With reference to Theorem 1.7, we deduce from equation (3.5) that $|\rho_0| + |\rho_1| + |\rho_2| + |\rho_3| = [r+s+t+u] < 1$, and then the proof of parts (i), (ii) follow. Also, from equation (3.5) we deduce for $[r+s+t+u] > 1$ that $|\rho_0| + |\rho_1| + |\rho_2| + |\rho_3| = 1 + \frac{3([r+s+t+u]-1)}{[r+s+t+u]} > 1$, and hence the proof of part (iii) follows. \square

Theorem 3.2. Assume that $[r+s+t+u] > 1$, and let $\{y_n\}_{n=-l}^\infty$ be a solution of equation (2.1) such that

$$\begin{aligned} & y_{-l}, y_{-l+2}, \dots, y_{-l+2n}, \dots, y_{-k}, y_{-k+2}, \dots, y_{-k+2n}, \dots, \\ & y_{-m+1}, y_{-m+3}, \dots, y_{-m+2n+1}, \dots, y_0 \geq \tilde{y}_2 \\ & \text{and} \\ & y_{-l+1}, y_{-l+3}, \dots, y_{-l+2n+1}, \dots, y_{-k+1}, y_{-k+3}, \dots, \\ & y_{-k+2n+1}, \dots, y_{-m}, y_{-m+2}, \dots, y_{-m+2n}, \dots, y_{-1} < \tilde{y}_2. \end{aligned} \tag{3.6}$$

Then $\{y_n\}_{n=-l}^\infty$ oscillates about $\tilde{y}_2 = \left(\frac{[r+s+t+u]-1}{2}\right)^{\frac{1}{3}}$ with a semicycle of length one.

Proof. Assume that (3.6) holds. Then

$$y_1 = \frac{ry_{-m} + sy_0 + ty_{-k} + uy_{-l}}{1 + y_{-k}y_{-l}(y_{-k} + y_{-l})} < \frac{ry_{-m} + sy_0 + ty_{-k} + uy_{-l}}{1 + 2\tilde{y}_2^3} < \frac{[r+s+t+u]\tilde{y}_2}{1 + ([r+s+t+u] - 1)} = \tilde{y}_2$$

and

$$y_2 = \frac{ry_{-m+1} + sy_1 + ty_{-k+1} + uy_{-l+1}}{1 + y_{-k+1}y_{-l+1}(y_{-k+1} + y_{-l+1})} \geq \frac{ry_{-m+1} + sy_1 + ty_{-k+1} + uy_{-l+1}}{1 + 2\tilde{y}_2^3} \geq \frac{[r+s+t+u]\tilde{y}_2}{1 + ([r+s+t+u] - 1)} = \tilde{y}_2$$

and hence the proof follows by induction. \square

Theorem 3.3. Assume that $[r+s+t+u] < 1$, then the equilibrium point $\tilde{y}_1 = 0$ of equation (2.1) is globally asymptotically stable.

Proof. We have shown in Theorem 3.3 that if $[r+s+t+u] < 1$ then the equilibrium point $\tilde{y}_1 = 0$ is locally asymptotically stable. It remains to show that $\tilde{y}_1 = 0$ is a global attractor. To this end, let $\{y_n\}_{n=-l}^\infty$ be a solution of equation (2.1). It suffices to show that $\lim_{n \rightarrow \infty} y_n = 0$. Since

$$0 \leq y_{n+1} = \frac{ry_{n-m} + sy_n + ty_{n-k} + uy_{n-l}}{1 + y_{n-k}y_{n-l}(y_{n-k} + y_{n-l})} \leq ry_{n-m} + sy_n + ty_{n-k} + uy_{n-l} < y_{n-k}.$$

Then we have $\lim_{n \rightarrow \infty} y_n = 0$. This completes the proof. \square

Theorem 3.4. Assume that $[r+s+t+u] > 1$, then equation (2.1) possesses an unbounded solution.

Proof. With the aid of Theorem 3.2, we have

$$\begin{aligned} y_{2n+2} &= \frac{ry_{-m+2n+1} + sy_{2n+1} + ty_{-k+2n+1} + uy_{-l+2n+1}}{1 + y_{-k+2n+1}y_{-l+2n+1}(y_{-k+2n+1} + y_{-l+2n+1})} > \frac{ry_{-m+2n+1} + sy_{2n+1} + ty_{-k+2n+1} + uy_{-l+2n+1}}{1 + 2\tilde{y}_2^3} \\ &> \frac{ry_{-m+2n+1} + sy_{2n+1} + ty_{-k+2n+1} + uy_{-l+2n+1}}{1 + ([r+s+t+u] - 1)} = \frac{ry_{-m+2n+1} + sy_{2n+1} + ty_{-k+2n+1} + uy_{-l+2n+1}}{[r+s+t+u]}, \end{aligned}$$

and

$$\begin{aligned} y_{2n+3} &= \frac{ry_{-m+2n+2} + sy_{2n+2} + ty_{-k+2n+2} + uy_{-l+2n+2}}{1 + y_{-k+2n+2} y_{-l+2n+2} (y_{-k+2n+2} + y_{-l+2n+2})} \leq \frac{ry_{-m+2n+2} + sy_{2n+2} + ty_{-k+2n+2} + uy_{-l+2n+2}}{1 + 2y_2^3} \\ &\leq \frac{ry_{-m+2n+2} + sy_{2n+2} + ty_{-k+2n+2} + uy_{-l+2n+2}}{1 + ([r+s+t+u] - 1)} = \frac{ry_{-m+2n+2} + sy_{2n+2} + ty_{-k+2n+2} + uy_{-l+2n+2}}{[r+s+t+u]}. \end{aligned}$$

From which it follows that

$$\lim_{n \rightarrow \infty} y_{2n} = \infty \quad \text{and} \quad \lim_{n \rightarrow \infty} y_{2n+1} = 0.$$

Hence, the proof of Theorem 3.4 is now completed. \square

Theorem 3.5. (i) If m is odd, and k, l are even, equation (2.1) has prime period two solution if $(r - [s+t+u]) < 1$ and has not prime period two solution if $(r - [s+t+u]) \geq 1$.

(ii) If m is even and k, l are odd, equation (2.1) has not prime period two solution.

(iii) If all m, k, l are even, equation (2.1) has prime period two solution.

(iv) If all m, k, l are odd, equation (2.1) has prime period two solution if $(r - [s+t+u]) > 1$, and has not prime period two solution if $(r - [s+t+u]) \leq 1$.

(v) If m, k are even and l is odd, equation (2.1) has not prime period two solution if $[r+s+t] + 1 > u$.

(vi) If m, k are odd and l is even, equation (2.1) has prime period two solution if $([r+t] - [s+u]) > 1$, and has not prime period two solution if $([r+t] - [s+u]) \leq 1$.

(vii) If m, l are odd and k is even, equation (2.1) has prime period two solution if $([r+u] - [s+t]) > 1$, and has not prime period two solution if $([r+u] - [s+t]) \leq 1$.

(viii) If m, l are even and k is odd, equation (2.1) has not prime period two solution.

Proof. Assume that there exists distinct positive solutions

$$\dots, \phi, \psi, \phi, \psi, \dots$$

of prime period two of equation (2.1).

(i) If m is odd, and k, l are even, then $y_{n+1} = y_{n-m}$ and $y_n = y_{n-k} = y_{n-l}$. It follows from equation (2.1) that

$$\phi = \frac{r\phi + [s+t+u]\psi}{1 + 2\psi^3}, \quad \psi = \frac{r\psi + [s+t+u]\phi}{1 + 2\phi^3}.$$

Consequently, we have

$$0 < 2\phi\psi(\phi + \psi) = 1 - (r - [s+t+u]). \quad (3.7)$$

We deduce that (3.7) is always true if $(r - [s+t+u]) < 1$ and hence equation (2.1) has prime period two solution. If $(r - [s+t+u]) \geq 1$, we have a contradiction, and hence equation (2.1) has not prime period two solution.

(ii) If m is even, and k, l are odd, then $y_n = y_{n-m}$, and $y_{n+1} = y_{n-k} = y_{n-l}$. It follows from equation (2.1) that

$$\phi = \frac{[t+u]\phi + [r+s]\psi}{1 + 2\phi^3}, \quad \psi = \frac{[t+u]\psi + [r+s]\phi}{1 + 2\psi^3}.$$

Consequently, we have

$$0 < 2(\phi + \psi)(\phi^2 + \psi^2) = -([r+s+t+u] + 1). \quad (3.8)$$

Since $[r+s+t+u] > 0$, we have a contradiction. Hence equation (2.1) has not prime period two solution.

(iii) If all m, k, l are even, then $y_n = y_{n-m} = y_{n-k} = y_{n-l}$. It follows from equation (2.1) that

$$\phi = \frac{[r+s+t+u]\psi}{1 + 2\psi^3}, \quad \psi = \frac{[r+s+t+u]\phi}{1 + 2\phi^3}.$$

Consequently, we get

$$0 < 2\phi\psi(\phi + \psi) = [r+s+t+u] + 1. \quad (3.9)$$

Since $[r+s+t+u] > 0$, the formula (3.9) is always true. Hence equation (2.1) has prime period two solution.

(iv) If all m, k, l are odd, then $y_{n+1} = y_{n-m} = y_{n-k} = y_{n-l}$. It follows from equation (2.1) that

$$\phi = \frac{r\psi + [s+t+u]\phi}{1+2\phi^3}, \quad \psi = \frac{r\phi + [s+t+u]\psi}{1+2\psi^3}.$$

Consequently, we get

$$0 < 2(\phi + \psi)(\phi^2 + \psi^2) = (r - [s+t+u]) - 1. \tag{3.10}$$

If $(r - [s+t+u]) > 1$, the formula (3.10) is always true, and hence equation (2.1) has prime period two solution. If $(r - [s+t+u]) \leq 1$, we have a contradiction and hence equation (2.1) has not prime period two solution.

(v) If m, k are even, and l is odd, then $y_n = y_{n-k} = y_{n-m}$, and $y_{n+1} = y_{n-l}$. It follows from equation (2.1) that

$$\phi = \frac{[r+s+t]\psi + u\phi}{1 + \psi^2\phi + \psi\phi^2}, \quad \psi = \frac{[r+s+t]\phi + u\psi}{1 + \phi^2\psi + \phi\psi^2}.$$

Consequently, we have

$$0 < \phi\psi(\phi + \psi) = u - ([r+s+t] + 1). \tag{3.11}$$

Since $[r+s+t] + 1 > u$, we have a contradiction. Hence equation (2.1) has not a prime period two solution.

(vi) If m, k are odd, and l is even, then $y_{n+1} = y_{n-m} = y_{n-k}$, and $y_n = y_{n-l}$. It follows from equation (2.1) that

$$\phi = \frac{[r+t]\phi + [s+u]\psi}{1 + \phi\psi(\phi + \psi)}, \quad \psi = \frac{[r+t]\psi + [s+u]\phi}{1 + \phi\psi(\phi + \psi)}.$$

Consequently, we have

$$0 < \phi\psi(\phi + \psi) = ([r+t] - [s+u]) - 1. \tag{3.12}$$

If $([r+t] - [s+u]) > 1$, the formula (3.12) is always true, and hence equation (2.1) has prime period two solution. If $([r+t] - [s+u]) \leq 1$, we have a contradiction. Hence equation (2.1) has not a prime period two solution.

(vii) If m, l are odd, and k is even, then $y_{n+1} = y_{n-m} = y_{n-l}$, and $y_n = y_{n-k}$. It follows from equation (2.1) that

$$\phi = \frac{[r+u]\phi + [s+t]\psi}{1 + \phi\psi(\phi + \psi)}, \quad \psi = \frac{[r+u]\psi + [s+t]\phi}{1 + \phi\psi(\phi + \psi)}.$$

Consequently, we have

$$0 < \phi\psi(\phi + \psi) = ([r+u] - [s+t]) - 1. \tag{3.13}$$

If $([r+u] - [s+t]) > 1$, the formula (3.13) is always true, and hence equation (2.1) has prime period two solution. If $([r+u] - [s+t]) \leq 1$, we have a contradiction. Hence equation (2.1) has not a prime period two solution.

(viii) If m, l are even, and k is odd, then $y_n = y_{n-m} = y_{n-l}$, and $y_{n+1} = y_{n-k}$. It follows from equation (2.1) that

$$\phi = \frac{[r+s+u]\psi + t\phi}{1 + \psi^2\phi + \psi\phi^2}, \quad \psi = \frac{[r+s+u]\phi + t\psi}{1 + \phi^2\psi + \phi\psi^2},$$

Consequently, we have

$$0 < \phi\psi(\phi + \psi) = t - ([r+s+u] + 1). \tag{3.14}$$

Since $[r+s+u] + 1 > t$, we have a contradiction. Hence equation (2.1) has not a prime period two solution. Hence the proof of Theorem (3.5) is now completed.

□

4. Numerical Examples

In order to illustrate the results of the previous section and to support our theoretical discussions, we consider some numerical examples in this section. These examples represent different types of qualitative behavior of solutions of equation (2.1).

Example 4.1. Figure 4.1 shows that the solution of equation (2.1) is bounded if $x_{-3} = 1$, $x_{-2} = 2$, $x_{-1} = 3$, $x_0 = 4$, $m = 1$, $k = 2$, $l = 3$, $r = 0.1$, $s = 0.2$, $t = 0.3$, $u = 0.25$, i.e. $[r+s+t+u] < 1$.

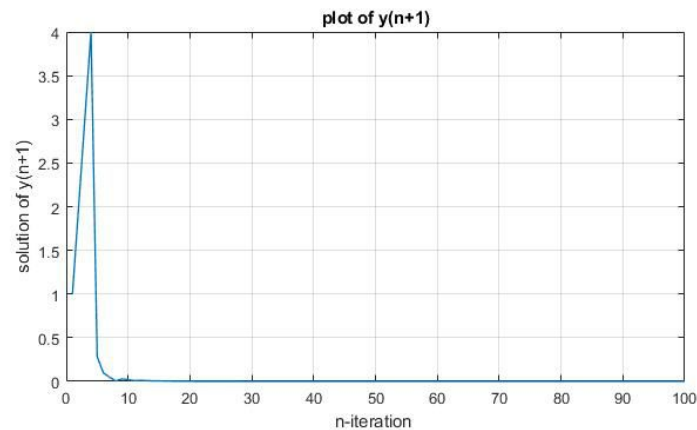


Figure 4.1: The solution of equation (2.1) is bounded.

Example 4.2. Figure 4.2 shows that the solution of equation (2.1) is unbounded if $x_{-3} = 1$, $x_{-2} = 2$, $x_{-1} = 3$, $x_0 = 4$, $m = 1$, $k = 2$, $l = 3$, $r = 1$, $s = 2$, $t = 3$, $u = 4$, i.e. $[r+s+t+u] > 1$.

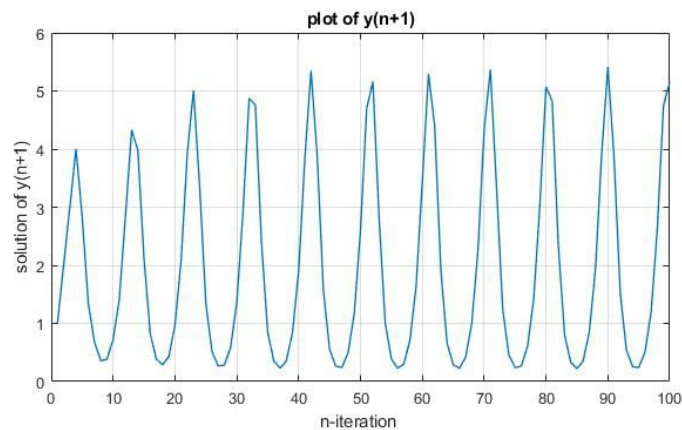


Figure 4.2: The solution of equation (2.1) is unbounded.

Example 4.3. Figure 4.3 shows that equation (2.1) is globally asymptotically stable if $x_{-4} = 1$, $x_{-3} = 2$, $x_{-2} = 3$, $x_{-1} = 4$, $x_0 = 5$, $m = 2$, $k = 3$, $l = 4$, $r = 0.1$, $s = 0.5$, $t = 0.2$, $u = 0.25$, i.e. $[r+s+t+u] < 1$.

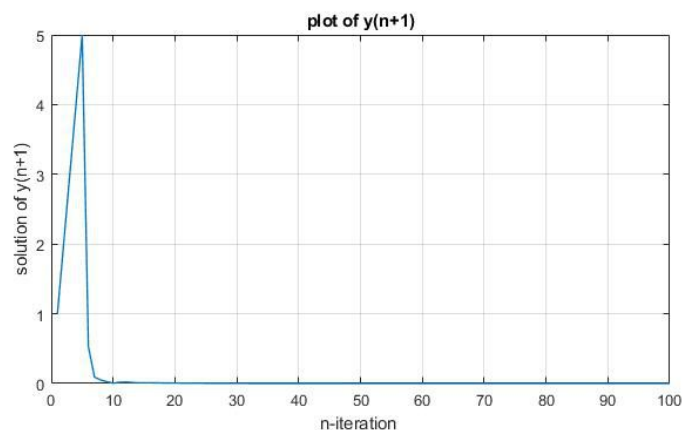


Figure 4.3: The solution of equation (2.1) is globally asymptotically stable.

Example 4. Figure 4.4 shows that equation (2.1) has no positive prime period two solutions if $x_{-3} = 1$, $x_{-2} = 2$, $x_{-1} = 3$, $x_0 = 4$, $m = 2$, $k = 1$, $l = 3$, $r = 100$, $s = 300$, $t = 400$, $u = 500$.

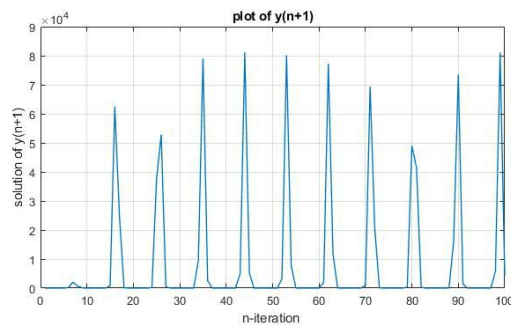


Figure 4.4: The solution of equation (2.1) has no positive prime period two solutions.

5. Conclusions

In this article, we have shown that equation (2.1) has two equilibrium points $\tilde{y}_1 = 0$ and $\tilde{y}_2 = \left(\frac{[r+s+t+u]-1}{2}\right)^{\frac{1}{3}}$. If $[r+s+t+u] < 1$, we have proved that $\tilde{y}_1 = 0$ is globally asymptotically stable, while if $[r+s+t+u] > 1$, the solution of equation (2.1) oscillates about the point $\tilde{y}_2 = \left(\frac{[r+s+t+u]-1}{2}\right)^{\frac{1}{3}}$ with a semicycle of length one. When $[r+s+t+u] > 1$, we have proved that the solution of equation (2.1) is unbounded. The periodicity of the solution of equation (2.1) has been discussed in details in Theorem 3.5.

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