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Research Article

D-Homothetic Deformations and Almost Paracontact Metric Manifolds

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Abstract: In this study, we determine some of the classes of almost paracontact metric structures which are invariant under D-homothetic deformations. We write the Riemannian curvature tensor, the Ricci tensor and the scalar curvature when the characteristic vector field is Killing. In addition, we give examples.

Keywords: Almost paracontact metric structure, D-homothetic deformation, Killing vector field.

1. Introduction

Differentiable manifolds having almost paracontact structures were introduced by [5] and after [11] many authors have made contribution, see [7, 9, 11–13] and references therein. Manifolds with almost paracontact metric structure were classified according to the Levi-Civita covariant derivative of the fundamental tensor. There are 2^{12} classes of almost paracontact metric manifolds. The defining relations and projections onto each subspace are given in [7, 13].

D-homothetic deformations of almost contact metric manifolds is extensively studied, see [1, 3] and references therein. For D-homothetic deformations of almost contact metric structures with B-metric, refer to [2]. D-homothetic deformations of almost paracontact metric structures were introduced in [11]. In [10], almost paracontact metric manifolds whose characteristic vector field is parallel are considered and their D-homothetic deformations are studied. Our aim is to investigate D-homothetic deformations of almost paracontact metric manifolds having arbitrary characteristic vector fields.

2. Preliminaries

Assume that M^{2n+1} is a smooth manifold having odd dimension. An ordered triple (φ, ξ, η) of an endomorphism, a vector field, a 1-form, respectively, with the properties below is called an almost

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paracontact structure on M

$$\varphi^2 = I - \eta \otimes \xi, \qquad \eta(\xi) = 1, \qquad \varphi(\xi) = 0,$$

there is a distribution $\mathbb{D} : p \in M \longrightarrow \mathbb{D}_p = Ker\eta$. M together with the almost paracontact structure is said to be an almost paracontact manifold. In addition, if M carries a semi-Riemannian metric g satisfying

$$g(\varphi(x), \varphi(y)) = -g(x,y) + \eta(x)\eta(y),$$

where $\mathfrak{X}(M)$ is the set of smooth vector fields on M and $x, y \in \mathfrak{X}(M)$, then M is called an almost paracontact metric manifold. The fundamental 2-form of the almost paracontact metric structure is given as

$$\Phi(x,y) = g(\varphi x, y).$$

We denote the vector fields and tangent vectors by letters x, y, z.

Consider the tensor F defined by

$$F(x,y,z) = g((\nabla_x \varphi)(y), z), \tag{1}$$

for all $x, y, z \in T_pM$, where T_pM is the tangent space at p, ∇ is the Levi-Civita covariant derivative of g. Then F satisfies

$$F(x,y,z) = -F(x,z,y), \tag{2}$$

$$F(x, \varphi y, \varphi z) = F(x,y,z) + \eta(y)F(x,z,\xi) - \eta(z)F(x,y,\xi). \tag{3}$$

The forms below are defined for any almost paracontact metric structure.

$$\theta(x) = g^{ij}F(e_i, e_j, x), \quad \theta^*(x) = g^{ij}F(e_i, \varphi e_j, x), \quad \omega(x) = F(\xi, \xi, x),$$

where $u \in T_pM$, $\{e_i, \xi\}$ is a basis for T_pM and the inverse of the matrix g_{ij} is g^{ij}.

Let $\mathcal{F}$ be the set of $(0,3)$ tensors over T_pM having properties (2), (3). $\mathcal{F}$ is the direct sum of four subspaces W_i, $i = 1, \ldots, 4$, where projections F^{W_i} we use are

$$F^{W_1}(x,y,z) = F(\varphi^2 x, \varphi^2 y, \varphi^2 z), \tag{4}$$

$$F^{W_2}(x,y,z) = -\eta(y)F(\varphi^2 x, \varphi^2 z, \xi) + \eta(z)F(\varphi^2 x, \varphi^2 y, \xi). \tag{5}$$

In addition, W_1 is a direct sum of four subspaces $\mathbb{G}_i$, $i = 1, \ldots, 4$, $W_2 = \mathbb{G}_5 \oplus \ldots \oplus \mathbb{G}_{10}$, and denote W_3 and W_4 by $\mathbb{G}_{11}$ and $\mathbb{G}_{12}$, respectively. A manifold with almost paracontact metric structure is said to be in the class $\mathbb{G}_i \oplus \mathbb{G}_j$, etc. if F belongs to $\mathbb{G}_i \oplus \mathbb{G}_j$ over T_pM for all $p \in M$. The defining relations of $\mathbb{G}_i$ and projections F^i onto each $\mathbb{G}_i$ are given in [7, 13]. We only write the classes and projections we use:

$$\mathbb{G}_5: \ F(x,y,z) = \frac{\theta_F(\xi)}{2n}\{g(\varphi x,\varphi z)\eta(y) - g(\varphi x,\varphi y)\eta(z)\} \tag{6}$$

$$\begin{aligned} \mathbb{G}_8: \ F(x,y,z) &= -\eta(y)F(x,z,\xi) + \eta(z)F(x,y,\xi), \\ F(x,y,\xi) &= F(y,x,\xi) = -F(\varphi x,\varphi y,\xi), \quad \theta_F(\xi) = 0 \end{aligned} \tag{7}$$

$$\begin{aligned} \mathbb{G}_9: \ F(x,y,z) &= -\eta(y)F(x,z,\xi) + \eta(z)F(x,y,\xi), \\ F(x,y,\xi) &= -F(y,x,\xi) = F(\varphi x,\varphi y,\xi) \end{aligned} \tag{8}$$

$$\begin{aligned} \mathbb{G}_{10}: \ F(x,y,z) &= -\eta(y)F(x,z,\xi) + \eta(z)F(x,y,\xi), \\ F(x,y,\xi) &= F(y,x,\xi) = F(\varphi x,\varphi y,\xi) \end{aligned} \tag{9}$$

$$\mathbb{G}_{11}: \ F(x,y,z) = \eta(x)F(\xi,\varphi y,\varphi z) \tag{10}$$

$$\mathbb{G}_{12}: \ F(x,y,z) = \eta(x)\{\eta(y)F(\xi,\xi,z) - \eta(z)F(\xi,\xi,y)\} \tag{11}$$

Some of the projections F^i onto each subspace $\mathbb{G}_i$ are

$$\begin{aligned} F^9(x,y,z) &= -\frac{1}{4}\eta(y)\left\{F(\varphi^2 x,\varphi^2 z,\xi) + F(\varphi x,\varphi z,\xi)\right. \\ &\left. -F(\varphi^2 z,\varphi^2 x,\xi) - F(\varphi z,\varphi x,\xi)\right\} + \frac{1}{4}\eta(z)\left\{F(\varphi^2 x,\varphi^2 y,\xi)\right. \\ &\left. +F(\varphi x,\varphi y,\xi) - F(\varphi^2 y,\varphi^2 x,\xi) - F(\varphi y,\varphi x,\xi)\right\}, \end{aligned} \tag{12}$$

$$\begin{aligned} F^{10}(x,y,z) &= -\frac{1}{4}\eta(y)\left\{F(\varphi^2 x,\varphi^2 z,\xi) + F(\varphi x,\varphi z,\xi)\right. \\ &\left. +F(\varphi^2 z,\varphi^2 x,\xi) + F(\varphi z,\varphi x,\xi)\right\} + \frac{1}{4}\eta(z)\left\{F(\varphi^2 x,\varphi^2 y,\xi)\right. \\ &\left. +F(\varphi x,\varphi y,\xi) + F(\varphi^2 y,\varphi^2 x,\xi) + F(\varphi y,\varphi x,\xi)\right\}, \end{aligned} \tag{13}$$

$$F^{11}(x,y,z) = \eta(x)F(\xi,\varphi^2 y,\varphi^2 z), \tag{14}$$

$$F^{12}(x,y,z) = \eta(x)\{\eta(y)F(\xi,\xi,\varphi^2 z) - \eta(z)F(\xi,\xi,\varphi^2 y)\}. \tag{15}$$

Note that ξ is Killing in any direct sum of $\mathbb{G}_1,\mathbb{G}_2,\mathbb{G}_3,\mathbb{G}_4,\mathbb{G}_5,\mathbb{G}_8,\mathbb{G}_9,\mathbb{G}_{11}$ and ξ is parallel in $\mathbb{G}_1$, $\mathbb{G}_2$, $\mathbb{G}_3$, $\mathbb{G}_4$, $\mathbb{G}_{11}$ and also in any direct sum of these classes [10].

For any almost paracontact metric sructure (φ, ξ, η, g) on a manifold M, consider the quadruple $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ where

$$\tilde{\varphi} = \varphi, \quad \tilde{\xi} = \frac{1}{t}\xi, \quad \tilde{\eta} = t\eta, \quad \tilde{g} = -tg + t(t+1)\eta \otimes \eta \tag{16}$$

for a positive constant t [11]. The structure $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is called a D-homothetic deformation of (φ, ξ, η, g). In [10], the Levi-Civita covariant derivative $\tilde{\nabla}$ of metric $\tilde{g}$ is obtained as

$$\begin{aligned} g(\tilde{\nabla}_x y, z) &= g(\nabla_x y, z) + \frac{(t+1)^2}{2t}\eta(z)\{-\eta(x)g(\nabla_\xi \xi, y) \\ &\quad -\eta(y)g(\nabla_\xi \xi, x) + g(\nabla_x \xi, y) + g(\nabla_y \xi, x)\} \\ &\quad -\frac{(t+1)}{2}\{\eta(x)(g(\nabla_y \xi, z) - g(\nabla_z \xi, y)) \\ &\quad +\eta(y)(g(\nabla_x \xi, z) - g(\nabla_z \xi, x)) \\ &\quad +\eta(z)(g(\nabla_x \xi, y) + g(\nabla_y \xi, x))\}. \end{aligned} \tag{17}$$

Also it is proved that the classes with parallel characteristic vector field does not change after D-homothetic deformations. Our aim is to study the invariance of remaining basic classes $\mathbb{G}_5$, $\mathbb{G}_6$, $\mathbb{G}_7$, $\mathbb{G}_8$, $\mathbb{G}_9$, $\mathbb{G}_{10}$, $\mathbb{G}_{12}$. We also write the curvature tensors of the deformed metric when ξ is Killing and we give examples.

3. Classes of Deformed Structures

Consider a D-homothetic deformation given by (16).

First let ξ be Killing. In this case (17) simplifies into

$$\begin{aligned} g(\tilde{\nabla}_x y, z) &= g(\nabla_x y, z) - (t+1)\{\eta(x)g(\nabla_y \xi, z) \\ &\quad +\eta(y)g(\nabla_x \xi, z)\}, \end{aligned} \tag{18}$$

since g is non-degenerate, (18) gives

$$\tilde{\nabla}_x y = \nabla_x y - (t+1)\{\eta(x)\nabla_y \xi + \eta(y)\nabla_x \xi\}. \tag{19}$$

The Proposition 3.1 yields from (19).

Proposition 3.1 *Let ξ be g-Killing. Then $\tilde{\xi}$ is $\tilde{g}$-Killing.*

Now we write the curvature tensors of the deformed metric $\tilde{g}$ for an almost paracontact metric structure with Killing characteristic vector field. If $\{e_1, \ldots, e_n, \varphi e_1, \ldots, \varphi e_n, \xi\}$ is a g-orthonormal frame, then $\{f_1, \ldots, f_{2n+1}\} = \{\frac{1}{\sqrt{t}}\varphi e_1, \ldots, \frac{1}{\sqrt{t}}\varphi e_n, \frac{1}{\sqrt{t}}e_1, \ldots, \frac{1}{\sqrt{t}}e_n, \frac{1}{t}\xi\}$ is $\tilde{g}$-orthonormal [10] and $\tilde{g}^{ij} = g^{ij}$. We use this basis in calculations.

If ξ is Killing, the Riemannian, the Ricci and the scalar curvatures of the deformed metric $\tilde{g}$ are evaluated by direct calculation.

$$\begin{aligned}
\tilde{R}(x,y)z &= R(x,y)z - (t+1)\eta(z)R(x,y)\xi \\
&-(t+1)\eta(x)\nabla_{\nabla_y z}\xi + (t+1)\eta(y)\nabla_{\nabla_x z}\xi \\
&+(t+1)^2\eta(x)\eta(z)\nabla_{\nabla_y \xi}\xi - (t+1)^2\eta(y)\eta(z)\nabla_{\nabla_x \xi}\xi \\
&+(t+1)g(\nabla_y\xi, z)\nabla_x\xi - (t+1)g(\nabla_x\xi, z)\nabla_y\xi \\
&-2(t+1)g(\nabla_x\xi, y)\nabla_z\xi - (t+1)\eta(y)\nabla_x\nabla_z\xi \\
&+(t+1)\eta(x)\nabla_y\nabla_z\xi,
\end{aligned} \tag{20}$$

$$\begin{aligned}
\tilde{Ric}(x,y) &= Ric(x,y) - (t+1)\eta(y)Ric(x,\xi) \\
&+(t+1)\eta(x)\sum_{i=1}^{n}\{g(\nabla_{\nabla_{e_i} y}\xi, e_i) - g(\nabla_{\nabla_{\varphi e_i} y}\xi, \varphi e_i)\} \\
&+(t+1)^2\eta(x)\eta(y)\sum_{i=1}^{n}\{-g(\nabla_{\nabla_{e_i}\xi}\xi, e_i) + g(\nabla_{\nabla_{\varphi e_i}\xi}\xi, \varphi e_i)\} \\
&-(t+1)\eta(x)div(\nabla_y\xi) + 2(t+1)g(\nabla_x\xi, \nabla_y\xi)
\end{aligned}$$

and

$$\tilde{s} = \frac{1}{t}\{-s + (t+1)\sum_{i=1}^{n}\{g(\nabla_{\varphi e_i}\xi, \nabla_{\varphi e_i}\xi) - g(\nabla_{e_i}\xi, \nabla_{e_i}\xi)\}\}.$$

Now let ξ be any vector field which is not necessarily Killing. We write the tensor $\tilde{F}$ of the deformed structure in terms of F defined by (1). Since

$$(\tilde{\nabla}_x\tilde{\varphi})(y) = \tilde{\nabla}_x(\varphi y) - \varphi(\tilde{\nabla}_x y) \tag{21}$$

and

$$\begin{aligned}
\tilde{F}(x,y,z) &= \tilde{g}\big((\tilde{\nabla}_x\tilde{\varphi})(y), z\big) \\
&= -tg\big((\tilde{\nabla}_x\tilde{\varphi})(y), z\big) \\
&\quad +t(t+1)\eta((\tilde{\nabla}_x\tilde{\varphi})(y))\eta(z),
\end{aligned} \tag{22}$$

replacing (21) in (22) and using (17) and the identity $g(\nabla_x \xi, y) = -F(x, \varphi y, \xi)$ yields

$$\begin{aligned} \tilde{F}(x,y,z) &= -tF(x,y,z) \\ &+\frac{t(t+1)}{2}\{\eta(x)\{-F(\varphi y,\varphi z,\xi)+F(z,y,\xi) \\ &-F(y,z,\xi)+F(\varphi z,\varphi y,\xi)\} \\ &+\eta(z)\{F(x,y,\xi)-F(\varphi y,\varphi x,\xi)\} \\ &+\eta(y)\{-F(x,z,\xi)+F(\varphi z,\varphi x,\xi)\}\}. \end{aligned} \tag{23}$$

Now we study the invariance of classes W_i, $i = 1,\ldots,4$ under a D-homothetic deformation. First note that for any almost paracontact metric structure in a direct sum of $W_1 \oplus W_3 = \mathbb{G}_1 \oplus \mathbb{G}_2 \oplus \mathbb{G}_3 \oplus \mathbb{G}_4 \oplus \mathbb{G}_{11}$, since ξ is parallel [10], the equation (23) implies $\tilde{F} = -tF$ and thus a D-homothetic deformation of any direct sum of $W_1 \oplus W_3$ is also in this class.

If ξ is any vector field, not necessarily parallel, from (4) and (23), we have

$$\tilde{F}^{W_1}(x,y,z) = \tilde{F}(\varphi^2 x,\varphi^2 y,\varphi^2 z) = -tF(\varphi^2 x,\varphi^2 y,\varphi^2 z) = -tF^{W_1}(x,y,z). \tag{24}$$

Thus $\tilde{F}^{W_1}$ is zero if and only if F^{W_1} is zero, that is, a deformed structure contains summands from the class W_1 if and only if the first structure has a summand from W_1.

By (5) and (23), we get

$$\begin{aligned} \tilde{F}^{W_2} &= \frac{t(t-1)}{2}F^{W_2}(x,y,z) \\ &+\frac{t(t+1)}{2}\{\eta(y)F(\varphi z,\varphi x,\xi)-\eta(z)F(\varphi y,\varphi x,\xi)\}. \end{aligned} \tag{25}$$

Define S as

$$S(x,y,z) = \frac{t(t+1)}{2}\{\eta(y)F(\varphi z,\varphi x,\xi)-\eta(z)F(\varphi y,\varphi x,\xi)\}. \tag{26}$$

Then it can be easily seen that $S^{W_2} = S$ and thus $S \in W_2$. In addition, we have $F^{W_2}(\varphi x,\varphi y,z) = \eta(z)F(\varphi x,\varphi y,\xi)$. So $F^{W_2} = 0$ if and only if $S = 0$. Thus a deformed structure has summands from the class W_2 if and only if the first structure has.

Consider the projection $F^{W_3} = F^{11}$. From (14) and (23), we have

$$\begin{aligned} \tilde{F}^{11}(x,y,z) &= -tF^{11}(x,y,z)+\frac{t(t+1)}{2}\eta(x)\{-F(\varphi y,\varphi z,\xi)+F(\varphi z,\varphi y,\xi) \\ &+F(\varphi^2 z,\varphi^2 y,\xi)-F(\varphi^2 y,\varphi^2 z,\xi)\}. \end{aligned} \tag{27}$$

Define

$$T(x,y,z) = \frac{t(t+1)}{2}\eta(x)\{-F(\varphi y,\varphi z,\xi)+F(\varphi z,\varphi y,\xi) + F(\varphi^2 z,\varphi^2 y,\xi)-F(\varphi^2 y,\varphi^2 z,\xi)\}. \quad (28)$$

It can be checked that T satisfies the defining relation (10) of $\mathbb{G}_{11}$, that is, $T^{11}=T$. Thus if $F^{11}=0$, or equivalently, if the first almost paracontact structure does not contain a summand from $\mathbb{G}_{11}$, and if $T\neq 0$, then the deformed structure contains a summand from $\mathbb{G}_{11}$ since $T\in\mathbb{G}_{11}$.

For the projection $F^{W_4}=F^{12}$, by using (23) and (15), we get

$$\tilde{F}^{12}(x,y,z) = t^2F^{12}(x,y,z). \quad (29)$$

Thus the deformed structure belongs to a direct sum containing $\mathbb{G}_{12}$ if and only if the first almost paracontact structure has summands from this class.

It is known that almost paracontact metric structures which belong to $\mathbb{G}_1$, $\mathbb{G}_2$, $\mathbb{G}_3$, $\mathbb{G}_4$, $\mathbb{G}_{11}$ or one of their direct sums are invariant under D-homothetic deformations. These are structures with parallel characteristic vector fields [10]. We investigate the invariance of remaining basic classes $\mathbb{G}_5$, $\mathbb{G}_6$, $\mathbb{G}_7$, $\mathbb{G}_8$, $\mathbb{G}_9$, $\mathbb{G}_{10}$, $\mathbb{G}_{12}$.

Theorem 3.2 *The classes $\mathbb{G}_i$, where $i=5,6,7,8,10,12$ are invariant under a D-homothetic deformation, $\mathbb{G}_9$ is not invariant.*

Proof Assume that $\{e_1,\dots,e_n,\varphi e_1,\dots,\varphi e_n,\xi\}$ is a g-orthonormal frame. Then

$\{f_1,\dots,f_{2n+1}\}=\{\frac{1}{\sqrt{t}}\varphi e_1,\dots,\frac{1}{\sqrt{t}}\varphi e_n,\frac{1}{\sqrt{t}}e_1,\dots,\frac{1}{\sqrt{t}}e_n,\frac{1}{t}\xi\}$

is $\tilde{g}$-orthonormal and $\tilde{g}^{ij}=g^{ij}$.

Let $(\varphi,\xi,\eta,g)\in\mathbb{G}_5$. By (23), for $i=1,\dots,n$,

$$\begin{aligned}\tilde{F}(f_i,f_i,\tilde{\xi}) &= \frac{1}{t^2}\tilde{F}(\varphi e_i,\varphi e_i,\xi)\\ &= \frac{t-1}{2t}F(\varphi e_i,\varphi e_i,\xi)-\frac{t+1}{2}F(e_i,e_i,\xi)\end{aligned}$$

and for $i=n+1,\dots,2n$,

$$\begin{aligned}\tilde{F}(f_i,f_i,\tilde{\xi}) &= \frac{1}{t^2}\tilde{F}(e_i,e_i,\xi)\\ &= \frac{t-1}{2t}F(e_i,e_i,\xi)-\frac{t+1}{2}F(\varphi e_i,\varphi e_i,\xi).\end{aligned}$$

Thus

$$
\begin{aligned}
\tilde{\theta}_{\tilde{F}}(\tilde{\xi}) &= \tilde{g}^{ij}F(f_i,f_i,\tilde{\xi}) \\
&= \sum_{i=1}^{n}\tilde{F}(\frac{1}{\sqrt{t}}\varphi e_i,\frac{1}{\sqrt{t}}\varphi e_i,\tilde{\xi}) - \sum_{i=1}^{n}\tilde{F}(\frac{1}{\sqrt{t}}e_i,\frac{1}{\sqrt{t}}e_i,\tilde{\xi}) \\
&= -\theta_F(\xi).
\end{aligned}
$$

From (6) and (23), we get that $\tilde{F}$ satisfies the defining relation (6).

Similarly, the class $\mathbb{G}_6$ is invariant.

Let $(\varphi,\xi,\eta,g)\in\mathbb{G}_8$. Then the defining conditions (7) hold. First we evaluate $\tilde{\theta}_{\tilde{F}}(\tilde{\xi})$. If $\{e_1,\dots,e_n,\varphi e_1,\dots,\varphi e_n,\xi\}$ is a g-orthonormal frame, then

$\{f_1,\dots,f_{2n+1}\}=\{\frac{1}{\sqrt{t}}\varphi e_1,\dots,\frac{1}{\sqrt{t}}\varphi e_n,\frac{1}{\sqrt{t}}e_1,\dots,\frac{1}{\sqrt{t}}e_n,\frac{1}{t}\xi\}$ is $\tilde{g}$-orthonormal and $\tilde{g}^{ij}=g^{ij}$. From (7) and (23), we have

$$
\begin{aligned}
\tilde{F}(\varphi e_i,\varphi e_i,\xi) &= -tF(\varphi e_i,\varphi e_i,\xi) \\
&\quad +\frac{t(t+1)}{2}\{F(\varphi e_i,\varphi e_i,\xi)-F(\varphi^2 e_i,\varphi^2 e_i,\xi)\} \\
&= -tF(\varphi e_i,\varphi e_i,\xi)+t(t+1)F(\varphi e_i,\varphi e_i,\xi) \\
&= t^2F(\varphi e_i,\varphi e_i,\xi)
\end{aligned}
$$

and

$$
\tilde{F}(e_i,e_i,\xi) = t^2F(e_i,e_i,\xi),
$$

thus

$$
\begin{aligned}
\tilde{\theta}_{\tilde{F}}(\tilde{\xi}) &= \tilde{g}^{ij}F(f_i,f_i,\tilde{\xi}) \\
&= \sum_{i=1}^{n}\tilde{F}(\frac{1}{\sqrt{t}}\varphi e_i,\frac{1}{\sqrt{t}}\varphi e_i,\tilde{\xi}) - \sum_{i=1}^{n}\tilde{F}(\frac{1}{\sqrt{t}}e_i,\frac{1}{\sqrt{t}}e_i,\tilde{\xi}) \\
&= \frac{1}{t^2}\left\{\sum_{i=1}^{n}t^2F(\varphi e_i,\varphi e_i,\xi)-\sum_{i=1}^{n}t^2F(e_i,e_i,\xi)\right\} \\
&= -\theta_F(\xi) \\
&= 0.
\end{aligned}
$$

In addition, from (7) and (23)

$$\begin{aligned}\tilde{F}(x,y,z) &= -tF(x,y,z) \\ &\quad + \frac{t(t+1)}{2}\{2F(x,y,\xi)\eta(z) - 2F(x,z,\xi)\eta(y)\} \\ &= -tF(x,y,z) + t(t+1)F(x,y,z) \\ &= t^2F(x,y,z)\end{aligned}$$

and

$$\begin{aligned}& -\tilde{\eta}(y)\tilde{F}(x,z,\tilde{\xi}) + \tilde{\eta}(z)\tilde{F}(x,y,\tilde{\xi}) \\ &= t^2F(x,y,z) \\ &= \tilde{F}(x,y,z).\end{aligned}$$

Also,

$$\tilde{F}(x,y,\tilde{\xi}) = t^2F(x,y,\tilde{\xi}) = t^2F(y,x,\tilde{\xi}) = \tilde{F}(y,x,\tilde{\xi}),$$

$$\tilde{F}(x,y,\tilde{\xi}) = t^2F(x,y,\tilde{\xi}) = -t^2F(\varphi y,\varphi x,\tilde{\xi}) = -\tilde{F}(\tilde{\varphi}y,\tilde{\varphi}x,\tilde{\xi}).$$

Thus the new structure satisfies (7).

A similar proof can be done for the class $\mathbb{G}_7$. In this case, $\tilde{\theta}^*_{\tilde{F}}(\tilde{\xi}) = \frac{1}{t}\theta^*_F(\xi)$.

Let $(\varphi,\xi,\eta,g) \in \mathbb{G}_{10}$. Then the defining relations (9) hold. From (23), $\tilde{F} = -tF$ and (13) implies $\tilde{F}^{10} = -tF = -tF^{10} = \tilde{F}$.

Let $(\varphi,\xi,\eta,g) \in \mathbb{G}_{12}$. By using the defining relation (11) and (23), $\tilde{F} = t^2F$ and from (15), $\tilde{F}^{12} = t^2F^{12} = t^2F = \tilde{F}$. Since $\tilde{F} = \tilde{F}^{12}$, the deformed structure is in $\mathbb{G}_{12}$.

Now we show that the class $\mathbb{G}_9$ is not invariant.

For an arbitrary structure, using (23), we have

$$\tilde{F}(\varphi x,\varphi z,\xi) = \frac{t(t-1)}{2}\{F(\varphi x,\varphi z,\xi)\} + \frac{t(t+1)}{2}\left\{F(\varphi^2 z,\varphi^2 x,\xi)\right\} \tag{30}$$

and

$$\tilde{F}(\varphi^2 x,\varphi^2 z,\xi) = \frac{t(t-1)}{2}\left\{F(\varphi^2 x,\varphi^2 z,\xi)\right\} - \frac{t(t+1)}{2}\{F(\varphi z,\varphi x,\xi)\}. \tag{31}$$

By using equations (12), (30) and (31), we get $\tilde{F}^9 = t^2F^9$.

Let $(\varphi,\xi,\eta,g) \in \mathbb{G}_9$. From (8), $\tilde{F}^9 = t^2F^9 = t^2F$ and also from (8) and (23),

$$\tilde{F}(x,y,z) = t^2F(x,y,z) - 2t(t+1)\eta(x)F(y,z,\xi).$$

The structure is invariant if and only if $\tilde{F} = \tilde{F}^9$, that is

$$t^2 F(x,y,z) = t^2 F(x,y,z) - 2t(t+1)\eta(x)F(y,z,\xi)$$

holds. This implies $F(y,z,\xi) = 0$. Then the defining relation (8) of $\mathbb{G}_9$ implies $F = 0$. Thus a nontrivial structure in $\mathbb{G}_9$ is not in the same class after deformation. □

In addition, we determine the class of the deformed structure if the first structure is in $\mathbb{G}_9$.

Proposition 3.3 *Assume that the first almost paracontact metric structure belongs to the class $\mathbb{G}_9$. Then the deformed structure is in $\mathbb{G}_9 \oplus \mathbb{G}_{11}$.*

Proof Since $M \in \mathbb{G}_9$, we have $F^{W_1} = F^{W_3} = F^{11} = F^{W_4} = F^{12} = 0$ and $F^{W_2} = F^9$. From (24) and (29), we get $\tilde{F}^{W_1} = \tilde{F}^{W_4} = \tilde{F}^{12} = 0$. By using the defining relation (8), it can be seen that the tensor S defined in (26) also satisfies the defining relation of $\mathbb{G}_9$. Thus the equation (25) implies that $\tilde{F}^{W_2} = \frac{t(t-1)}{2}F^9 + S^9$, that is, the deformed structure contains a summand from $\mathbb{G}_9$ and no other summand from W_2. In addition, by using (8), the tensor T given in (28) is

$$T(x,y,z) = 2t(t+1)\eta(x)\{-F(\varphi y, \varphi z, \xi)\},$$

which is nonzero for a nontrivial structure in $\mathbb{G}_9$, otherwise (8) implies $F = 0$. From (27), $\tilde{F}^{11} = T \neq 0$.

To sum up, the deformed structure is in $\mathbb{G}_9 \oplus \mathbb{G}_{11}$. □

Proposition 3.4 *Normal almost paracontact manifolds are invariant under D-homothetic deformations.*

Proof Let the first almost paracontact metric structure be normal. Then

$$F(x,y,\varphi z) + F(\varphi x,y,z) + \eta(z)F(x,\varphi y,z) = 0. \tag{32}$$

(32) implies

$$F(x,\varphi y,\xi) = -F(\varphi x,y,\xi), \tag{33}$$

see [13]. Then by (23), (32) and (33), we get

$$\tilde{F}(x,y,\tilde{\varphi}z) + \tilde{F}(\tilde{\varphi}x,y,z) + \tilde{\eta}(z)\tilde{F}(x,\tilde{\varphi}y,z) = 0.$$

As a result, the deformed structure is also normal. □

Example 3.5 *Let L be Lie algebra having basis $\{e_1,e_2,e_3\}$ whose only nonzero bracket is*

$$[e_1,e_2] = \alpha e_3,$$

together with the semi-Riemannian metric satisfying $g(e_1,e_1) = -g(e_2,e_2) = g(e_3,e_3) = 1$ *and* $g(e_i,e_j) = 0$ *for* $i \neq j$. *Let* $\varphi(e_1) = e_2$, $\varphi(e_2) = e_1$, $\varphi(e_3) = 0$, $e_3 = \xi$ *and* $\eta = e^3$, *where* e^3 *is the metric dual of* e_3. *It is known that* (L,φ,ξ,η,g) *is an almost paraconact metric manifold of class* $\mathbb{G}_5$. *The nonzero covariant derivatives are*

$$\nabla_{e_1}e_2 = -\nabla_{e_2}e_1 = \frac{\alpha}{2}e_3, \quad \nabla_{e_1}e_3 = \nabla_{e_3}e_1 = \frac{\alpha}{2}e_2, \quad \nabla_{e_2}e_3 = \nabla_{e_3}e_2 = \frac{\alpha}{2}e_1.$$

The Ricci tensor is

$$Ric(x,y) = sg(x,y) - 2s\eta(x)\eta(y),$$

where s *is the scalar curvature given by* $s = \alpha^2/2$, *that is,* L *is an* η*-Einstein manifold, see [13]. Then from (20),*

$$\begin{aligned} \tilde{Ric}(x,y) &= Ric(x,y) - (t+1)\eta(y)Ric(x,e_3) \\ &\quad -2(t+1)\frac{\alpha^2}{4}\{x_1y_1 - x_2y_2 - t\eta(x)\eta(y)\}, \end{aligned}$$

where $x = x_1e_1 + x_2e_2 + x_3e_3$ *and* $y = y_1e_1 + y_2e_2 + y_3e_3$. *It can be checked that*

$$\tilde{Ric}(x,y) = \frac{\alpha^2}{2}\tilde{g}(x,y) - \alpha^2\tilde{\eta}(x)\tilde{\eta}(y),$$

that is the deformed manifold is also η*-Einstein.*

Example 3.6 *Consider the nilpotent Lie algebra* $\mathfrak{g}_1$ *given in [4] with basis* $\{e_1,\ldots,e_5\}$, *whose nonzero brackets are*

$$[e_1,e_2] = e_5, [e_3,e_4] = e_5.$$

Assume that g *is the metric such that* $\{e_1,\ldots,e_5\}$ *is orthonormal and* $\epsilon_i = g(e_i,e_i) = \pm 1$. *The nonzero covariant derivatives are evaluated in [8] by Kozsul's formula:*

$$\nabla_{e_1}e_2 = \frac{1}{2}e_5, \quad \nabla_{e_1}e_5 = -\frac{1}{2}\epsilon_2\epsilon_5e_2,$$

$$\nabla_{e_2}e_1 = -\frac{1}{2}e_5, \quad \nabla_{e_2}e_5 = \frac{1}{2}\epsilon_1\epsilon_5e_1,$$

$$\nabla_{e_3}e_4 = \frac{1}{2}e_5, \quad \nabla_{e_3}e_5 = -\frac{1}{2}\epsilon_4\epsilon_5e_4,$$

$$\nabla_{e_4}e_3 = -\frac{1}{2}e_5, \quad \nabla_{e_4}e_5 = \frac{1}{2}\epsilon_3\epsilon_5e_3,$$

$$\nabla_{e_5}e_1 = -\frac{1}{2}\epsilon_2\epsilon_5e_2, \quad \nabla_{e_5}e_2 = \frac{1}{2}\epsilon_1\epsilon_5e_1, \quad \nabla_{e_5}e_3 = -\frac{1}{2}\epsilon_4\epsilon_5e_4, \quad \nabla_{e_5}e_4 = \frac{1}{2}\epsilon_3\epsilon_5e_3.$$

Consider now the structure (φ,ξ,η,g) *defined by* $g(e_1,e_1)=g(e_2,e_2)=-g(e_3,e_3)=-g(e_4,e_4)=g(e_5,e_5)=1$, $\xi=e_5$, $\eta=e^5$, *whose endomorphism is given via basis elements as follows.*

$\varphi(e_1)=e_3$, $\varphi(e_2)=e_4$, $\varphi(e_3)=e_1$, $\varphi(e_4)=e_2$, $\varphi(e_5)=0$. *Nonzero structure constants of* F *are*

$$F(e_1,e_4,e_5)=-F(e_1,e_5,e_4)=-F(e_2,e_3,e_5)=F(e_2,e_5,e_3)=1/2,$$

$$-F(e_3,e_5,e_2)=F(e_3,e_2,e_5)=-F(e_4,e_1,e_5)=F(e_4,e_5,e_1)=1/2,$$

$$-F(e_5,e_1,e_4)=F(e_5,e_4,e_1)=F(e_5,e_2,e_3)=-F(e_5,e_3,e_2)=1.$$

Note that $\xi=e_5$ *is Killing [8] and this structure is in the class* $\mathbb{G}_9\oplus\mathbb{G}_{11}$ *[6]. We determine the class of the deformed structure after a D-homothetic deformation. Proposition 3.1 implies that* $\tilde{\xi}$ *is Killing, so* $\tilde{F}^6=\tilde{F}^7=\tilde{F}^{10}=\tilde{F}^{12}=0$. *Also since* $\tilde{F}^{W_1}=-tF^{W_1}$ *and* F^{W_1} *vanishes,* $\tilde{F}^{W_1}$ *also vanishes. It can be checked that this structure satisfies*

$$F(\varphi y,\varphi z,\xi)=-F(\varphi z,\varphi y,\xi)=F(\varphi^2y,\varphi^2z,\xi)$$

and thus

$$\begin{aligned}\tilde{F}^{11}(x,y,z) &= -tF^{11}(x,y,z)+\frac{t(t+1)}{2}\eta(x)\{-F(\varphi y,\varphi z,\xi)+F(\varphi z,\varphi y,\xi)\\ &\quad +F(\varphi^2z,\varphi^2y,\xi)-F(\varphi^2y,\varphi^2z,\xi)\}\\ &= -2t(t+1)\eta(x)F(\varphi y,\varphi z,\xi)\\ &= t(t+1)x_5\{y_2z_3-y_3z_2+y_4z_1-y_1z_4\}\neq 0.\end{aligned}$$

In addition, by direct calculation

$$\begin{aligned}F^9(x,y,z) &= \eta(y)F(\varphi z,\varphi x,\xi)-\eta(z)F(\varphi y,\varphi x,\xi)\\ &= -\frac{1}{2}y_5\{x_1z_4-x_2z_3+x_3z_2-x_4z_1\}\\ &\quad +\frac{1}{2}z_5\{x_1y_4-x_2y_3+x_3y_2-x_4y_1\}\end{aligned}$$

and

$$\begin{aligned}\tilde{F}^{W_2} &= \frac{t(t-1)}{2}F^{W_2}(x,y,z)\\ &+\frac{t(t+1)}{2}\{\eta(y)F(\varphi z,\varphi x,\xi)-\eta(z)F(\varphi y,\varphi x,\xi)\}\\ &= \frac{t(t-1)}{2}F^{9}(x,y,z)\\ &+\frac{t(t+1)}{2}\left\{-\frac{1}{2}y_5\{x_1z_4-x_2z_3+x_3z_2-x_4z_1\}\right.\\ &\left.+\frac{1}{2}z_5\{x_1y_4-x_2y_3+x_3y_2-x_4y_1\}\right\}\\ &= t^2F^9(x,y,z)\neq 0\end{aligned}$$

As a result the deformed structure is also in $\mathbb{G}_9\oplus\mathbb{G}_{11}$*. So we obtain infinitely many examples of structures of type* $\mathbb{G}_9\oplus\mathbb{G}_{11}$ *by D-homothetic deformation. Note that although an almost paracontact structure of class* $\mathbb{G}_9$ *is not invariant, a direct sum containing the class* $\mathbb{G}_9$ *may be invariant.*

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Declaration of Ethical Standards

The author declares that the materials and methods used in her study do not require ethical committee and/or legal special permission.

Conflicts of Interest

The author declares no conflict of interest.

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