SAKARYA ÜNİVERSİTESİ
FEN BİLİMLERİ ENSTİTÜSÜ DERGİSİ

Sakarya University Journal of Science
SAUJS

ISSN 1301-4048 e-ISSN 2147-835X Period Bimonthly Founded 1997 Publisher Sakarya University
http://www.saujs.sakarya.edu.tr/

Title: Some Spectral Properties of Schrödinger Operators on Semi Axis

Authors: İbrahim ERDAL

Recieved: 2023-01-20 00:00:00

Accepted: 2023-02-24 00:00:00

Article Type: Research Article

Volume: 27
Issue: 3
Month: June
Year: 2023
Pages: 542-549

How to cite
İbrahim ERDAL; (2023), Some Spectral Properties of Schrödinger Operators on Semi Axis. Sakarya University Journal of Science, 27(3), 542-549, DOI: 10.16984/saufenbilder.1240115
Access link
https://dergipark.org.tr/en/pub/saufenbilder/issue/78131/1240115

Sakarya University Journal of Science 27(3), 542-549, 2023

Some Spectral Properties of Schrödinger Operators on Semi Axis

İbrahim ERDAL [*1]

Abstract

The main aim of this work is to investigate some spectral properties of Schrödinger operators on semi axis. We first present the Schrödinger equation with a piecewise continuous potential function $\boldsymbol{q}$ so that the problem differs from the classical Schrödinger problems. Then, we get the Wronskian of two specific solution of the given equation which helps us to create the sets of eigenvalues and spectral singularities. The rest of the paper deals with eigenvalues and spectral singularities. By the help of the analytical properties of Jost solutions and resolvent operator of the Schrödinger operators, we provide sufficient conditions guarenteeing finiteness of eigenvalues and spectral singularities with finite multiplicities.

Keywords: Schrödinger operators, eigenvalues, spectral singularities, resolvent operator

1. INTRODUCTION

In this study, we take into consideration the following one dimensional time independent Schrödinger equation on the semi axis

$$-y'' + q(x)y = \lambda^2 y, \quad 0 \le x < \infty \qquad (1)$$

with boundary condition

$$y(0) = 0, \qquad (2)$$

where λ is a spectral parameter and q is a complex valued potential function.

Before starting to examine our problem, let us briefly give an outline about the studies in the literature on the spectral theory of differential operators without any discontinuity. The spectral analysis of Schrödinger operators or differential operators were introduced by Naimark [1]. He showed that some poles of the kernel of resolvent operator of Schrödinger operator are not the eigenvalues of the operator. Naimark proved that these poles are a mathematical obstacle for the completeness of the eigenvectors and are imbedded in the continuous spectrum as well. These poles which are called spectral singularities by Schwartz [2] are very significant in physics and mathematics especially in quantum theory and applied mathematics.

Another technique for the treatment of the spectral theory of Schrödinger operator was presented by Marchenko [3]. He exposed that equation (1) has a bounded solution fulfilling the condition

$$\lim_{x\to\infty} e(x,\lambda)e^{-i\lambda x} = 1,$$

* Corresponding author: ierdal@ankara.edu.tr (I. ERDAL)
[1] Ankara University, Faculty of Science, Department of Mathematics
ORCID: https://orcid.org/0000-0002-4445-2389

where

$\lambda \in \overline{\mathbb{C}}_{+} := \{\lambda \in \mathbb{C}: Im\lambda \geq 0\}$.

$e(x,\lambda)$ is called the Jost solution of Schrödinger equation (1) and it has an image

$$e(x,\lambda) = e^{i\lambda x} + \int_x^\infty K(x,t)e^{i\lambda t}dt, \lambda \in \overline{\mathbb{C}}_{+} \quad (3)$$

under the condition

$$\int_0^\infty x|q(x)|dx < \infty, \quad (4)$$

where $K(x,t)$ is identified by the potential function q [3, 4]. Also $K(x,t)$ satisfies

$$|K(x,t)| \leq c\sigma\left(\frac{x+t}{2}\right), \quad (5)$$

$$|K_x(x,t)| \leq \frac{1}{4}\left|q\left(\frac{x+t}{2}\right)\right| + c\sigma\left(\frac{x+t}{2}\right), \quad (6)$$

where $c > 0$ is a constant and

$$\sigma(x) = \int_0^\infty |q(t)|dt.$$

Up to the present, a wide range of problems connected to the spectral theory of Schrödinger and Sturm-Liouville operators with spectral singularities have been examined [5-12].

On the other hand, spectral theory of Schrödinger and Sturm-Liouville operators with discontinuities inside an interval, namely impulsive operators were studied in detail recently [13-16]. However, in all studies, the discontinuity point is not releated to the potential function or coefficient of the operator, is related to the interval in which the operator is defined.

In light of the advancements on this theory, in this work, we present a second-order one dimensional time independent Schrödinger operator on the semi axis with piecewise continuous complex valued potential function. Since the potential function q behaves as an impulsive condition, the spectral analysis of Schrödinger operator getting more diffucult. Nevertheless, by specifying the resolvent of mentioned operator, we can give some useful spectral results.

2. STATEMENT OF THE PROBLEM

Let L denote the one dimensional time independent Schrödinger operator on the semi axis in $L^2[0,\infty)$ by the equation

$$-y'' + q(x)y = \lambda^2 y, \ \ 0 \leq x < \infty \quad (7)$$

with the boundary condition

$$y(0) = 0 \quad (8)$$

and piecewise continuous potential function

$$q(x) = \begin{cases} 0, & 0 \leq x \leq 1 \\ v(x), & 1 < x < \infty \end{cases}$$

where λ is a spectral parameter and $v \neq 0$ is a complex valued function and satisfies the condition

$$\int_1^\infty x|v(x)|dx < \infty. \quad (9)$$

It can be easily seen that $cos\lambda x$ and $sin\lambda x$ are the linearly independent solutions of equation (7) for $0 \leq x \leq 1$.

On the other part, (7) accepts another solution

$$\hat{e}(x,\lambda) = e^{-i\lambda x} + \frac{1}{2i\lambda}\int_{x_0}^x e^{i(x-t)\lambda}q(t)\hat{e}(t,\lambda)dt + \frac{1}{2i\lambda}\int_x^\infty e^{i(x-t)\lambda}q(t)\hat{e}(t,\lambda)dt \quad (10)$$

for $\lambda \in \overline{\mathbb{C}}_{+}$ in $(1,\infty)$ fulfilling the condition

$$\lim_{x\to\infty} \hat{e}(x,\lambda)e^{i\lambda x} = 1,$$

where x_0 is a positive real number. From (3) and (10), it is easy to see that

$$W[e(x,\lambda), \hat{e}(x,\lambda)] = -2i\lambda, \ \ \lambda \in \overline{\mathbb{C}}_{+},$$

where $W[y_1, y_2]$ denotes the Wronskian of the solutions y_1 and y_2 of equation (7).

By means of linearly independent solutions cosλx, sinλx and e(x,λ), $\hat{e}(x,\lambda)$ of (7) in the intervals $[0,1]$ and $(1,\infty)$ respectively, we can state two solutions of boundary value problem (7)-(8) as

$$E(x,\lambda) = \begin{cases} A(\lambda)cos\lambda x + B(\lambda)sin\lambda x, & x \in [0,1] \\ e(x,\lambda), & x \in (1,\infty) \end{cases}$$

and

$$F(x,\lambda) = \begin{cases} sin\lambda x, & x \in [0,1] \\ C(\lambda)e(x,\lambda) + D(\lambda)\hat{e}(x,\lambda), & x \in (1,\infty) \end{cases}$$

for $\lambda \in \overline{\mathbb{C}}_+$ and λ dependent arbitrary coefficients A, B, C, D. Using the continuity of the solutions $E(x,\lambda)$ and $F(x,\lambda)$ during the semi axis, we get uniquely

$$A(\lambda) = \cos\lambda\, e(1,\lambda) - \frac{sin\lambda}{\lambda} e'(1,\lambda) \qquad (11)$$

$$B(\lambda) = sin\lambda e(1,\lambda) + \frac{cos\lambda}{\lambda} e'(1,\lambda) \qquad (12)$$

$$C(\lambda) = -\frac{1}{2i\lambda}[sin\lambda\hat{e}'(1,\lambda) - \\ -\lambda cos\lambda\hat{e}(1,\lambda)] \qquad (13)$$

$$D(\lambda) = \frac{1}{2i\lambda}[sin\lambda e'(1,\lambda) - \lambda cos\lambda e(1,\lambda) \qquad (14)$$

for all $\lambda \in \overline{\mathbb{C}}_+$. With the help of solutions $E(x,\lambda)$, $F(x,\lambda)$ and (11)-(14), we can introduce the following Lemma and Theorem.

Lemma 1. The following equations are valid for all $\lambda \in \overline{\mathbb{C}}_+$:

$$W[E,F](x,\lambda) = \lambda A(\lambda), \quad x \to \infty,$$

$$W[E,F](x,\lambda) = \lambda A(\lambda), \quad x \to 0.$$

Proof. Using the definition of Wronskian of two solutions E and F, for $x \in (1,\propto)$, we obtain

$$\begin{aligned} W[E,F](x,\lambda) &= \\ &= e(x,\lambda)[C(\lambda)e'(x,\lambda) + D(\lambda)\hat{e}'(x,\lambda)] \\ &\quad -e'(x,\lambda)[C(\lambda)e(x,\lambda) + D(\lambda)\hat{e}(x,\lambda)] \\ &= D(\lambda)W[e(x,\lambda),\hat{e}(x,\lambda)] \\ &= -2i\lambda D(\lambda) \\ &= \lambda A(\lambda). \end{aligned}$$

Similarly, for $x \in [0,1]$, we get

$$W[E,F](x,\lambda) = \lambda A(\lambda).$$

Theorem 2. The resolvent operator of L has the representation

$$\Re_{\lambda^2}(L)\varphi(x) = \int_0^\infty G(x,t;\lambda)\varphi(t)dt,$$

for $\varphi \in L^2[0,\infty)$ where

$$G(x,t;\lambda) = \begin{cases} \frac{F(t,\lambda)E(x,\lambda)}{W[E,F](x,\lambda)}, & 0 \le t \le x \\ \frac{F(x,\lambda)E(t,\lambda)}{W[E,F](x,\lambda)}, & x < t < \infty \end{cases}$$

is the Green function for $\lambda \in \overline{\mathbb{C}}_+$.

Proof. Let us take into account the following nonhomogeneous Schrödinger equation:

$$-y'' + q(x)y - \lambda^2 y = \varphi(x), x \in [0,\infty) \qquad (15)$$

Take the advantage of two linearly independent solutions of homogeneous part of Schrödinger equation (15), the general solution of (15) can be written as

$$y(x,\lambda) = c_1(x)E(x,\lambda) + c_2(x)F(x,\lambda).$$

By means of variation of constants, we arrive at

$$c_1(x) = \alpha_1 + \int_0^x \frac{\varphi(t)F(t,\lambda)}{W[E,F](x,\lambda)}dt$$

and

$$c_2(x) = \alpha_2 + \int_x^\infty \frac{\varphi(t)E(t,\lambda)}{W[E,F](x,\lambda)}dt.$$

The rest of the proof is clear.

Theorem 3. Under the condition (9), the function $A(\lambda)$ fulfills the following asymptotic equation

$$A(\lambda) = 1 + o(1),\ \lambda \in \overline{\mathbb{C}}_+,\ |\lambda| \to \infty. \quad (16)$$

Proof. It is obtained from (11) that

$$\begin{aligned} A(\lambda) &= \left(cos\lambda e^{i\lambda}\right)\left(e(1,\lambda)e^{-i\lambda}\right) \\ &\quad - \left(sin\lambda e^{i\lambda}\right)\left(\frac{e'(1,\lambda)e^{-i\lambda}}{\lambda}\right) \\ &= e^{i\lambda}(cos\lambda - isin\lambda) + o(1) \\ &= 1 + o(1), \end{aligned}$$

for , $\lambda \in \overline{\mathbb{C}}_+,\ |\lambda| \to \infty$.

3. EIGENVALUES AND SPECTRAL SINGULARITIES

This section contains the primary results for the spectral analysis of Schrodinger operator L. Firstly, we introduce the sets of eigenvalues and spectral singularities of Schrödinger operator L as

$$\sigma_d(L) = \{\mu = \lambda^2 : Im\lambda > 0,\ \ A(\lambda) = 0\}$$

and

$$\sigma_{ss}(L) = \{\mu = \lambda^2 : Im\lambda = 0, \lambda \neq 0, A(\lambda) = 0\},$$

respectively.

At present, we describe the sets

$$P_1 = \{\lambda : \lambda \in \mathbb{C}_+, A(\lambda) = 0\}$$

and

$$P_2 = \{\lambda : \lambda \in \mathbb{R} \setminus \{0\}, A(\lambda) = 0\}.$$

Hence, we can write again these sets

$$\sigma_d(L) = \{\mu:\ \mu = \lambda^2, \lambda \in P_1\}$$

and

$$\sigma_{ss}(L) = \{\mu:\ \mu = \lambda^2, \lambda \in P_2\}.$$

Lemma 4. Assume (9).

(i) The set P_1 has at most countable number of elements and P_1 is bounded. If the limit points of this set exist, they are only in a bounded subinterval of the real axis.

(ii) Linear Lebesgue measure of the set P_2 is zero and P_2 is compact.

Proof. Asymptotic equation (16) shows that $A(\lambda)$ can not equal to zero for adequately large $\lambda \in \overline{\mathbb{C}}_+$. Hence, the boundedness of P_1 and P_2 comes from (16). Since $A(\lambda)$ is analytic in $\mathbb{C}_+$, the set P_1 has at most countably many elements and limit points of P_1 are only in a bounded subinterval of the real axis. Finally, by means of the uniqueness theorem of the analytic functions [17], we obtain that linear Lebesgue measure of the the set P_2 is zero and P_2 is a closed set.

The following Theorem is a straight conclusion of Lemma 4.

Theorem 5. Under condition (9),

(i) The set of eigenvalues of L has finite number of elements and it is bounded. If the limit points of this set exist, they are only in a bounded subinterval of the real axis.

(ii) The set of spectral singularities of L is compact and its linear Lebesgue measure is zero.

Proof. The proof is obtained similarly to the proof of Lemma 4.

At the moment, we continue by imposing the extra stipulation

$$\int_1^\infty \exp(\varepsilon x)\,|v(x)|dx < \infty \qquad (17)$$

on function v to ensure the finiteness of the sets of spectral singularities and eigenvalues.

Theorem 6. If the inequality (17) is satisfied for some $\varepsilon > 0$, then the Schrödinger operator L has a finite number of spectral singularities and eigenvalues, and each of them has finite multiplicity.

Proof. Using (5),(6) and (17), we obtain that

$$|K(1,t)| \leq c_1 exp\left(-\varepsilon\frac{1+t}{2}\right),\ c_1 > 0$$

and

$$\left|\int_1^\infty K_x(1,t)e^{i\lambda t}dt\right|$$
$$\leq 2e^{Im\lambda}\int_1^\infty |v(u)|\,e^{\varepsilon u}e^{-u(\varepsilon+2Im\lambda)}du +$$
$$+c_2\int_1^\infty e^{-\varepsilon/2}e^{-t(\varepsilon/2+Im\lambda)}dt,$$

where c_1, c_2 are arbitrary constants and $u=\frac{1+t}{2}$. From the last inequality and (17), we arrive at $\left|\int_1^\infty K_x(1,t)e^{i\lambda t}dt\right| < \infty$ for $Im\lambda > -\frac{\varepsilon}{2}$. Thus, $A(\lambda)$ has an analytic continuation from the real axis to the lower half plane $Im\lambda > -\frac{\varepsilon}{2}$. Hence the sets $\sigma_d(L)$ and $\sigma_{ss}(L)$ don't have any limit points on the real axis. With the help of Theorem 5, we see that this sets are bounded and have at most countable number of elements. Moreover, the uniqueness theorem of the analytic functions [17] give us, all roots of $A(\lambda)$ in $\overline{\mathbb{C}}_+$ are of finite multiplicities.

Now, let us specify the all limit points of P_1 and P_2 by P_3 and P_4, respectively. Also the set of all roots of $A(\lambda)$ with infinite multiplicity in $\overline{\mathbb{C}}_+$ is denoted by P_5.

By the help of uniqueness theorem, we can present the following Lemma.

Lemma 7. (i) $P_3 \subset P_2,\ P_4 \subset P_2,\ P_5 \subset P_2,$ $P_3 \subset P_5, P_4 \subset P_5.$

(ii) $\mu(P_3) = \mu(P_4) = \mu(P_5) = 0,$

where μ is linear Lebesgue measure.

Proof. Proof of Lemma is clear from the boundary uniqueness theorem of analytic functions in [17].

Theorem 8. If

$$\int_1^\infty e^{\varepsilon x^\delta}\,|v(x)|dx < \infty, \qquad (18)$$

for some $\varepsilon > 0$ and $\frac{1}{2} \leq \delta < 1$, then $P_5 = \emptyset$.

Proof. From (11), we get

$$A(\lambda) = \left(cos\lambda e^{i\lambda}\right)\left(e(1,\lambda)e^{-i\lambda}\right) - $$
$$-\left(\frac{sin\lambda}{\lambda}e^{i\lambda}\right)\left(e'(1,\lambda)e^{-i\lambda}\right) \qquad (19)$$

where $J_1(\lambda) = e(1,\lambda)e^{-i\lambda}$, $J_3(\lambda) = cos\lambda e^{i\lambda}$, $J_2(\lambda) = e'(1,\lambda)e^{-i\lambda}$ and $J_4(\lambda) = \frac{sin\lambda}{\lambda}e^{i\lambda}$.
It follows from (5),(6) and (18) that

$$\left|\frac{d^m}{d\lambda^m}J_1(\lambda)\right| \leq c_3\int_1^\infty t^m e^{-\varepsilon(t/2)^\delta}dt, \qquad (20)$$

$$\left|\frac{d^m}{d\lambda^m}J_2(\lambda)\right| \leq c_4\int_1^\infty t^m e^{-\varepsilon(t/2)^\delta}dt, \qquad (21)$$

$$\left|\frac{d^m}{d\lambda^m}J_3(\lambda)\right| \leq c_5 2^m, \qquad (22)$$

$$\left|\frac{d^m}{d\lambda^m}J_4(\lambda)\right| \leq c_6 2^m, \qquad (23)$$

where c_3, c_4, c_5, c_6 are arbitrary constants. Using (19)-(23), we see that

$$\left|\frac{d^m}{d\lambda^m}A(\lambda)\right| \leq \sum_{s=0}^m \binom{m}{s}\left|\frac{d^{m-s}}{d\lambda^{m-s}}J_1\right|\left|\frac{d^s}{d\lambda^s}J_3\right|$$

$$+\sum_{s=0}^{m}\binom{m}{s}\left|\frac{d^{m-s}}{d\lambda^{m-s}}J_2\right|\left|\frac{d^s}{d\lambda^s}J_3\right|$$

$$\leq c_7 2^m\int_1^\infty t^m e^{-\varepsilon(t/2)^\delta}dt, \quad (24)$$

for $m=1,2,\dots$, where c_7 is arbitrary constant.

Now, we can write

$$\left|\frac{d^m}{d\lambda^m}A(\lambda)\right|\leq A_m$$

for $m=1,2,\dots$ and $\lambda\in\overline{\mathbb{C}}_+,\ |\lambda|<N$, where

$$A_m=c_7 2^m\int_1^\infty t^m e^{-\varepsilon(t/2)^\delta}dt.$$

Since $A(\lambda)$ can't be zero, Pavlov's theorem [18] gives that

$$\int_0^k \ln \mathrm{Z}(s)\,d\mu(P_5,s)>-\infty, \quad (25)$$

where $\mathrm{Z}(s)=inf\left\{\frac{A_m s^m}{m!}:m=0,1,2,\dots\right\}$ and $\mu(P_5,s)$ is the linear Lebesgue measure of the $s-$neighbourhood of P_5. Using the definition of gamma function, we can express

$$A_m\leq K\alpha^m m^{\frac{m(1-\delta)}{\delta}}m!,$$

where K is a constant. Thus, we get

$$\mathrm{Z}(s)\leq K\alpha exp\left\{-\frac{1-\delta}{\delta}e^{-1}\alpha^{-\frac{\delta}{1-\delta}}s^{-\frac{\delta}{1-\delta}}\right\}.$$

From the last equality and (25), we obtain

$$\int_0^k s^{-\delta/(1-\delta)}\ d\mu(P_5,s)<\infty.$$

Since $\delta/(1-\delta)\geq 1$, the previous inequality gives that $P_5=\emptyset$.

Since $P_5=\emptyset$, it is easy to get the following Theorem.

Theorem9. Assume (18). Then the Schrödinger operator L has countably many number of spectral singularities and eigenvalues and each of them has finite multiplicity.

4. CONCLUSIONS

In this work, we discussed some primary spectral problems of one dimensional time independent Schrödinger operators on the semi axis. In particular, we presented the solutions of Schrödinger equations and found the resolvent of Schrödinger operators. In the last part, we investigated the structure and finiteness of spectral singularities and eigenvalues with the help of uniqueness theorems.

Funding
The author (s) has no received any financial support for the research, authorship or publication of this study.

Authors' Contribution
The authors contributed equally to the study.

The Declaration of Conflict of Interest/ Common Interest
No conflict of interest or common interest has been declared by the authors.

The Declaration of Ethics Committee Approval
This study does not require ethics committee permission or any special permission.

The Declaration of Research and Publication Ethics
The authors of the paper declare that they comply with the scientific, ethical and quotation rules of SAUJS in all processes of the paper and that they do not make any falsification on the data collected. In addition, they declare that Sakarya University Journal of Science and its editorial board have no responsibility for any ethical violations that may be encountered, and that this study has not been evaluated in any academic publication environment other than Sakarya University Journal of Science.

REFERENCES

[1] M. A. Naimark, “Investigation of the spectrum and the expansion in eigenfunctions of a non-selfadjoint operators of second order on a semi-axis,” American Mathematical Society Translations: Series 2. vol. 2, no.16, pp. 103-193, 1960.

[2] J. T. Schwartz, “Some non-selfadjoint operators,” Communications on Pure and Applied Mathematics, vol. 13, pp. 609-639, 1960.

[3] V. A. Marchenko, “Sturm-Liouville Operators and Applications Operator Theory: Advances and Applications,” vol. 22, Birkhauser, Basel 1986.

[4] B. M. Levitan, I. S. Sargsjan, “Sturm-Liouville and Dirac Operators,” Kluwer Academic Publishers, 1991.

[5] E. Bairamov, A. O. Celebi, “Spectral analysis of nonselfadjoint Schrödinger operators with spectral parameter in boundary conditions,” Facta Universitatis, Series: Mathematics and Informatics, vol. 13, pp. 79-94, 1998.

[6] E. Bairamov, O. Cakar, A. M. Krall, “An eigenfunction expansion for a quadratic pencil of a Schrödinger operator with spectral singularities,” Journal of Differential Equations, vol. 151, pp. 268-289, 1999.

[7] M. Adıvar, E. Bairamov, “ Spectral singularities of the nonhomogeneous Sturm-Liouville equations,” Applied Mathematics Letters, vol. 15, no.7, pp. 825-832, 2002.

[8] E. Bairamov, E. Kir, ”Spectral properties of a finite system of Sturm-Liouville differential operators,” Indian Journal of Pure and Applied Mathematics, vol. 35, no.2, pp. 249-256, 2004.

[9] G. Sh. Guseinov, “On the concept of spectral singularities,” Pramana-Journal of Physics, vol. 73, no.3, pp. 587-603, 2009.

[10] A. M. Krall, E. Bairamov, O. Cakar, “Spectrum and spectral singularities of a quadratic pencil of a Schrödinger operator with a general boundary condition,” Journal of Differential Equations, vol. 151, no.2, pp. 252-267, 1999.

[11] A. Mostafazadeh, “Optical spectral singularities as treshold resonances,” Physical Review A Third Series-83:045801, 2011.

[12] E. Bairamov, S. Cebesoy, “Spectral singularities of the matrix Schrödinger equations,” Hacettepe Journal of Mathematics and Statistics, vol. 45, no.4, pp. 1007-1014, 2016.

[13] S. Cebesoy, “Examination of eigenvalues and spectral singularities of a discrete Dirac operator with an interaction point,” Turkish Journal of Mathematics, vol. 46, no.1, pp. 157-166, 2022.

[14] Ş. Yardımcı, İ. Erdal, “Investigation of an impulsive Sturm-Liouville operator on semi axis,” Hacettepe Journal of Mathematics and Statistics, vol. 48, no.5, pp. 1409-1416, 2019.

[15] Y. Aygar, G. G. Özbey, “Scattering analysis of a quantum impulsive boundary value problem with spectral parameter,” Hacettepe Journal of Mathematics and Statistics, vol. 51, no.1, pp. 142-155, 2022.

[16] T. Köprübaşi, Y. Aygar Küçükevcilioğlu, “Discrete impulsive Sturm-Liouville equation with hyperbolic eigenparameter,” Turkish

Journal of Mathematics, vol. 46, no.2, pp. 387-396, 2022.
[17] E. P. Dolzhenko, “Boundary value uniqueness theorems for analytic functions,” Mathematical Notes, vol. 26, pp. 437-442, 1979.

[18] B. S. Pavlov, “The non-selfadjoint Schrödinger operators,” Mathematical Physics, vol. 1, pp. 87-114, 1967.