Mathematical Sciences and Applications E-Notes

https://doi.org/10.36753/mathenot.1243583
11 (1) 56-66 (2023) - Research Article
ISSN: 2147-6268

On the Qualitative Behavior of the Difference Equation $\delta_{m+1} = \omega + \zeta \frac{f(\delta_m, \delta_{m-1})}{\delta_{m-1}^{\beta}}$

Mehmet Gümüş* and Şeyma Irmak Eğilmez

Abstract
In this paper, we aim to investigate the qualitative behavior of a general class of non-linear difference equations. That is, the prime period two solutions, the prime period three solutions and the stability character are examined. We also use a new technique introduced in [1] by E. M. Elsayed and later developed by O. Moaaz in [2] to examine the existence of periodic solutions of these general equations. Moreover, we use homogeneous functions for the investigation of the dynamics of the aforementioned equations.

Keywords: *Homogeneous function; difference equation; periodicity; qualitative behavior; stability.*

AMS Subject Classification (2020): *Primary: 39A05 ; Secondary: 39A21; 39A23; 39A30.*

****Corresponding author***

1. Introduction

Since the emergence of the difference equation theory, many pioneering studies have been carried out that will benefit both the development of the theory and other applied sciences (see [3], [4], [5], [6], [7]). The use of difference equations is not only in theory but also in many applied sciences outside the field of mathematics in terms of applying mathematical models of physical phenomena to daily life. Especially in mathematical biology, ecology, and economics, different mathematical models are needed to study populations, population growth and the spread of epidemics (see [8], [9], [10]). Therefore, the difference equations create mathematical models that can be applied to the basic living conditions of physical phenomena. At the same time, it can be said that the solution of even the simplest problem encountered in differential equations, which is another branch of applied mathematics, is more complex and more difficult than the difference equations. For this reason, difference equations have been used as an approach in the mathematical modelling of many physical, chemical and biological phenomena that can

Received : *27-01-2023,* ***Accepted :*** *11-03-2023*

*(**Cite as** "M. Gümüş, Ş. I. Eğilmez, On the Qualitative Behavior of the Difference Equation $\delta_{m+1} = \omega + \zeta \frac{f(\delta_m, \delta_{m-1})}{\delta_{m-1}^{\beta}}$, Math. Sci. Appl. E-Notes, 11(1) (2023), 56-66")*

express with ordinary and partial differential equations (ODE and PDE) and in equations that are difficult to solve analytically [11].

In recent years, difference equation research has attracted great interest from researchers. In particular, applications of higher-order non-linear difference equations have influenced many researchers (see [12], [13], [14], [15], [16], [17], [18], [19]). It is very important to examine especially the oscillation, the asymptotic behavior and the stability character of the solutions of the general classes of the higher-order non-linear difference equations, which have a complex structure and contain different states. However, there are not many articles and books that handle with the qualitative studies of non-linear difference equations. Therefore, there are many aspects of non-linear difference equations that need to be investigated and developed.

In [1], Elsayed introduced a new method for the prime period two solutions and the prime period three solutions of the rational difference equation

$$\delta_{m+1} = \mu + \phi \frac{\delta_m}{\delta_{m-1}} + \gamma \frac{\delta_{m-1}}{\delta_m},\ m = 0, 1, \ldots$$

where the parameters $\mu, \phi, \gamma \in \mathbb{R}^+$ and initial values $\delta_{-1}, \delta_0 \in \mathbb{R}^+$. Besides, the global convergence and the boundedness nature have been investigated.

In [20], Moaaz *et al.* examined the dynamical behaviors of solutions of a general class difference equation

$$z_{m+1} = g(z_m, z_{m-1}),\ m = 0, 1, \ldots$$

where the initial conditions $z_{-1}, z_0 \in \mathbb{R}$ and g is a continuous homogeneous function with degree zero. Namely, the stability, the oscillation and the periodicity character have been investigated.

In [21], Moaaz studied the global dynamics of solutions of the following general class of difference equations

$$\delta_{m+1} = g(\delta_{m-l}, \delta_{m-k}),\ m = 0, 1, \ldots$$

where l, k are positive integers, the initial conditions $\delta_{-\rho}, \delta_{-\rho+1}, \ldots, \delta_0 \in \mathbb{R}$ for $\rho = \max\{l, k\}$ and g is a continuous homogeneous real function of degree γ. That is, the global attractiveness, the periodic character, and the stability nature have been investigated. Furthermore, the author investigated the periodic solutions with used the new method [1].

In [22], Moaaz *et al.* studied the existence and the non-existence of periodic solutions of some non-linear difference equations. Especially, they investigated the existence of periodic solutions of the difference equation

$$\omega_{m+1} = \gamma \omega_{m-1} F(\omega_m, \omega_{m-1}),\ m = 0, 1, \ldots$$

where the parameter $\gamma \in \mathbb{R}^+$, the initial values $\omega_{-1}, \omega_0 \in \mathbb{R}^+$ and F is a homothetic function, namely there exists a strictly increasing function $F_1 : \mathbb{R} \to \mathbb{R}$ and $F_2 : \mathbb{R}^2 \to \mathbb{R}$ are homogenous function with degree ρ, such that $F = F_1(F_2)$ and also studied the following second-order difference equation

$$\omega_{m+1} = \mu + \eta \frac{\omega_{m-1}^{\rho}}{h(\omega_m, \omega_{m-1})},\ m = 0, 1, \ldots$$

where $\rho \in \mathbb{R}^+$, the parameters $\mu, \eta \in \mathbb{R}$, the initial values $\omega_{-1}, \omega_0 \in \mathbb{R}$ and h is a continuous homogeneous function with degree ρ.

In [23], Abdelrahman investigated the dynamical behavior of solutions of the general class of difference equations

$$\omega_{m+1} = h(\omega_m, \omega_{m-1}, \ldots, \omega_{m-k}),\ m = 0, 1, \ldots$$

where $h : (0, \infty)^{k+1} \to (0, \infty)$ is a continuously homogeneous function of degree zero and k is positive integer. That is, the stability, the periodicity and the oscillation nature have been examined.

The aim of this paper is to investigate the global behavior of solutions, that is, the prime period two solutions, the prime period three solutions and the stability character of a new general class of the second-order difference equation

$$\delta_{m+1} = \omega + \zeta \frac{f(\delta_m, \delta_{m-1})}{\delta_{m-1}^{\beta}},\ m = 0, 1, \ldots \tag{1.1}$$

where the parameters $\omega, \zeta \in \mathbb{R}$, the initial conditions $\delta_{-1}, \delta_0 \in \mathbb{R}$ and $f : (0, \infty)^2 \to (0, \infty)$ is a continuous homogeneous function with degree β. Also, in particular, the two periodic solutions and the three periodic solutions are examined by using the new method [1, 2]. In addition, we specify the new sufficient conditions for the stability character of the positive equilibrium point.

2. Preliminaries

In the following, we give some basic definitions and theorems that we will benefit from in this paper.

Assume that J be an interval of real numbers and let the initial conditions for every $z_{-1}, z_0 \in J$. If $h : J \times J \to J$ be a continuously differentiable function, then the difference equation

$$z_{n+1} = h(z_n, z_{n-1}),\ n \in \mathbb{N} \tag{2.1}$$

has a unique positive solution $\{z_n\}_{n=-1}^{\infty}$.

Definition 2.1. [24] (Periodicity) Let t be a positive integer. Then, the solution $\{x_n\}_{n=-1}^{\infty}$ of Eq.(2.1) is said to be periodic with period t if

$$x_{n+t} = x_n,\ n = 0, 1, \ldots$$

where t is the smallest integer.

Theorem 2.1. *[24] (The Linearized Stability Theorem)*
(i) If both roots of the quadratic equation

$$\lambda^2 - p\lambda - q = 0 \tag{2.2}$$

lie in the open unit disk $|\lambda| < 1$, then the equilibrium point $\overline{x}$ of Eq.(2.1) is locally asymptotic stable.

(ii) If at least one of the roots of Eq.(2.2) has absolute value greater than one, then the equilibrium point $\overline{x}$ of Eq.(2.1) is unstable.

Theorem 2.2. *[5] (Clark Theorem) Assume that $\rho_0, \rho_1 \in \mathbb{R}$ and $k \in \{0, 1, \ldots\}$. Then, the difference equation*

$$\delta_{m+1} + \rho_0\delta_m + \rho_1\delta_{m-k} = 0,\ m = 0, 1, \ldots$$

is the asymptotic stability if

$$|\rho_0| + |\rho_1| < 1.$$

Definition 2.2. [25] (Homogeneous Function) Assume that $f : \mathbb{R}_+^n \to \mathbb{R}$ is called a homogeneous function with degree k if for every $x \in \mathbb{R}_+^n$ and every $\lambda > 0$

$$f(\lambda x) = \lambda^k f(x).$$

Theorem 2.3. *[25] (Euler's Homogeneous Function Theorem) Assume*

$$f : \mathbb{R}_+^n \to \mathbb{R}$$

is a continuous function and also differentiable on $\mathbb{R}_+^n$. Then, f is homogeneous function with degree k if only if for every $x \in \mathbb{R}_+^n$

$$kf(x) = \sum_{i=1}^{n} D_i f(x) x_i.$$

Corollary 2.1. *[25] Let $f : \mathbb{R}_+^n \to \mathbb{R}$ be continuous function, and also differentiable on $\mathbb{R}_+^n$. If f is homogeneous function with degree k, then $D_j f(x)$ is homogeneous with degree $k - 1$.*

3. Asymptotic behavior of solutions of the non-linear difference equation (1.1)

In this section, we will examine the two periodic solutions, the three periodic solutions and the stability character of the second-order non-linear difference equation (1.1).

Here, we investigate the stability character of the positive equilibrium point of Eq.(1.1). From the definition equilibrium point, we obtain that

$$\begin{aligned}\bar{\delta} &= \omega + \zeta\frac{f(\bar{\delta}, \bar{\delta})}{\bar{\delta}^{\beta}} \\ &= \omega + \zeta\frac{\bar{\delta}^{\beta} f(1, 1)}{\bar{\delta}^{\beta}}.\end{aligned}$$

Thus, the positive equilibrium point is

$$\bar{\delta} = \omega + \zeta f(1,1).$$

Now, let's define the function $f : (0,\infty)^2 \to (0,\infty)$ by

$$f(u,v) = \omega + \zeta \frac{f(u,v)}{v^{\beta}}.$$

Thus, we obtain that

$$\frac{\partial f}{\partial u}(u,v) = \zeta \frac{f_u(u,v) v^{\beta}}{(v^{\beta})^2},$$

and

$$\frac{\partial f}{\partial v}(u,v) = \zeta \frac{f_v(u,v) v^{\beta} - \beta v^{\beta-1} f(u,v)}{(v^{\beta})^2}.$$

In the next theorem, the locally asymptotic stability for Eq.(1.1) will be investigated.

Theorem 3.1. *The equilibrium point of Eq.(1.1) $\bar{\delta} = \omega + \zeta f(1,1)$ is locally asymptotically stable if*

$$|f_u(1,1)| + |f_v(1,1) - \beta f(1,1)| < \left| \frac{\omega + \zeta f(1,1)}{\zeta} \right|.$$

Proof. Using Euler's Homogeneous Function Theorem, and from Corollary (2.1), we can easily obtain that

$$\begin{aligned} f_u(\bar{\delta},\bar{\delta}) &= \zeta \frac{f_u(\bar{\delta},\bar{\delta})\bar{\delta}^{\beta}}{\bar{\delta}^{2\beta}} \\ &= \zeta \frac{\bar{\delta}^{2\beta-1} f_u(1,1)}{\bar{\delta}^{2\beta}} \\ &= \zeta \frac{f_u(1,1)}{\bar{\delta}}, \end{aligned}$$

and

$$\begin{aligned} f_v(\bar{\delta},\bar{\delta}) &= \zeta \frac{f_v(\bar{\delta},\bar{\delta})\bar{\delta}^{\beta} - \beta v^{\beta-1} f(\bar{\delta},\bar{\delta})}{\bar{\delta}^{2\beta}} \\ &= \zeta \frac{\bar{\delta}^{2\beta-1} f_v(1,1) - \beta \bar{\delta}^{2\beta-1} f(1,1)}{\bar{\delta}^{2\beta}} \\ &= \zeta \frac{f_v(1,1) - \beta f(1,1)}{\bar{\delta}}. \end{aligned}$$

Hence, by using Clark Theorem, we obtain that

$$\left| \zeta \frac{f_u(1,1)}{\bar{\delta}} \right| + \left| \zeta \frac{f_v(1,1) - \beta f(1,1)}{\bar{\delta}} \right| < 1.$$

Since $\bar{\delta} = \omega + \zeta f(1,1)$, we find

$$\left| \zeta \frac{f_u(1,1)}{\omega + \zeta f(1,1)} \right| + \left| \zeta \frac{f_v(1,1) - \beta f(1,1)}{\omega + \zeta f(1,1)} \right| < 1,$$

and so

$$|f_u(1,1)| + |f_v(1,1) - \beta f(1,1)| < \left| \frac{\omega + \zeta f(1,1)}{\zeta} \right|.$$

The proof is completed. □

In the following theorem, the two periodic solutions of Eq.(1.1) will be examined.

Theorem 3.2. *Eq.(1.1) has the prime period two solution*

$$\ldots,\sigma,\mu,\sigma,\mu,\ldots$$

if and only if

$$\omega=\zeta\frac{\phi^{\beta+1}f\left(1,\frac{1}{\phi}\right)-f\left(\frac{1}{\phi},1\right)}{(1-\phi)} \tag{3.1}$$

where $\phi=\frac{\sigma}{\mu}$, $\phi\in\mathbb{R}-\{0,\pm1\}$.

Proof. Suppose that Eq.(1.1) has a prime period two solution in the following form

$$\ldots,\sigma,\mu,\sigma,\mu,\ldots.$$

Let's define $\delta_{m-(2s+1)}=\sigma$ and $\delta_{m-2s}=\mu$ for $s=0,1,2,\ldots$. From the definition of the periodicity, we can rewrite the following equalities

$$\sigma=\omega+\zeta\frac{f(\mu,\sigma)}{\sigma^{\beta}},$$

and

$$\mu=\omega+\zeta\frac{f(\sigma,\mu)}{\mu^{\beta}}.$$

Therefore, we obtain that

$$\sigma=\omega+\zeta\frac{\sigma^{\beta}f\left(\frac{\mu}{\sigma},1\right)}{\sigma^{\beta}}\Rightarrow\sigma=\omega+\zeta f\left(\frac{1}{\phi},1\right), \tag{3.2}$$

and

$$\mu=\omega+\zeta\frac{\sigma^{\beta}f\left(1,\frac{\mu}{\sigma}\right)}{\mu^{\beta}}\Rightarrow\mu=\omega+\zeta\phi^{\beta}f\left(1,\frac{1}{\phi}\right). \tag{3.3}$$

Now, by using the fact $\sigma-\phi\mu=0$, we get

$$0=\sigma-\phi\mu=\omega+\zeta f\left(\frac{1}{\phi},1\right)-\phi\left(\omega+\zeta\phi^{\beta}f\left(1,\frac{1}{\phi}\right)\right),$$

and so

$$\omega(1-\phi)=\zeta\phi^{\beta+1}f\left(1,\frac{1}{\phi}\right)-\zeta f\left(\frac{1}{\phi},1\right).$$

Therefore, we find

$$\omega=\zeta\frac{\phi^{\beta+1}f\left(1,\frac{1}{\phi}\right)-f\left(\frac{1}{\phi},1\right)}{(1-\phi)}.$$

Thus, from (3.2) and (3.3), we obtain that

$$\begin{aligned}\sigma&=\zeta\frac{\phi^{\beta+1}f\left(1,\frac{1}{\phi}\right)-f\left(\frac{1}{\phi},1\right)}{(1-\phi)}+\zeta f\left(\frac{1}{\phi},1\right)\\&=\zeta\frac{\phi^{\beta+1}f\left(1,\frac{1}{\phi}\right)-\phi f\left(\frac{1}{\phi},1\right)}{(1-\phi)},\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}\mu&=\zeta\frac{\phi^{\beta+1}f\left(1,\frac{1}{\phi}\right)-f\left(\frac{1}{\phi},1\right)}{(1-\phi)}+\zeta\phi^{\beta}f\left(1,\frac{1}{\phi}\right)\\&=\zeta\frac{\phi^{\beta}f\left(1,\frac{1}{\phi}\right)-f\left(\frac{1}{\phi},1\right)}{(1-\phi)}.\end{aligned} \tag{3.5}$$

Secondly, suppose (3.1) holds. Let's choose the initial conditions

$$\delta_{-1} = \sigma \text{ and } \delta_0 = \mu,$$

where σ, μ defined as (3.2) and (3.3), respectively. Therefore, we see that

$$\begin{aligned}
\delta_1 &= \omega + \zeta \frac{f(\delta_0, \delta_{-1})}{\delta_{-1}^{\beta}} \\
&= \zeta \frac{\phi^{\beta+1} f\left(1, \frac{1}{\phi}\right) - f\left(\frac{1}{\phi}, 1\right)}{(1-\phi)} + \zeta \frac{f(\mu, \sigma)}{\sigma^{\beta}} \\
&= \zeta \frac{\phi^{\beta+1} f\left(1, \frac{1}{\phi}\right) - f\left(\frac{1}{\phi}, 1\right)}{(1-\phi)} + \zeta \frac{\sigma^{\beta} f\left(\frac{\mu}{\sigma}, 1\right)}{\sigma^{\beta}} \\
&= \zeta \frac{\phi^{\beta+1} f\left(1, \frac{1}{\phi}\right) - f\left(\frac{1}{\phi}, 1\right)}{(1-\phi)} + \zeta f\left(\frac{1}{\phi}, 1\right) \\
&= \zeta \frac{\phi^{\beta+1} f\left(1, \frac{1}{\phi}\right) - \phi f\left(\frac{1}{\phi}, 1\right)}{(1-\phi)} = \sigma
\end{aligned}$$

and

$$\begin{aligned}
\delta_2 &= \omega + \zeta \frac{f(\delta_1, \delta_0)}{\delta_0^{\beta}} \\
&= \zeta \frac{\phi^{\beta+1} f\left(1, \frac{1}{\phi}\right) - f\left(\frac{1}{\phi}, 1\right)}{(1-\phi)} + \zeta \frac{f(\sigma, \mu)}{\mu^{\beta}} \\
&= \zeta \frac{\phi^{\beta+1} f\left(1, \frac{1}{\phi}\right) - f\left(\frac{1}{\phi}, 1\right)}{(1-\phi)} + \zeta \frac{\sigma^{\beta} f\left(1, \frac{\mu}{\sigma}\right)}{\mu^{\beta}} \\
&= \zeta \frac{\phi^{\beta+1} f\left(1, \frac{1}{\phi}\right) - f\left(\frac{1}{\phi}, 1\right)}{(1-\phi)} + \zeta \phi^{\beta} f\left(1, \frac{1}{\phi}\right) \\
&= \zeta \frac{\phi^{\beta} f\left(1, \frac{1}{\phi}\right) - f\left(\frac{1}{\phi}, 1\right)}{(1-\phi)} = \mu.
\end{aligned}$$

Then, by induction, we can obtain for all $m \geq 0$

$$\delta_{2m-1} = \sigma \text{ and } \delta_{2m} = \mu.$$

Hence, Eq.(1.1) has a prime period two solution. The proof is completed. □

In the following theorem, the three periodic solutions of Eq.(1.1) will be investigated.

Theorem 3.3. *Eq.(1.1) has the prime period three solution $\{\delta_m\}_{m=-1}^{\infty}$ where*

$$\delta_m = \begin{cases} \sigma, & \text{for } m = 3z-1 \\ \mu, & \text{for } m = 3z \\ \rho, & \text{for } m = 3z+1 \end{cases}, \quad z = 0, 1, \ldots$$

if and only if

$$\begin{aligned}
\eta\left(\omega + \zeta \frac{f(\psi, \eta)}{\eta^{\beta}}\right) &= \omega + \zeta \frac{f(1, \psi)}{\psi^{\beta}} \\
\psi\left(\omega + \zeta \frac{f(\psi, \eta)}{\eta^{\beta}}\right) &= \omega + \zeta f(\eta, 1)
\end{aligned} \tag{3.6}$$

where $\eta = \frac{\mu}{\sigma}$ and $\psi = \frac{\rho}{\sigma}$, $\eta, \psi \in \mathbb{R} - \{0, \pm 1\}$.

Proof. Suppose that Eq.(1.1) has a prime period three solution in the following form

$$\ldots, \sigma, \mu, \rho, \sigma, \mu, \rho, \ldots .$$

From Eq.(1.1), we find

$$\sigma = \omega + \zeta \frac{f(\rho, \mu)}{\mu^{\beta}},$$

$$\mu = \omega + \zeta \frac{f(\sigma, \rho)}{\rho^{\beta}},$$

and

$$\rho = \omega + \zeta \frac{f(\mu, \sigma)}{\sigma^{\beta}}.$$

Since f is a homogeneous function with degree β, we find

$$\sigma = \omega + \zeta \frac{\sigma^{\beta} f(\psi, \eta)}{\mu^{\beta}} \Rightarrow \sigma = \omega + \zeta \frac{f(\psi, \eta)}{\eta^{\beta}},$$

$$\mu = \omega + \zeta \frac{\sigma^{\beta} f(1, \psi)}{\rho^{\beta}} \Rightarrow \mu = \omega + \zeta \frac{f(1, \psi)}{\psi^{\beta}},$$

and

$$\rho = \omega + \zeta \frac{\sigma^{\beta} f(\eta, 1)}{\sigma^{\beta}} \Rightarrow \rho = \omega + \zeta f(\eta, 1).$$

Therefore, we can obtain that

$$\eta = \frac{\mu}{\sigma} = \frac{\omega + \zeta \frac{f(1,\psi)}{\psi^{\beta}}}{\omega + \zeta \frac{f(\psi,\eta)}{\eta^{\beta}}},$$

and

$$\psi = \frac{\rho}{\sigma} = \frac{\omega + \zeta f(\eta, 1)}{\omega + \zeta \frac{f(\psi,\eta)}{\eta^{\beta}}}.$$

Hence, we find

$$\eta \left(\omega + \zeta \frac{f(\psi, \eta)}{\eta^{\beta}} \right) = \omega + \zeta \frac{f(1, \psi)}{\psi^{\beta}},$$

$$\psi \left(\omega + \zeta \frac{f(\psi, \eta)}{\eta^{\beta}} \right) = \omega + \zeta f(\eta, 1).$$

Secondly, assume that (3.6) holds. Let's choose the initial values for all $\eta, \psi \in \mathbb{R} - \{0, 1\}$

$$\delta_{-1} = \omega + \zeta \frac{f(\psi, \eta)}{\eta^{\beta}}$$

and

$$\delta_0 = \omega + \zeta \frac{f(1, \psi)}{\psi^{\beta}}.$$

Thus, we obtain that

$$\begin{aligned}
\delta_1 &= \omega + \zeta \frac{f(\delta_0, \delta_{-1})}{\delta_{-1}^{\beta}} \\
&= \omega + \zeta \frac{f\left(\omega + \zeta \frac{f(1,\psi)}{\psi^{\beta}}, \omega + \zeta \frac{f(\psi,\eta)}{\eta^{\beta}}\right)}{\left(\omega + \zeta \frac{f(\psi,\eta)}{\eta^{\beta}}\right)^{\beta}} \\
&= \omega + \zeta \frac{f\left(\eta\left(\omega + \zeta \frac{f(\psi,\eta)}{\eta^{\beta}}\right), \omega + \zeta \frac{f(\psi,\eta)}{\psi^{\beta}}\right)}{\left(\omega + \zeta \frac{f(\psi,\eta)}{\eta^{\beta}}\right)^{\beta}} \\
&= \omega + \zeta \frac{\left(\omega + \zeta \frac{f(\psi,\eta)}{\psi^{\beta}}\right)^{\beta} f(\eta, 1)}{\left(\omega + \zeta \frac{f(\psi,\eta)}{\psi^{\beta}}\right)^{\beta}} \\
&= \omega + \zeta f(\eta, 1) = \rho,
\end{aligned}$$

$$\begin{aligned}
\delta_2 &= \omega + \zeta \frac{f(\delta_1, \delta_0)}{\delta_0^{\beta}} \\
&= \omega + \zeta \frac{f\left(\omega + \zeta f(\eta, 1), \omega + \zeta \frac{f(1,\psi)}{\psi^{\beta}}\right)}{\left(\omega + \zeta \frac{f(1,\psi)}{\psi^{\beta}}\right)^{\beta}} \\
&= \omega + \zeta \frac{f\left(\psi\left(\omega + \zeta \frac{f(\psi,\eta)}{\eta^{\beta}}\right), \eta\left(\omega + \zeta \frac{f(\psi,\eta)}{\eta^{\beta}}\right)\right)}{\left(\eta\left(\omega + \zeta \frac{f(\psi,\eta)}{\eta^{\beta}}\right)\right)^{\beta}} \\
&= \omega + \zeta \frac{\left(\omega + \zeta \frac{f(\psi,\eta)}{\eta^{\beta}}\right)^{\beta} f(\psi, \eta)}{\left(\eta\left(\omega + \zeta \frac{f(\psi,\eta)}{\eta^{\beta}}\right)\right)^{\beta}} \\
&= \omega + \zeta \frac{f(\psi, \eta)}{\eta^{\beta}} = \sigma
\end{aligned}$$

and

$$\begin{aligned}
\delta_3 &= \omega + \zeta \frac{f(\delta_2, \delta_1)}{\delta_1^{\beta}} \\
&= \omega + \zeta \frac{f\left(\omega + \zeta \frac{f(\psi,\eta)}{\eta^{\beta}}, \omega + \zeta f(\eta, 1)\right)}{(\omega + \zeta f(\eta, 1))^{\beta}} \\
&= \omega + \zeta \frac{f\left(\omega + \zeta \frac{f(\psi,\eta)}{\eta^{\beta}}, \psi\left(\omega + \zeta \frac{f(\psi,\eta)}{\eta^{\beta}}\right)\right)}{\left(\psi\left(\omega + \zeta \frac{f(\psi,\eta)}{\eta^{\beta}}\right)\right)^{\beta}} \\
&= \omega + \zeta \frac{\left(\omega + \zeta \frac{f(\psi,\eta)}{\eta^{\beta}}\right)^{\beta} f(1, \psi)}{\left(\psi\left(\omega + \zeta \frac{f(\psi,\eta)}{\eta^{\beta}}\right)\right)^{\beta}} \\
&= \omega + \zeta \frac{f(1, \psi)}{\psi^{\beta}} = \mu.
\end{aligned}$$

Then, by induction, we can obtain that for all $m \geq 0$

$$\delta_{3m+1} = \rho, \ \delta_{3m+2} = \sigma \text{ and } \delta_{3m+3} = \mu.$$

Hence, Eq.(1.1) has a prime period three solution. The proof is completed. $\square$

4. Conclusions and suggestions

Mathematical models are of great importance in the natural sciences, including biology, ecology, engineering sciences and genetics (see [8], [9], [10]). Mathematical models are developed to explain a system, study the effects of its various components, and make predictions about their behavior. Discrete models treat time or system states as discrete. Mathematical models can be created with the help of difference equations. In this respect, each study in the field of difference equation theory is very valuable both in terms of its own importance and has applications in other disciplines.

In this work, we introduced a new general class of non-linear difference equations. we investigated the qualitative behavior of solutions of the introduced second-order non-linear difference equations. In other words, we dealt with the two periodic solutions, the three periodic solutions and the stability character of given difference equations. In particular, we obtained the periodic solutions using the new method. Finally, we obtained new sufficient conditions for local asymptotic stability for the given difference equations. We can say that the results we have obtained here have gathered and developed many previous studies under one roof. Contributing to the theory of difference equations introduced with the help of homogeneous functions [20–23, 26] in this article has been one of the main aims.

It can be suggested to those who do research in this field that research can be done in the equations established with the help of homogeneous functions. Difference equations created with these functions are very convenient and useful for researching general classes of difference equations.

In our future studies, we will aim to investigate some general classes of difference equations formed by homogeneous functions of different degrees.

Article Information

Acknowledgements: The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Author's contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the author.

Supporting/Supporting Organizations: No grants were received from any public, private or non-profit organizations for this research.

Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism detected.

Availability of data and materials: Not applicable.

References

[1] Elsayed, E. M.: *New method to obtain periodic solutions of period two and three of a rational difference equation*. Nonlinear Dynamics. **79** (1), 241-250 (2014).

[2] Moaaz, O.: *Comment on "new method to obtain periodic solutions of period two and three of a rational difference equation" [Nonlinear Dyn 79:241–250]*. Nonlinear Dyn. **88**, 1043-1049 (2017).

[3] Elaydi, S.: An introduction to difference equations. 3rd ed. Springer-Verlag. New York (2005).

[4] Kelley, W. G., Peterson, A. C.: Difference equations: An introduction with applications. Academic Press. New York (1991).

[5] Kocić, V., Ladas, G.: Global behavior of non-linear difference equations of higher-order with applications. Kluwer Academic Publishers. Dordrecht (1993).

[6] Levin, S. A., May, R. M.: *A note on difference-delay equations.* Theoretical Population Biology. **9** (2), 178-187 (1976).

[7] Mickens, R. E.: Difference equations, theory and applications. Van Nostrand Rheinhold. (1990).

[8] Allen, L. J. S.: An introduction to mathematical biology. Pearson/Prentice Hall. New Jersey (2007).

[9] Murray, J. D.: Mathematical biology I: An introduction. 3rd ed. Springer. (2002).

[10] Pielou, E. C.: An introduction to mathematical ecology. Wiley Interscience. New York (1969).

[11] Oztepe, G. S.: *An investigation on the Lasota-Wazewska model with a piecewise constant argument.* Hacettepe Journal of Mathematics and Statistics. **50** (5), 1500-1508 (2021).

[12] Abo-Zeid, R.: *Global attractivity of a higher-order difference equation.* Discrete Dynamics in Nature and Society. **2012**, 930410 (2012).

[13] Abo-Zeid, R.: *Global behavior of a higher-order difference equation.* Mathematica Slovaca. **64** (4), 931-940 (2014).

[14] Belhannache, F., Touafek, N., Abo-Zeid, R.: *Dynamics of a third-order rational difference equation.* Bulletin Mathematique de La Societe Des Sciences Mathematiques de Roumanie. **59(107)** (1), 13-22 (2016).

[15] Gumus, M.: *Global dynamics of solutions of a new class of rational difference equations.* Konuralp Journal of Mathematics. **7** (2), 380-387 (2019).

[16] Gumus, M.: *Analysis of periodicity for a new class of non-linear difference equations by using a new method.* Electron. J. Math. Anal. Appl. **8**, 109-116 (2020).

[17] Halim, Y., Touafek, N.,Yazlik, Y.: *Dynamic behavior of a second-order non-linear rational difference equation.* Turkish Journal of Mathematics. **39** (6), 1004-1018 (2015).

[18] Touafek, N., Halim, Y.: *Global attractivity of a rational difference equation.* Mathematical Sciences Letters. **2** (3), 161-165 (2013).

[19] Yalçınkaya, I.: *On the difference equation* $x_{n+1} = \alpha + x_{n-m}/x_n^k$. Discrete Dynamics in Nature and Society. **2008**, 805460 (2008).

[20] Moaaz, O., Chalishajar, D., Bazighifan, O.: *Some qualitative behavior of solutions of general class of difference equation.* Mathematics. **7**, 585 (2019).

[21] Moaaz, O.: *Dynamics of difference equation* $x_{n+1} = f(x_{n-l}, x_{n-k})$. Advances in Difference Equations. **2018**, 447 (2018).

[22] Moaaz, O., Mahjoub, H., Muhib, A.: *On the periodicity of general class of difference equations.* Axioms. 9, 75 (2020).

[23] Abdelrahman, M. A. E.: *On the difference equation* $z_{m+1} = f(z_m, z_{m-1}, \ldots, z_{m-k})$. Journal of Taibah University for Science. **13** (1), 1014-1021 (2019).

[24] Kulenović, M. R. S., Ladas, G.: Dynamics of second-order rational difference equations. Chapman & Hall/CRC. (2001).

[25] Border, K. C.: *Euler's theorem for homogeneous function.* Caltech Division of The Humanities and Social Sciences. **27**, 16-34 (2017).

[26] Boulouh, M., Touafek, N., Tollu, D. T.: *On the behavior of the solutions of an abstract system of difference equations*. Journal of Applied Mathematics and Computing. **68**, 2937-2969 (2022).

Affiliations

MEHMET GÜMÜŞ
ADDRESS: Zonguldak Bülent Ecevit University, Faculty of Science, Department of Mathematics, Farabi Campus, 67100, Zonguldak, TURKEY.
E-MAIL: m.gumus@beun.edu.tr
ORCID ID:0000-0002-7447-479X

ŞEYMA IRMAK EĞILMEZ
ADDRESS: Zonguldak Bülent Ecevit University, Faculty of Science, Department of Mathematics, Farabi Campus, 67100, Zonguldak, TURKEY.
E-MAIL: e.seymairmak@gmail.com
ORCID ID:0000-0003-1781-5399