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# A DIOPHANTINE EQUATION INCLUDING FIBONACCI AND FIBONOMIAL COEFFICIENTS

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Abstract. In this paper, we solve the equation

$$\sum_{k=0}^{m} \begin{bmatrix} 2m+1 \\ k \end{bmatrix}_F \pm F_t = F_n,$$

under weak assumptions. Here, $F_n$ is $n^{th}$ Fibonacci number and $\begin{bmatrix} \cdot \\ \cdot \end{bmatrix}_F$ denotes Fibonomial coefficient.

## 1. Introduction

For $n \geq 2$, the Fibonacci sequence $\{F_n\}$ is defined by recurrence relation

$$F_n = F_{n-1} + F_{n-2}$$

with $F_0 = 0$ and $F_1 = 1$. A few terms of Fibonacci sequence are $0, 1, 1, 2, 3, 5, 8, 13, ....$ Its Binet formula is known as

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

where $\alpha$ and $\beta$ are the roots of the characteristic equation $x^2 - x - 1 = 0$.

The Fibonacci companion sequence $\{L_n\}$ is known as the Lucas sequence which satisfies the same recurrence relations with Fibonacci sequence and the initials $L_0 = 2$, $L_1 = 1$. A few terms of Lucas sequence are $2, 1, 3, 4, 7, 11, 18, 29, \ldots$. The Binet formula of $n^{th}$ Lucas number is

$$L_n = \alpha^n + \beta^n.$$

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Another concept of the paper is Fibonomial coefficient. For $n \geq k > 0$, the number

$$\frac{F_n F_{n-1} \dots F_{n-k+1}}{F_1 F_2 \dots F_k}$$

is known as Fibonomial coefficient inspired by the binomial coefficient and denoted by $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_F$. Also, for $k = 0$, it is defined by $\left[\begin{smallmatrix} n \\ 0 \end{smallmatrix}\right]_F = 1$. It is interesting that this coefficient always gets integer values for $n, k \in \mathbb{Z}^+$.

The Diophantine equation

$$n! + 1 = m^2 \tag{1}$$

is known as Brocard-Ramanujan Diophantine equation. It is known that $m = 4, 5$ and 7 are the solutions of the this equation. These are not full solutions of the equation (1). Berndt and Galway [1] showed that there are no further solutions with $m \leq 10^9$. The Brocard-Ramanujan equation is still open problem. A number of mathematicians have contributed several generalizations and results regarding this Diophantine equation. For example, Grossman and Luca [4] proved that the equation

$$F_n = m_1! + m_2! + \dots + m_k!$$

has finitely many positive integers $n$ for fixed $k$. Moreover, the case $k \leq 2$ was determined. The case $k = 3$ was solved by Bollman, Hernandez and Luca in [2]. Luca and Siksek [9] found all factorials expressible as the sum of at least three Fibonacci numbers. Marques handled the different versions of the Brocard-Ramanujan equation including Fibonacci and Fibonomial coefficient (for the details see [10], [11], [12]). In what follows, Szalay [13] solved the equation

$$G_{n_1} G_{n_2} \dots G_{n_k} + 1 = G_m^2$$

where the sequence $\{G_n\}$ is either Fibonacci sequence or the Lucas sequence or the sequence of balancing numbers, respectively. Recently, the author [5] proved that the solutions of the equation

$$\sum_{k=0}^{m} \begin{bmatrix} 2m+1 \\ k \end{bmatrix}_F \pm 1 = F_n \tag{2}$$

are $(m, n) = (1, 3)\,,\ (3, 14)$ according to the sign $-$. If the sing is $+$, then there is no solution.

In this paper we focus on the generalization of the equation (2). Our result is following,

**Theorem 1.** *Let $n$ and $t$ are positive integers such that $n \equiv t \pmod 2$ or $n - t = 1, 3$. The solutions of the equations*

$$\sum_{k=0}^{m} \begin{bmatrix} 2m+1 \\ k \end{bmatrix}_F \pm F_t = F_n \tag{3}$$

*are*

| $m$ | 1 | 1 | 2 | 2 |
|---|---|---|---|---|
| $n$ | 5 | 6 | 9 | 10 |
| $t$ | 3 | 5 | 7 | 9 |

*according to the sign* $+$. *If the sign is* $-$, *then the solutions are*

| $m$ | 1 | 1 | 2 | 2 | 4 | 4 |
|---|---|---|---|---|---|---|
| $n$ | 3 | $1,2$ | 7 | 6 | 22 | 10 |
| $t$ | $1,2$ | 3 | 6 | 7 | 10 | 22 |

.

## 2. Preliminary

Before going further, we give several lemmas to prove our theorem.

**Definition 1.** *A primitive divisor* $p$ *of* $F_n$ *is a prime factor of* $F_n$ *which does not divide* $\prod_{j=1}^{n-1} F_j$.

For example, we know that $29 \mid F_{14}$, but $29 \nmid \prod_{j=1}^{13} F_j$. Here, 29 is a primitive divisor of $F_{14}$. The following lemma guarantees the existence of a primitive divisor for the Fibonacci sequence.

**Lemma 1.** *A primitive divisor* $p$ *of* $F_n$ *exists whenever* $n \geq 13$ *(see [3]).*

We present several identities regarding Fibonacci and Lucas numbers that we will use them later.

**Lemma 2.** *We have the following*
*i. For any* $k \geq 0$, *then* $F_k L_k = F_{2k}$.
*ii. For any* $k \geq 0$, *then* $F_{k+3} - F_k = 2F_{k+1}$ *and* $F_{k+3} + F_k = 2F_{k+2}$.
*iii. Let* $n$ *and* $t$ *are positive integers such that* $n \equiv t \pmod 2$. *Then*

$$F_n \mp F_t = \begin{cases} F_{\frac{n\mp t}{2}} L_{\frac{n\pm t}{2}}, & \text{if } n \equiv t \pmod 4 \\ F_{\frac{n\pm t}{2}} L_{\frac{n\mp t}{2}} & \text{if } n \not\equiv t \pmod 4 \end{cases}$$

*holds.*
*iv. For any* $k \geq 0$, *then* $3 \mid F_{4k}$.

*Proof.* (i) can be proven easily by using Binet formulas of the sequence Fibonacci and Lucas. By the recurrence of Fibonacci sequence, we have $F_{n+3} - F_n = F_{n+2} + F_{n+1} - F_n = 2F_{n+1}$. This proves (ii).

*iii.* Assume that $n \equiv t \pmod 4$, then we have followings,

$$\begin{aligned} F_{\frac{n+t}{2}} L_{\frac{n-t}{2}} &= \frac{\alpha^{(n+t)/2} - \beta^{(n+t)/2}}{\alpha - \beta} (\alpha^{(n-t)/2} + \beta^{(n-t)/2}) \\ &= \frac{1}{\alpha - \beta} \left( \alpha^n - \beta^n + (\alpha\beta)^{(n-t)/2} (\alpha^t - \beta^t) \right) \\ &= F_n + F_t. \end{aligned}$$

*iv.* We refer the book of Koshy [8] (Theorem 16.1, p. 196).

□

**Lemma 3.** *Let $m$ be a positive integer, then the identity*

$$\sum_{k=0}^{m} \begin{bmatrix} 2m+1 \\ k \end{bmatrix}_F = \prod_{k=1}^{m} L_{2k}$$

*holds (see [7]).*

The proof of the following lemma is given in [6] as Lemma 2.3.

**Lemma 4.** *For integers $s > t > 1$, the equation*

$$F_r = F_s + F_t$$

*is satisfied only for $r - 1 = s = t + 1$.*

## 3. Proof

Now, we will investigate the solutions of the equation (3) in two different cases $n \equiv t \pmod 2$ and $n \not\equiv t \pmod 2$.

3.1. **The case $n \equiv t \pmod 2$.** In this case, Lemma 2 (iii) yields the following equations

$$\sum_{k=0}^{m} \begin{bmatrix} 2m+1 \\ k \end{bmatrix}_F = F_{\frac{n-t}{2}} L_{\frac{n+t}{2}} \tag{4}$$

or

$$\sum_{k=0}^{m} \begin{bmatrix} 2m+1 \\ k \end{bmatrix}_F = F_{\frac{n+t}{2}} L_{\frac{n-t}{2}}. \tag{5}$$

We will consider the equation (4). Lemma 3 yields that

$$L_2 L_4 \dots L_{2m} = F_{\frac{n-t}{2}} L_{\frac{n+t}{2}}.$$

If we multiply both sides with $F_2 F_4 \dots F_{2m}$ and $F_{\frac{n+t}{2}}$, then we have

$$\begin{aligned} (L_2 L_4 \dots L_{2m})(F_2 F_4 \dots F_{2m}) F_{\frac{n+t}{2}} &= (F_2 L_2)(F_4 L_4) \dots (F_{2m} L_{2m}) F_{\frac{n+t}{2}} \\ &= F_4 F_8 \dots F_{4m} F_{\frac{n+t}{2}} \\ &= (F_2 F_4 \dots F_{2m}) F_{\frac{n-t}{2}} (F_{\frac{n+t}{2}} L_{\frac{n+t}{2}}) \\ &= (F_2 F_4 \dots F_{2m}) F_{\frac{n-t}{2}} F_{n+t} \end{aligned}$$

where we use the fact $F_n L_n = F_{2n}$ that given in Lemma 2 (i). That is to say, we get

$$F_4 F_8 \dots F_{4m} F_{\frac{n+t}{2}} = (F_2 F_4 \dots F_{2m}) F_{\frac{n-t}{2}} F_{n+t}. \tag{6}$$

Assume that $m \geq 5$. Since $4m \geq 20$, we can use the Primitive Divisor Theorem (PDT). If $4m > n + t$, then there exists a prime $p$ dividing $F_{4m}$ does not divide $F_2,\ F_4,\ \dots F_{2m},\ F_{\frac{n-t}{2}},\ F_{n+t}$. So, the equation (6) does not hold. Similarly, if

$4m < n+t$, then there exist a prime $p$ such that $p \mid F_{n+t}$, but $p \nmid F_i$ where $i = 4, 8, \ldots 4m, \frac{n+t}{2}$. Since the inequalities $4m > n+t$ and $4m < n+t$ are not true, then we get $4m = n+t$. After simplifying the equation (6), then the equation

$$F_4 F_8 \ldots F_{4m-4} = F_2 F_4 \ldots F_{2m-2} F_{2m-t}$$

follows where we use $(n-t)/2 = 2m-t$ (because $4m = n+t$).

If $m$ is even integer, then we have

$$F_{2m} F_{2m+4} \ldots F_{4m-4} = F_{2m-t} F_2 F_6 \ldots F_{2m-6} F_{2m-2}. \tag{7}$$

If $m$ is odd integer, then

$$F_{2m+2} F_{2m+6} \ldots F_{4m-4} = F_{2m-t} F_2 F_6 \ldots F_{2m-8} F_{2m-4} \tag{8}$$

follows. Since $4m-4 \geq 16$, then we apply PDT again. If $2m-t \geq 4m-4$, then we have $4-t \geq 2m$ which is not possible as $m \geq 5$ and $t$ is positive integer. If $4m-4 > 2m-t$, then there exists a prime $p$ dividing $F_{4m-4}$. But $p \nmid F_j$ where $j = 2, 6, \ldots, 2m-6, 2m-2, 2m-t$ for the equation (7). Since we arrive at a similar contradiction for the equation (8), we omit it. We get the similar calculations for the equation (5).

Therefore, $m \leq 4$. So, we have

$$\sum_{k=0}^{1} \begin{bmatrix} 3 \\ k \end{bmatrix}_F = 3 = F_n \mp F_t, \quad \sum_{k=0}^{4} \begin{bmatrix} 9 \\ k \end{bmatrix}_F = 17766 = F_n \mp F_t$$

$$\sum_{k=0}^{2} \begin{bmatrix} 5 \\ k \end{bmatrix}_F = 21 = F_n \mp F_t, \quad \sum_{k=0}^{5} \begin{bmatrix} 11 \\ k \end{bmatrix}_F = 2185218 = F_n \mp F_t.$$

Namely, we investigate the solutions of the equations

$$F_n \mp F_t = 3 \tag{9}$$

$$F_n \mp F_t = 21 \tag{10}$$

$$F_n \mp F_t = 17766 \tag{11}$$

and

$$F_n \mp F_t = 2185218 \tag{12}$$

Assume that the sign is $-$. We focus on the equations (9) and (10). Since $F_4 = 3$ and $F_8 = 21$, by Lemma 4 there is one solution of the equation $F_n - F_t = F_4 = 3$ which is $(n,t) = (5,3)$. Similarly, the only solution of the equation $F_n - F_t = F_8 = 21$ is $(n,t) = (9,7)$ . Consider the equation (11). By Lemma 2 (iii), we have

$$F_n - F_t = F_{\frac{n \mp t}{2}} L_{\frac{n \pm t}{2}} = 17766 = 2 \cdot 3^3 \cdot 7 \cdot 47.$$

Since 17766 is not the product of two Fibonacci and Lucas number, the equation (11) has no solution. Similarly, there is no solution of the equation (12).

If the sign is $+$ for the equations (9), (10), (11) and (12), then we obtain the solutions given in the Table 1 below.

TABLE 1. Solutions of the equations $F_n + F_t = F_j$

| | $(n,t)$ |
|---|---|
| $F_n + F_t = 3$ | $(3,2),(2,3),(3,1),(1,3)$ |
| $F_n + F_t = 21$ | $(6,7),(7,6)$ |
| $F_n + F_t = 17766$ | $(22,10),(10,22)$ |
| $F_n + F_t = 2185218$ | no solution |

3.2. **The case** $n \not\equiv t \pmod 2$**.** In this case, we will solve the equation (6) under the conditions $n - t = 1$ or $n - t = 3$.

Firstly, we will deal with the case $n - t = 1$. This yields $F_n \mp F_t$ is a Fibonacci number. Then the equation (3) turns to

$$\sum_{k=0}^{m} \begin{bmatrix} 2m+1 \\ k \end{bmatrix}_F = F_x \tag{13}$$

where $F_n \mp F_t = F_x$ and $x \in \mathbb{Z}^+$. After multiplying both sides with $F_2F_4 \dots F_{2m}$, we have

$$F_4F_8 \dots F_{4m} = F_2F_4 \dots F_{2m}F_x.$$

Assume that $m \geq 4$. Since $4m \geq 16$, we can use PDT which yields $4m = x$. Then,

$$F_4F_8 \dots F_{4m-4} = F_2F_4 \dots F_{2m}$$

follows which is a contradiction. Because left hand side of the equation is obviously bigger than right hand side. Now, assume $m \leq 3$.

If $m = 1$, then we have the equation $F_x = F_n \mp F_t = 3 = F_4$. We obtain $x = 4$. So, the pairs $(n,t) = (3,2),(2,3),(3,1),(1,3)$ are the solutions of the equation $F_n + F_t = 3$. The pair $(n,t) = (6,5)$ is the only solution of the equation $F_n - F_t = 3$ by Lemma 4.

If $m = 2$, then we get $F_x = F_n \mp F_t = 21 = F_8$. We get $x = 8$. The solutions of $F_n + F_t = 21$ are $(n,t) = (6,7),(7,6)$. The pair $(n,t) = (10,9)$ is the only solution of the equation $F_n - F_t = 21$.

If $m = 3$, then we obtain $L_2L_4L_6 = 378$ which is not a Fibonacci number. That is, the equation (3.2) is not satisfied.

Now, assume that $n - t = 3$. Firstly, we handle the equation

$$\sum_{k=0}^{m} \begin{bmatrix} 2m+1 \\ k \end{bmatrix}_F + F_t = F_n.$$

By Lemma 2 (ii), we have

$$\sum_{k=0}^{m} \begin{bmatrix} 2m+1 \\ k \end{bmatrix}_F = L_2L_4 \dots L_{2m} = F_n - F_t = 2F_{n-2}.$$

After multiplying both sides with $F_2F_4\dots F_{2m}$, we get

$$F_4F_8\dots F_{4m} = F_2F_4\dots F_{2m}2F_{n-2}. \tag{14}$$

Assume that $m \geq 4$. By PDT, there exists a primitive divisor $p$ such that $p \mid F_{4m}$. If $p = 2$, then $p$ can not be a primitive divisor since, at least, $2 \mid F_6$. This yields that $p \neq 2$. If $4m > n-2$, then $p \nmid F_2F_4\dots F_{2m}2F_{n-2}$ which is not possible. We get the similar contradiction if $n-2 > 4m$. So, we deduce $4m = n-2$. Then the equation (14) reduces to

$$L_2L_4\dots L_{2m-2} = 2F_{2m}. \tag{15}$$

where we use the fact $F_{2n} = F_nL_n$. This is not possible for $m \geq 4$. Because left hand side of the equation (15) is greater than right hand side. Now, we assume $m \leq 3$. So, we have to solve the following equation

$$2F_{n-2} = q$$

where $q \in \{L_2 = 3, L_2L_4 = 21, L_2L_4L_6 = 378\}$. Obviously, there is no solution. We arrive at similar contradictions for the equation

$$\sum_{k=0}^{m} \begin{bmatrix} 2m+1 \\ k \end{bmatrix}_F = L_2L_4\dots L_{2m} = F_n + F_t = 2F_{n-1}.$$

We do not give its details.

Finally, we complete the proof.

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