Communications in Advanced Mathematical Sciences
Vol. 6, No. 2, 104-114, 2023
Research Article
e-ISSN: 2651-4001
DOI:10.33434/cams.1242905

Nonlinear Approximation by q-Favard-Szász-Mirakjan Operators of Max-Product Kind

Döne Karahan[1]*, Ecem Acar[2]

Abstract
In this study, nonlinear q-Favard-Szász-Mirakjan operators of max-product kind are defined and approximation properties of these operators are investigated. Classical approximation and A-statistical approximation theorems are given.

Keywords: Favard-Szász-Mirakjan operators, Modulus of continuity, Nonlinear max-product operators, q-integers
2010 AMS: 41A30, 41A46, 41A25

[1] *Harran University, Faculty of Science and Letter, Department of Mathematics, Şanlıurfa, Turkey, ORCID: 0000-0001-6644-5596*
[2] *Harran University, Faculty of Science and Letter, Department of Mathematics, Şanlıurfa, Turkey, ORCID: 0000-0002-2517-5849*
***Corresponding author**: dkarahan@harran.edu.tr
Received: 26 January 2023, **Accepted:** 28 June 2023, **Available online:** 30 June 2023
How to cite this article: D.Karahan, E. Acar, *Nonlinear Approximation by q-Favard-Szász-Mirakjan Operators of Max-Product Kind*, Commun. Adv. Math. Sci., (6)2 (2023) 104-114.

1. Introduction

The approximation of functions by using linear positive operators introduced via q-Calculus and (p,q)-Calculus is currently under intensive research. Firstly, generalizations of Bernstein polynomials based on the q-integers has been investigated by Lupas [1] and Phillips [2]. Later, generalized q-Bernstein operators and the q-generalization of other operators were studied in [3]-[8]. Also, in recent years, a nonlinear modification of the classical Bernstein polynomial has been introduced by Bede and Gal [9]. All the max-product operators are nonlinear and piecewise rational, and they present, for many subclasses of functions, essentially better approximation properties than the classical linear operators. In [10]-[13], Favard-Szász-Mirakjan operator of max-product kind and Bernstein operator of max-product kind were studied. Duman constructed a nonlinear approximation operator by modifying the q-Bernstein polynomial in [14].

In this study, we define nonlinear q-Favard-Szász-Mirakjan operators of max-product kind. But, before that the classical Favard-Szász-Mirakjan operators (see [15]) and its q-generalization (see [16]) are given respectively by

$$S_n(f,x) = e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right) \tag{1.1}$$

and

$$S_{n,q}(f,x) = E_q\left(-[n]_q x\right)\sum_{k=0}^{\infty}\frac{([n]_q x)^k}{[k]_q!}f\left(\frac{[k]_q}{[n]_q}\right), \tag{1.2}$$

where $n \in \mathbb{N}$, f is bounded, $f \in C[0,+\infty)$, $x \in [0,+\infty)$, $q \in (0,1)$ and $E_q(x) = \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} \frac{x^n}{[n]_q!}$.

The aim of this paper is to study the nonlinear approximation properties of q-Favard-Szász-Mirakjan operators of max-product kind.

We first recall some basic definitions in q-calculus. Let parameter q be a positive real number and n a non-negative integer. $[n]_q$ denotes a q integer, defined by

$$[n]_q = \begin{cases} \frac{1-q^n}{1-q}, & q \neq 1 \\ n, & q = 1. \end{cases}$$

Let $q > 0$ be given. We define a q-factorial, $[n]_q!$ of $k \in \mathbb{N}$, as

$$[n]_q! = \begin{cases} [1]_q[2]_q...[n]_q, & n = 1,2,... \\ 1, & n = 0. \end{cases}$$

The q-binomial coefficient $\begin{bmatrix} n \\ r \end{bmatrix}_q$ by

$$\begin{bmatrix} n \\ r \end{bmatrix}_q = \frac{[n]_q!}{[n-r]_q![r]_q!}.$$

2. Construction of the Operators

The approximation properties of the classical Favard-Szasz-Mirakjan operators of max-product kind were investigated in [9]. In this section, we construct nonlinear q-Favard-Szász-Mirakjan operators of max-product kind. We consider the operations "$\vee$" (maximum) and "." (product) over the interval $[0,+\infty)$. Then $([0,+\infty), \vee, .)$ has a semiring structure and is called "max-product algebra" (see, for instance [13]).

Let $C_+[0,+\infty) := \{f : [0,+\infty) \to [0,+\infty) : f \ \ \textit{is continuous on} \ \ [0,+\infty)\}$. We define nonlinear q-Favard-Szász-Mirakjan operators of max-product kind as follows:

$$F_{n,q}^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{\infty} s_{n,k}(x,q) f\left(\frac{[k]_q}{[n]_q}\right)}{\bigvee_{k=0}^{\infty} s_{n,k}(x,q)}, \tag{2.1}$$

where $n \in \mathbb{N}$, $f \in C_+[0,+\infty)$, $x \in [0,+\infty)$, $q \in (0,1)$ and $s_{n,k}(x,q)$ is given by

$$s_{n,k}(x,q) = \frac{([n]_q x)^k}{[k]_q!}. \tag{2.2}$$

Since it easy to check that $F_{n,q}^{(M)}(f)(0) - f(0) = 0$ for all n, notice that in the notations, proofs and statements of all approximation results in fact we always may suppose that $x > 0$.

Since $f \in C_+[0,+\infty)$ and $s_{n,k}(x,q)$ is positive for all $x \in [0,+\infty)$, $F_{n,q}^{(M)}(f)(x)$ is a positive operator. Now, we show that $F_{n,q}^{(M)}(f)(x)$ is not linear operator on $C_+[0,+\infty)$.

Let $f, g \in C_+[0,+\infty)$. Then, by definition we see that

$$f \leq g \implies F_{n,q}^{(M)}(f)(x) \leq F_{n,q}^{(M)}(g)(x). \tag{2.3}$$

Thus, $F_{n,q}^{(M)}(f)(x)$ is increasing with respect to $f \in C_+[0,+\infty)$. Besides, for any $f, g \in C_+[0,+\infty)$ we have

$$F_{n,q}^{(M)}(f+g)(x) \leq F_{n,q}^{(M)}(f)(x) + F_{n,q}^{(M)}(g)(x). \tag{2.4}$$

In general, $\omega_1(f,\delta)$, $\delta > 0$ denote the modulus of continuity of $f \in C_+[0,+\infty)$ defined by

$$\omega_1(f,\delta) = \sup\{|f(x) - f(y)| : x,y \in [0,+\infty), |x-y| \leq \delta\}.$$

Now, using (2.3), (2.4) and also applying Corollary 2.3 in [11] or Corollary 3 in [13], we have the following inequality:

$$|F_{n,q}^{(M)}(f)(x) - f(x)| \leq \left(1 + \frac{1}{\delta_n} F_{n,q}^{(M)}(\varphi_x)(x)\right) \omega_1(f,\delta_n), \tag{2.5}$$

where $n \in \mathbb{N}$, $f \in C_+[0,+\infty)$, $x \in [0,+\infty)$, $q \in (0,1)$ and $\varphi_x(t) = |x-t|$.

3. Auxiliary Results

For each $k, j \in \{0,1,2,...\}$ and $x \in \left[\frac{[j]_q}{[n]_q}, \frac{[j+1]_q}{[n]_q}\right]$, let us denote

$$M_{k,n,j}(x,q) = \frac{s_{n,k}(x,q)\left|\frac{[k]_q}{[n]_q} - x\right|}{s_{n,j}(x,q)}, \tag{3.1}$$

$$m_{k,n,j}(x,q) = \frac{s_{n,k}(x,q)}{s_{n,j}(x,q)}. \tag{3.2}$$

It can easily see that if $k \geq j+1$ then

$$M_{k,n,j}(x,q) = \frac{s_{n,k}(x,q)\left(\frac{[k]_q}{[n]_q} - x\right)}{s_{n,j}(x,q)},$$

and if $k \leq j-1$ then

$$M_{k,n,j}(x,q) = \frac{s_{n,k}(x,q)\left(x - \frac{[k]_q}{[n]_q}\right)}{s_{n,j}(x,q)}.$$

Lemma 3.1. *Let $q \in (0,1)$. For all $k, j \in \{0,1,2,...\}$ and $x \in \left[\frac{[j]_q}{[n]_q}, \frac{[j+1]_q}{[n]_q}\right]$, we get*

$$m_{k,n,j}(x,) \leq 1. \tag{3.3}$$

Proof. We consider two cases: (i) $k \geq j$ and (ii) $k < j$.

Case (i). From (3.2), we have

$$\frac{m_{k,n,j}(x,q)}{m_{k+1,n,j}(x,q)} = \frac{[k+1]_q}{[n]_q}\frac{1}{x}.$$

Since the function $h(x) = \frac{1}{x}$ is non-increasing on $\left[\frac{[j]_q}{[n]_q}, \frac{[j+1]_q}{[n]_q}\right]$, from here we get

$$\begin{aligned}\frac{m_{k,n,j}(x,q)}{m_{k+1,n,j}(x,q)} &= \frac{[k+1]_q}{[n]_q}\frac{[n]_q}{[j+1]_q} \\ &= \frac{[k+1]_q}{[j+1]_q} \geq 1\end{aligned}$$

which immediately implies

$$m_{j,n,j}(x,q) \geq m_{j+1,n,j}(x,q) \geq m_{j+2,n,j}(x,q) \geq ... \quad .$$

Case (ii) We get

$$\frac{m_{k,n,j}(x,q)}{m_{k-1,n,j}(x,q)} = \frac{[n]_q}{[k]_q}x \geq \frac{[n]_q}{[k]_q}\frac{[j]_q}{[n]_q} = \frac{[j]_q}{[k]_q} \geq 1,$$

which immediately implies

$$m_{j,n,j}(x,q) \geq m_{j-1,n,j}(x,q) \geq m_{j-2,n,j}(x,q) \geq ... \geq m_{0,n,j}(x,q).$$

Since $m_{j,n,j}(x,q) = 1$ the proof of the lemma is finished. □

Lemma 3.2. *Let $q \in (0,1)$, $j \in \{1,2,...\}$ and $x \in \left[\frac{[j]_q}{[n]_q}, \frac{[j+1]_q}{[n]_q}\right]$.*

(i) If $k \in \{j+1, j+2, ...\}$ is such that $[k+1]_q - \sqrt{q^k[k+1]_q} \geq [j+1]_q$, then $M_{k,n,j}(x,q) \geq M_{k+1,n,j}(x,q)$.

(ii) If $k \in \{1,2,...,j-1\}$ is such that $[k]_q - \sqrt{q^{k-1}[k]_q} \leq [j]_q$, then $M_{k-1,n,j}(x,q) \leq M_{k,n,j}(x,q)$.

Proof. *(i)* Let $k \in \{j+1, j+2, ...\}$ and $[k+1]_q - \sqrt{q^k [k+1]_q} \geq [j+1]_q$. Then, we can write that

$$\frac{M_{k,n,j}(x,q)}{M_{k+1,n,j}(x,q)} = \frac{[k+1]_q}{[n]_q} \frac{1}{x} \frac{\frac{[k]_q}{[n]_q} - x}{\frac{[k+1]_q}{[n]_q} - x}.$$

Since the $g(x) = \frac{1}{x} \frac{\frac{[k]_q}{[n]_q} - x}{\frac{[k+1]_q}{[n]_q} - x}$ clearly is decreasing on the interval $\left[\frac{[j]_q}{[n]_q}, \frac{[j+1]_q}{[n]_q}\right]$, we have

$$g(x) \geq g\left(\frac{[j+1]_q}{[n]_q}\right) = \frac{[n]_q}{[j+1]_q} \frac{\frac{[k]_q}{[n]_q} - \frac{[j+1]_q}{[n]_q}}{\frac{[k+1]_q}{[n]_q} - \frac{[j+1]_q}{[n]_q}}$$
$$= \frac{[n]_q}{[j+1]_q} \frac{[k]_q - [j+1]_q}{[k+1]_q - [j+1]_q}.$$

Since the condition $[k+1]_q - \sqrt{q^k [k+1]_q} \geq [j+1]_q$ is equivalent to $[k+1]_q - \sqrt{[k+1]_q^2 - [k]_q [k+1]_q} \geq [j+1]_q$ which implies that $[k+1]_q \left([k]_q - [j+1]_q\right) \geq [j+1]_q \left([k+1]_q - [j+1]_q\right)$.

So, we achieve that

$$\frac{M_{k,n,j}(x,q)}{M_{k+1,n,j}(x,q)} \geq 1,$$

which proves Lemma 3.2 *(i)*.

(ii) Let $k \in \{1, 2, ..., j-1\}$ and $[k]_q - \sqrt{q^{k-1} [k]_q} \leq [j]_q$. Then, we can write that

$$\frac{M_{k,n,j}(x,q)}{M_{k-1,n,j}(x,q)} = \frac{[n]_q}{[k]_q} x \frac{x - \frac{[k]_q}{[n]_q}}{x - \frac{[k+1]_q}{[n]_q}}.$$

Since the $h(x) = x \frac{x - \frac{[k]_q}{[n]_q}}{x - \frac{[k+1]_q}{[n]_q}}$ clearly is increasing on the interval $\left[\frac{[j]_q}{[n]_q}, \frac{[j+1]_q}{[n]_q}\right]$, we have

$$h(x) \geq h\left(\frac{[j]_q}{[n]_q}\right) = \frac{[j]_q}{[n]_q} \frac{\frac{[j]_q}{[n]_q} - \frac{[k]_q}{[n]_q}}{\frac{[j]_q}{[n]_q} - \frac{[k-1]_q}{[n]_q}}$$
$$= \frac{[j]_q}{[n]_q} \frac{[j]_q - [k]_q}{[j]_q - [k-1]_q}.$$

Since the condition $[k]_q + \sqrt{q^{k-1} [k+1]_q} \leq [j]_q$ is equivalent to $[k]_q - \sqrt{[k]_q^2 - [k]_q [k1]_q} \leq [j]_q$ which implies that $[j]_q \left([j]_q - [k]_q\right) \geq [k]_q \left([j]_q - [k-1]_q\right)$.

So, we achieve that

$$\frac{M_{k,n,j}(x,q)}{M_{k-1,n,j}(x,q)} \geq 1$$

which proves Lemma 3.2 *(ii)*. □

Lemma 3.3. *Let $q \in (0,1)$, $j \in \{0, 1, 2, ...\}$ and $x \in \left[\frac{[j]_q}{[n]_q}, \frac{[j+1]_q}{[n]_q}\right]$. We get*

$$\bigvee_{k=0}^{\infty} s_{n,k}(x,q) = s_{n,j}(x,q).$$

Proof. Firstly, we show that for fixed $n \in \mathbb{N}$ and $0 \leq k$ we get

$$0 \leq s_{n,k+1}(x,q) \leq s_{n,k}(x,q) \iff x \in \left[0, \frac{[k+1]_q}{[n]_q}\right].$$

Indeed, from $s_{n,k}(x,q)=\frac{([n]_q x)^k}{[k]_q!}$ we have

$$0\leq s_{n,k+1}(x,q)\leq s_{n,k}(x,q)$$

$$0\leq \frac{([n]_q x)^{k+1}}{[k+1]_q!}\leq \frac{([n]_q x)^k}{[k]_q!},$$

which after simplifications is obviously equivalent to

$$0\leq x\leq \frac{[k+1]_q}{[n]_q}.$$

So, if we take $k=0,1,2,...$, then we achieve that

$$s_{n,1}(x,q)\leq s_{n,0}(x,q) \iff x\in\left[0,\frac{[1]_q}{[n]_q}\right],$$

$$s_{n,2}(x,q)\leq s_{n,1}(x,q) \iff x\in\left[0,\frac{[2]_q}{[n]_q}\right],$$

$$s_{n,3}(x,q)\leq s_{n,2}(x,q) \iff x\in\left[0,\frac{[3]_q}{[n]_q}\right],$$

so on,

$$s_{n,k+1}(x,q)\leq s_{n,k}(x,q) \iff x\in\left[0,\frac{[k+1]_q}{[n]_q}\right],$$

and so on.

From above inequalities, we can easily write:

$$if \quad x\in\left[0,\frac{[1]_q}{[n]_q}\right] \quad then \quad s_{n,k}(x,q)\leq s_{n,0}(x,q), \quad for \quad all \quad k=0,1,2,...,$$

$$if \quad x\in\left[\frac{[1]_q}{[n]_q},\frac{[2]_q}{[n]_q}\right] \quad then \quad s_{n,k}(x,q)\leq s_{n,1}(x,q), \quad for \quad all \quad k=0,1,2,...,$$

$$if \quad x\in\left[\frac{[2]_q}{[n]_q},\frac{[3]_q}{[n]_q}\right] \quad then \quad s_{n,k}(x,q)\leq s_{n,2}(x,q), \quad for \quad all \quad k=0,1,2,...,$$

and so on, as a result, we obtain

$$if \quad x\in\left[\frac{[j]_q}{[n]_q},\frac{[j+1]_q}{[n]_q}\right] \quad then \quad s_{n,k}(x,q)\leq s_{n,j}(x,q), \quad for \quad all \quad k=0,1,2,...,$$

which completes the proof of Lemma 3.3. □

4. Approximation Results

Theorem 4.1. *Let $f:[0,+\infty)\to[0,+\infty)$ be bounded and continuous on $[0,+\infty)$ and $q\in(0,1)$. Then we get the following estimation*

$$\left|F_{n,q}^{(M)}(f)(x)-f(x)\right|\leq 8\omega_1\left(f;\frac{\sqrt{x}}{\sqrt{[n]_q}}\right), \tag{4.1}$$

where $n\in\mathbb{N}$, $x\in[0,+\infty)$ and

$$\omega_1(f,\delta)=\sup\{|f(x)-f(y)|:x,y\in[0,+\infty),|x-y|\leq\delta\}.$$

Proof. Taking $q = q_n \in (0,1)$ such that $\lim_n q_n = 1$, we deduce $\lim_n [n]_{q_n} = \infty$. From (2.5), we have

$$|F_{n,q}^{(M)}(f)(x) - f(x)| \leq \left(1 + \frac{1}{\delta_n} F_{n,q}^{(M)}(\varphi_x)(x)\right) \omega_1(f, \delta_n), \tag{4.2}$$

where $\varphi_x(t) = |x-t|$. Thus, it is enough to estimate

$$A_{n,q}(x) := F_{n,q}^{(M)}(\varphi_x)(x) = \frac{\bigvee_{k=0}^{\infty} s_{n,k}(x,q) \left| \frac{[k]_q}{[n]_q} - x \right|}{\bigvee_{k=0}^{\infty} s_{n,k}(x,q)},$$

where $x \in [0, +\infty)$. Let $x \in \left[\frac{[j]_q}{[n]_q}, \frac{[j+1]_q}{[n]_q}\right]$, where $j \in \{0,1,2,...\}$ is fixed,
arbitrary. By Lemma 3.3 we can easily achieve

$$A_{n,q}(x) = max\{M_{k,n,j}(x,q) : x \in \left[\frac{[j]_q}{[n]_q}, \frac{[j+1]_q}{[n]_q}\right], k = 0,1,...\}.$$

Firstly, we show that for $j = 0$ and $k = 0,1,2,...$ we obtain $A_{n,q}(x) \leq \frac{\sqrt{x}}{\sqrt{[n]_q}}$ for all $x \in \left[0, \frac{1}{[n]_q}\right]$.

Indeed, for $j = 0$ we get $M_{k,n,0}(x,q) = \frac{([n]_q x)^k}{[k]_q!} \left|\frac{[k]_q}{[n]_q} - x\right|$ which for $k = 0$ gives $M_{k,n,0}(x,q) = x = \sqrt{x}\sqrt{x} \leq \sqrt{x}\frac{1}{\sqrt{[n]_q}}$. Furthermore, for any $k = 1,2,...$ we have $\frac{1}{[n]_q} \leq \frac{[k]_q}{[n]_q}$ and we obtain

$$M_{k,n,0}(x,q) \leq \frac{([n]_q x)^k}{[k]_q!} \frac{[k]_q}{[n]_q} = \sqrt{x}\frac{[n]_q^{k-1} x^{k-\frac{1}{2}}}{[k-1]_q} \leq \sqrt{x}\frac{[n]_q^{k-1}}{[k-1]_q [n]^{k-\frac{1}{2}}{}_q} \leq \frac{\sqrt{x}}{\sqrt{[n]_q}}.$$

Now we claim that for each $M_{k,n,j}(x,q)$ when $j = 1,2,...$ and $k = 0,1,2,...$ the following inequality

$$M_{k,n,j}(x,q) \leq \frac{4\sqrt{x}}{\sqrt{[n]_q}}, \quad \forall x \in \left[\frac{[j]_q}{[n]_q}, \frac{[j+1]_q}{[n]_q}\right], \tag{4.3}$$

which immediately will imply that

$$A_{n,q}(x) \leq \frac{4\sqrt{x}}{\sqrt{[n]_q}}, \quad \forall x \in [0,\infty), n \in \mathbb{N},$$

and taking $\delta_n = \frac{4\sqrt{x}}{\sqrt{[n]_q}}$ in (4.2) we complete the proof of Theorem 4.1.

In order to prove (4.3) we consider the following three cases: 1) $k = j$, 2) $k \geq j+1$, 3) $k \leq j-1$.

Case 1) If $k = j$ then from (3.1) $M_{j,n,j}(x,q) = \left|\frac{[j]_q}{[n]_q} - x\right|$. Since $x \in \left[\frac{[j]_q}{[n]_q}, \frac{[j+1]_q}{[n]_q}\right]$ we can easily see that $M_{j,n,j}(x,q) \leq \frac{1}{[n]_q}$.
Since $j \geq 1$ we have $x \geq \frac{1}{[n]_q}$ which implies

$$M_{j,n,j}(x,q) \leq \frac{1}{[n]_q} = \frac{1}{\sqrt{[n]_q}} \frac{1}{\sqrt{[n]_q}} \leq \sqrt{x}\frac{1}{\sqrt{[n]_q}}.$$

Case 2) Subcase a) We suppose that $k \geq j+1$ and $[k+1]_q - \sqrt{q^k [k+1]_q} < [j+1]_q$. We have from Lemma 3.1 that

$$M_{k,n,j}(x,q) = m_{k,n,j}(x,q)\left(\frac{[k]_q}{[n]_q} - x\right) \leq \frac{[k]_q}{[n]_q} - \frac{[j]_q}{[n]_q}.$$

By hypothesis, since

$$q[k]_q - \sqrt{q^k [k+1]_q} < q[j]_q,$$

we have

$$M_{k,n,j}(x,q) \leq \frac{[k]_q}{[n]_q} - \frac{[k]_q - \sqrt{q^{k-2}[k+1]_q}}{[n]_q} = \frac{\sqrt{q^{k-2}[k+1]_q}}{[n]_q}.$$

Since $k \geq 2$ and $q \in (0,1)$, we obtain

$$M_{k,n,j}(x,q) \leq \frac{\sqrt{[k+1]_q}}{[n]_q}.$$

But we necessarily have $k \leq 3j$. Indeed, if we suppose that $k > 3j$, then because $g(k) = [k+1]_q - \sqrt{q^k[k+1]_q}$ is increasing with respect to k. Indeed, we can write that

$$\begin{aligned}
g(k+1)-g(k) &= [k+2]_q - [k+1]_q + \sqrt{q^k[k+1]_q} - \sqrt{q^{k+1}[k+2]_q} \\
&\geq [k+2]_q - [k+1]_q + \sqrt{q^k[k+1]_q} - \sqrt{q^k[k+2]_q} \\
&= q^{k+1} - q^{\frac{k}{2}}\left(\sqrt{[k+1]_q} - \sqrt{[k+2]_q}\right) \\
&= q^{k+1} - \frac{q^{k+1}q^{\frac{k}{2}}}{\sqrt{[k+1]_q} - \sqrt{[k+2]_q}} \\
&= q^{k+1}\left(1 - \frac{q^{\frac{k}{2}}}{\sqrt{[k+1]_q} - \sqrt{[k+2]_q}}\right) \\
&\geq q^{k+1}\left(1 - \frac{1}{\sqrt{[k+1]_q} - \sqrt{[k+2]_q}}\right) \\
&> 0.
\end{aligned}$$

Hence, we get that $[j+1]_q > [k+1]_q - \sqrt{q^k[k+1]_q} > [3j+1]_q - \sqrt{q^{3j}[3j+1]_q}$ which implies the obvious contradiction $[3j+1]_q - [j+1]_q < \sqrt{q^{3j}[3j+1]_q}$ is to equivalent $q^{j+1}[2j]_q < \sqrt{q^{3j}[3j+1]_q}$.

As a result, we achieve

$$\begin{aligned}
M_{k,n,j}(x,q) &\leq \frac{\sqrt{[k+1]_q}}{[n]_q} \leq \frac{\sqrt{[3j+1]_q}}{[n]_q} \\
&\leq \frac{\sqrt{[4j]_q}}{[n]_q} = \sqrt{(1+q^j)(1+q^{2j})}\frac{\sqrt{[j]_q}}{[n]_q} \\
&\leq \sqrt{(1+q^j)(1+q^{2j})}\frac{\sqrt{x}}{\sqrt{[n]_q}} \leq 2\frac{\sqrt{x}}{\sqrt{[n]_q}},
\end{aligned}$$

taking into account that $\sqrt{x} \geq \frac{\sqrt{[j]_q}}{[n]_q}$.

Subcase b) We suppose that $k \geq j+1$ and $[k+1]_q - \sqrt{q^k[k+1]_q} \geq [j+1]_q$. Since, the function $g(k) = [k+1]_q - \sqrt{q^k[k+1]_q}$ is increasing with respect to k, it follows that there exits $\overline{k} \in \{0,1,2,...\}$, of maximum value, such that

$$[\overline{k}+1]_q - \sqrt{q^{\overline{k}}[\overline{k}+1]_q} < [j+1]_q.$$

Let $\tilde{k} = \overline{k}+1$. Then for all $k \geq \tilde{k}$ we have

$$[k+1]_q - \sqrt{q^k[k+1]_q} \geq [j+1]_q$$

and

$$M_{\tilde{k},n,j}(x,q) = m_{\tilde{k},n,j}(x,q)\left(\frac{[\tilde{k}]_q}{[n]_q} - x\right) \leq \frac{[\overline{k}+1]_q}{[n]_q} - \frac{[j]_q}{[n]_q}.$$

Since

$$[j]_q \geq [\overline{k}+1]_q - q^j - \sqrt{q^{\overline{k}}[\overline{k}+1]_q},$$

we can see that

$$M_{\tilde{k},n,j}(x,q) \leq \frac{[\bar{k}+1]_q}{[n]_q} - \frac{[\bar{k}+1]_q - q^j - \sqrt{q^{\bar{k}}[\bar{k}+1]_q}}{[n]_q}$$
$$= \frac{q^j + \sqrt{q^{\bar{k}}[\bar{k}+1]_q}}{[n]_q}$$
$$\leq \frac{1+\sqrt{[\bar{k}+1]_q}}{[n]_q} \leq \frac{2\sqrt{[\bar{k}+1]_q}}{[n]_q}$$
$$\leq 4\frac{\sqrt{x}}{\sqrt{[n]_q}}.$$

The last above inequality follows from the fact that

$$[\bar{k}+1]_q - \sqrt{q^{\bar{k}}[\bar{k}+1]_q} < [j+1]_q,$$

necessarily implies $\bar{k} \leq 3j$ (see the similar reasoning in the above Subcase a)). Also, we get $\tilde{k} \geq j+1$. Indeed, this is a consequence of the fact that g is increasing function and because it is easy to see that $g(j) \leq [j+1]_q$.

By Lemma 3.2, (i) it follows that

$$M_{\bar{k}+1,n,j}(x,q) \geq M_{\bar{k}+2,n,j}(x,q) \geq ...$$

So, we achieve $M_{k,n,j}(x,q) \leq 4\frac{\sqrt{x}}{\sqrt{[n]_q}}$ for any $k \in \{\bar{k}+1, \bar{k}+2, ...\}$.

Case 3) Subcase a) We suppose that $k \leq j-1$ and $[k]_q + \sqrt{q^{k-1}[k]_q} \geq [j]_q$. We have from Lemma 3.1 that

$$M_{k,n,j}(x,q) = m_{k,n,j}(x,q)\left(x - \frac{[k]_q}{[n]_q}\right) \leq \frac{[j+1]_q}{[n]_q} - \frac{[k]_q}{[n]_q} = \frac{[j]_q + q^j}{[n]_q} - \frac{[k]_q}{[n]_q}$$

By hypothesis, we get

$$M_{k,n,j}(x,q) \leq \frac{[k]_q + \sqrt{q^{k-1}[k]_q} + q^j}{[n]_q} - \frac{[k]_q}{[n]_q}$$
$$= \frac{\sqrt{q^{k-1}[k]_q} + q^j}{[n]_q} \leq \frac{\sqrt{[k]_q}+1}{[n]_q}$$
$$\leq \frac{\sqrt{[j-1]_q}+1}{[n]_q} = \frac{1}{\sqrt{[n]_q}}\frac{\sqrt{[j-1]_q}+1}{\sqrt{[n]_q}}$$
$$\leq \frac{1}{\sqrt{[n]_q}}\frac{2\sqrt{[j]_q}}{\sqrt{[n]_q}} \leq 2\frac{\sqrt{x}}{\sqrt{[n]_q}}.$$

Subcase b) We suppose that $k \leq j-1$ and $[k]_q + \sqrt{q^{k-1}[k]_q} < [j]_q$. Let $\bar{k} \in \{0,1,2,...\}$ be the minimum value such that $[\bar{k}]_q + \sqrt{q^{\bar{k}-1}[\bar{k}]_q} \geq [j]_q$. Then $\tilde{k} = \bar{k}-1$ satisfies $[\bar{k}-1]_q + \sqrt{q^{\bar{k}-2}[\bar{k}-1]_q} < [j]_q$. Also we have

$$M_{\bar{k}-1,n,j}(x,q) = m_{\bar{k}-1,n,j}(x,q)\left(x - \frac{[\bar{k}-1]_q}{[n]_q}\right) \leq \frac{[j+1]_q}{[n]_q} - \frac{[\bar{k}-1]_q}{[n]_q}$$
$$= \frac{[j]_q + q^j}{[n]_q} - \frac{[\bar{k}-1]_q}{[n]_q}.$$

Since $[\bar{k}]_q + \sqrt{q^{\bar{k}-1}[\bar{k}]_q} \geq [j]_q$, we obtain

$$M_{\bar{k}-1,n,j}(x,q) \leq \frac{[\bar{k}]_q + \sqrt{q^{\bar{k}-1}[\bar{k}]_q} + q^j}{[n]_q} - \frac{[\bar{k}-1]_q}{[n]_q}$$
$$= \frac{q^{\bar{k}-1} + \sqrt{q^{\bar{k}-1}[\bar{k}]_q} + q^j}{[n]_q} \leq \frac{2+\sqrt{[\bar{k}]_q}}{[n]_q}$$
$$\leq 3\frac{\sqrt{[j]_q}}{[n]_q} \leq 3\frac{\sqrt{x}}{\sqrt{[n]_q}}.$$

By Lemma 3.2, (ii) it follows that

$$M_{\bar{k}-1,n,j}(x,q) \geq M_{\bar{k}-2,n,j}(x,q) \geq \ldots \geq M_{0,n,j}(x,q).$$

So, we achieve $M_{k,n,j}(x,q) \leq \frac{\sqrt{x}}{\sqrt{[n]_q}}$ for any $k \leq j-1$ and $x \in \left[\frac{[j]_q}{[n]_q}, \frac{[j+1]_q}{[n]_q}\right]$.

Collecting all the above estimates we have (4.3), which completes the proof of Theorem 4.1. □

5. A-Statistical Approximation

In this section, we will give an A-statistical approximation theorem for the (2.1) operators. Firstly, we have to replace a fixed $q \in (0,1)$ consider in the previous sections with an appropriate sequence (q_n) whose terms are in the interval $(0,1)$. This idea was first used by Philips [2] for the q-Bernstein polynomials.

Let $(q_n)$is a real sequence satisfying the following conditions,

$$0 < q_n < 1 \quad for \quad every \quad n \in \mathbb{N}, \tag{5.1}$$

$$st_A - \lim_n q_n = 1 \tag{5.2}$$

and

$$st_A - \lim_n q_n^n = 1. \tag{5.3}$$

Note that the notations in (5.2) and (5.3) denote the A-statistical limit of (q_n) where $A = [a_{jn}]$, $(j,n \in \mathbb{N})$ is an infinite non-negative regular summability matrix, i.e., $a_{jn} \geq 0$ for every $j,n \in \mathbb{N}$ and $\lim_j \sum_{n=1}^{\infty} a_{jn}x_n = L$ provided that, for a given sequence (x_n), we say that (x_n) is A-statistically convergent to a number L if, for every $\varepsilon > 0$, $\lim_j \sum_{n:|x_n-L|\geq\varepsilon}^{\infty} a_{jn}x_n = 0$ (see [17]). We should remark that this method of convergence generalizes both the classical convergence and the concept of statistical convergence which first introduced by Fast [18]. We give the following A-statistical approximation theorem.

Theorem 5.1. *Let $A = [a_{nj}]$ be a non-negative regular summability matrix and (q_n) be a sequence satisfying (5.1)-(5.3). Then for every $f \in C_+[0,\infty)$ we have*

$$st_A - \lim_n \left(\sup_{x\in[0,\infty)} \left| F_{n,q}^{(M)}(f)(x) - f(x) \right| \right) = 0. \tag{5.4}$$

Proof. Let $f \in C_+[0,\infty)$. Replacing q with (q_n), taking supremum over $x \in [0,\infty)$ and using the monotonicity of the modulus of continuity, we achieve from Theorem 4.1 that

$$E_n := \sup_{x\in[0,\infty)} \left| F_{n,q}^{(M)}(f)(x) - f(x) \right| \leq 8\omega_1\left(f; \frac{\sqrt{x}}{\sqrt{[n]_q}}\right), \tag{5.5}$$

holds for every $n \in \mathbb{N}$. Then, let we prove

$$st_A - \lim_n E_n = 0.$$

From (5.1)-(5.3), we get

$$st_A - \lim_n \frac{1}{\sqrt{[n]_{q_n}}} = 0.$$

So we can write

$$st_A - \lim_n \omega_1 \left(f; \frac{\sqrt{x}}{\sqrt{[n]_q}} \right) = 0. \tag{5.6}$$

So, the proof of Theorem 5.1 follows from (5.1)-(5.6) immediately. □

We should note that the A-statistical approximation result in Theorem 5.1 includes the classical approximation by choosing $A = I$ the identity matrix.

Article Information

Acknowledgements: The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Author's contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the author.

Supporting/Supporting Organizations: No grants were received from any public, private or non-profit organizations for this research.

Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism detected.

Availability of data and materials: Not applicable.

References

[1] A. Lupas, *A q-analogue of the Bernstein operator*, in Seminar on Numerical and Statistical Calculus, University of Cluj-Napoca, **9** (1987), 85–92.

[2] G.M. Philips, *Bernstein polynomials based on the q-integers*, Ann. Numer. Math., **4** (1997), 511-518.

[3] S. Ostrovska, *q-Bernstein polynomials and their iterates*, J. Approximation Theory, **123**(2) (2003), 232-255.

[4] S. Ostrovska, *On the Lupas q-analogue of the Bernstein operator*, Rocky Mountain. J. Math., **36**(5) (1997), 1615-1629.

[5] H. Oruc, N. Tuncer, *On the convergence and iterates of q-Bernstein polynomials*, J. Approx. Theory, **117** (2002), 301-313.

[6] M.A. Siddigue, R.R. Aqrawal, N. Gupta, *On a class of new Bernstein operators*, Advanced Studies in Contemporary Mathematics, 2015.

[7] D. Karahan, A. Izgi, *On approximation properties of generalized q-Bernstein operators*, Num. Funct. Anal. Opt., **39** (2018), 990-998.

[8] D. Karahan, A. Izgi, *On approximation properties of (p,q)-Bernstein operators*, Eur. J. of Pure and App. Math., **11** (2018), 457-467.

[9] B. Bede, L. Coroianu, S.G. Gal, *Approximation by max-product type operators*, Springer International Publishing Switzerland, 2016.

[10] B. Bede, L. Coroianu, S.G. Gal, *Approximation by truncated Favard-Szász-Mirakjan operator of max-product kind*, Demonstratio Math. **44** (2011), 105-122.

[11] B. Bede, L. Coroianu, S.G. Gal, *Approximation and shape preserving properties of the Bernstein operator of max-product kind*, Int. J. Math Sci. Art. ID 590589, (2009), 26pp.

[12] B. Bede, L. Coroianu, S.G. Gal, *Approximation and shape preserving properties of the nonlinear Favard-Szasz-Mirakjan operator of max-product kind*, Filomat, **24**(3) (2010), 55-72.

[13] B. Bede, S.G. Gal, *Approximation by nonlinear Bernstein and Favard-Szász-Mirakyan operators of max-product kind*, J. Concrete and Applicable Math., **8**(2) (2010), 193–207.

[14] O. Duman, *Nonlinear Approximation: q-Bernstein operators of max-product kind*, Intelligent Mathematics II: Applied Mathematics and Approximation Theory, vol 441. Springer.

[15] J. Favard, *Sur les multiplicateurs d'interpolation*, J. Math. Pures Appl. Ser., **9**(23) (1944), 219–247.

[16] N.I. Mahmudov, *Approximation by the q-Szasz-Mirakjan operators*, Abstr. Appl. Anal., **2012**, Article ID 754217, 16 pages, 2012.

[17] G.H. Hardy, *Divergent Series*, Clarendon Press, Oxford, 1949.

[18] H. Fast, *Sur la convergence statistique*, Colloquium Math. **2** (1951), 241-244.