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Berezin Radius Inequalities of Functional Hilbert Space Operators

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Abstract

We investigate new upper bounds for the Berezin radius and Berezin norm of 2×2 operator matrices using the Cauchy-Buzano inequality, and we propose a required condition for the equality case in the triangle inequalities for the Berezin norms. We also show various Berezin radius inequalities for matrices with 2×2 operators.

1. Introduction

Let $\mathbb{L}(\mathscr{H})$ denote the C^*-algebra of all bounded linear operators defined on a complex Hilbert space $(\mathscr{H},\langle .,.\rangle)$. Throughout the paper, we work on functional Hilbert space (FHS), which are complete inner product spaces made up of complex-valued functions defined on a non-empty set Υ with bounded point evaluation. If $\langle Px,x\rangle>0$ for all $x\in\mathscr{H}$, then an operator $P\in\mathbb{L}(\mathscr{H})$ is called positive. Recall that a functional Hilbert space $\mathscr{H}=\mathscr{H}(\Upsilon)$ is a complex Hilbert space on a (nonempty) Υ, which has the property that point evaluations are continuous for each $\omega\in\Upsilon$ there is a unique element $k_\omega\in\mathscr{H}$ such that $f(\omega)=\langle f,k_\omega\rangle$, for all $f\in\mathscr{H}$. The family $\{k_\omega:\omega\in\Upsilon\}$ is called the reproducing kernel $\mathscr{H}$. If $\{e_n\}_{n\geq 0}$ is an orthonormal basis for FHS, the reproducing kernel is shown by $k_\omega=\sum_{n=0}^{\infty}\overline{e_n(\omega)}e_n(z)$; (see, [1]). For $\omega\in\Upsilon$, $\widehat{k}_\omega=\frac{k_\omega}{\|k_\omega\|}$ is called the normalized reproducing kernel. For $P\in\mathbb{L}(\mathscr{H})$, the function $\widetilde{P}$ defined on Υ by $\widetilde{P}(\omega)=\left\langle P\widehat{k}_\omega,\widehat{k}_\omega\right\rangle$ is the Berezin symbol (or Berezin transform) of P. The Berezin symbol firstly has been introduced by Berezin in [2]. The Berezin set and Berezin radius (or number) of the operator P are defined by

$$\operatorname{Ber}(P)=\left\{\widetilde{P}(\omega):\omega\in\Upsilon\right\}\quad\text{and}\quad\operatorname{ber}(P)=\sup\left\{\left|\widetilde{P}(\omega)\right|:\omega\in\Upsilon\right\},$$

respectively. In some recent works, several Berezin radius inequalities have been examined by authors in [3–10].
We also define the so-called Berezin norm of operators $P\in\mathscr{B}(\mathscr{H})$ as follows:

$$\|P\|_{\text{Ber}}:=\sup_{\varepsilon\in\Upsilon}\left\|P\widehat{k}_\omega\right\|.$$

It is obvious that $\|P\|_{\text{Ber}}$ determines a new operator norm in $\mathbb{L}(\mathscr{H}(\Upsilon))$. It is also obvious that $\operatorname{ber}(P)\leq\|P\|_{\text{Ber}}\leq\|P\|$.
A significant inequality for $\operatorname{ber}(P)$ is the power inequality stating that

$$\operatorname{ber}(P^n)\leq\operatorname{ber}^n(P)$$

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for $n = 1, 2, \ldots$; more generally, if P is not nilpotent, then

$$C_1 \mathrm{ber}^n(P) \leq \mathrm{ber}(P^n) \leq C_2 \mathrm{ber}^n(P),$$

for some constant $C_1, C_2 > 0$. In a FHS, the Berezin range of an operator P is a subset of numerical range of P, i.e.,

$$\mathrm{Ber}(P) \subseteq W(P).$$

Hence $\mathrm{ber}(P) \leq w(P)$.
The numerical range and numerical radius of $P \in \mathbb{L}(\mathscr{H})$ are denoted by

$$W(P) = \{\langle Px, x\rangle : x \in \mathscr{H} \text{ and } \|x\| = 1\} \text{ and } w(P) = \sup\{|\langle Px, x\rangle| : x \in \mathscr{H} \text{ and } \|x\| = 1\},$$

respectively. The absolute value of positive operator is denoted by $|P| = (P^*P)^{\frac{1}{2}}$. The numerical range has several intriguing features. For example, it is usually assumed that an operator's spectrum is confined in the closure of its numerical range. For an illustration of how this and other numerical radius inequalities were addressed in those sources, we urge the reader read [11–14].
It is well known that

$$\frac{\|P\|}{2} \leq w(P) \leq \|P\|$$

and

$$\mathrm{ber}(P) \leq w(P) \leq \|P\|$$

for any $P \in \mathbb{L}(\mathscr{H})$. Also shown by Karaev in [15] are the Berezin range and Berezin radius of operators, which are numerical features of operators on the FHS. In 2022, Huban et al. [16, 17] have proved the following results:

$$\mathrm{ber}(P) \leq \frac{1}{2}\||P| + |P^*|\|_{\mathrm{ber}} \leq \frac{1}{2}\left(\|P\|_{\mathrm{ber}} + \|P\|_{\mathrm{ber}}^{\frac{1}{2}}\right) \tag{1.1}$$

and

$$\mathrm{ber}^{2r}(P) \leq \frac{1}{2}\left\||P|^{2r} + |P^*|^{2r}\right\|_{\mathrm{ber}} \quad \text{where} \quad r \geq 1.$$

Başaran et al. [18] have showed the following inequality:

$$\mathrm{ber}^2(P) \leq \frac{1}{2}\left\||P|^2 + |P^*|^2\right\|_{\mathrm{ber}}. \tag{1.2}$$

The direct sum of two copies of $\mathscr{H}$ is denoted by $\mathscr{H}^2 = \mathscr{H} \oplus \mathscr{H}$. If $P, R, S, T \in \mathbb{L}(\mathscr{H})$, then the operator matrix $A = \begin{bmatrix} P & R \\ S & T \end{bmatrix}$ can be considered as an operator in $\mathbb{L}(\mathscr{H} \oplus \mathscr{H})$, which is defined by $Ax = \begin{pmatrix} Px_1 + Rx_2 \\ Sx_1 + Tx_2 \end{pmatrix}$ for every vector $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathscr{H} \oplus \mathscr{H}$. In 2018, Bakherad, has proved Berezin radius inequalities of block matrix of the form $\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}$ (see, [19–21]). As one can see in [22, 23], where operator norm and numerical radius inequalities were researched and implemented, operator matrices and their properties and inequalities have attracted a lot of attention in the literature.
In order to complete the inequalities for the Berezin number, the numerical radius, and the operator norm, we offer Berezin number inequalities for operator matrices in this study. Our work consists of three sections. In the first section, It has been given introduction. In the second section, it has been given known lemmas. In the third section, it has obtained new upper bounds for the Berezin radius and Berezin norm of 2×2 operator matrices by Cauchy-Buzano inequality. Also, it has been given a necessary condition for the equality case in the triangle inequalities for the Berezin norms. Finally, it has been proven some Berezin radius inequalities for 2×2 operator matrices.

2. Main Results

2.1. Auxiliary theorems

To start our work, we need the following lemmas. The first lemma is found by Bakherad in [19].

Lemma 2.1. *Let $P \in \mathbb{L}(\mathscr{H}_1(\Upsilon))$, $R \in \mathbb{L}(\mathscr{H}_2(\Upsilon), \mathscr{H}_1(\Upsilon))$, $S \in \mathbb{L}(\mathscr{H}_1(\Upsilon), \mathscr{H}_2(\Upsilon))$, $T \in \mathbb{L}(\mathscr{H}_2(\Upsilon))$. Then the following statements holds:*

$$\mathrm{ber}\left(\begin{bmatrix} P & 0 \\ 0 & T \end{bmatrix}\right) = \max\{\mathrm{ber}(P), \mathrm{ber}(T)\} \tag{2.1}$$

and

$$\mathrm{ber}\left(\begin{bmatrix} 0 & R \\ S & 0 \end{bmatrix}\right) = \frac{1}{2}(\|R\| + \|S\|). \tag{2.2}$$

The second lemma arises from the spectral theorem for positive operators and Jensen inequality.

Lemma 2.2 ([24]). *Let $P \in \mathbb{L}(\mathscr{H})$ be a positive operator, and $u \in \mathscr{H}$ is an unit vector. Then*

$$\langle Pu,u\rangle^r \leq \langle P^r u,u\rangle \quad \text{for all} \quad r \geq 1. \tag{2.3}$$

Aujla and Silva provide the third lemma, a norm inequality involving convex functions of positive operators (see [25]).

Lemma 2.3. *let f be a non-negative convex function on $[0,\infty)$, and let $P,R \in \mathbb{L}(\mathscr{H})$ be a positive operators. Then*

$$\left\| f\left(\frac{P+R}{2}\right)\right\| \leq \left\|\frac{f(P)+f(R)}{2}\right\|. \tag{2.4}$$

In particular, if $r \geq 1$, then

$$\left\| \left(\frac{P+R}{2}\right)^r\right\| \leq \left\|\frac{P^r+R^r}{2}\right\|. \tag{2.5}$$

The fourth lemma is shown by Kittaneh (see [24]).

Lemma 2.4. *Let $P \in \mathbb{L}(\mathscr{H})$ and let $u,v \in \mathscr{H}$ be any vectors. Then*

$$|\langle Pu,v\rangle|^2 \leq \langle |P|u,v\rangle \langle |P^*|v,v\rangle. \tag{2.6}$$

The fifth lemma may be found in [26, Lemma 2.2], and it is based on the corollary of Cauchy-Buzano inequality.

Lemma 2.5. *Let $u,v,e \in \mathscr{H}$ and $\|e\| = 1$. Then*

$$|\langle u,e\rangle\langle e,v\rangle|^2 \leq \frac{1}{4}\left(3\|u\|^2\|v\|^2 + \|u\|\|v\| + |\langle u,v\rangle|\right). \tag{2.7}$$

For any positive operators A,B, the classic Arithmetic Mean-Geometric Mean inequality, often known as AM-GM inequality, states that $\sqrt{AB} \leq \frac{A+B}{2}$.

2.2. Some Berezin radius inequalities

Our findings are presented in this section. Now, we can the prove the first theorem.

Theorem 2.6. *Let $P,R \in \mathbb{L}(\mathscr{H})$. Then*

$$\operatorname{ber}\left(\begin{bmatrix} P & R \\ R & P \end{bmatrix}\right) = \max\{\operatorname{ber}(P+R), \operatorname{ber}(P-R)\} \tag{2.8}$$

In particular,

$$\operatorname{ber}\left(\begin{bmatrix} 0 & R \\ R & 0 \end{bmatrix}\right) = \operatorname{ber}(R).$$

Proof. Assume $U = \frac{1}{\sqrt{2}}\begin{bmatrix} I & I \\ -I & I \end{bmatrix}$ and $T = \begin{bmatrix} P & R \\ R & P \end{bmatrix}$. It can easy that

$$U^*TU = \begin{bmatrix} P-R & 0 \\ 0 & P+R \end{bmatrix}.$$

Using the inequality (2.1) and $\operatorname{ber}(U^*TU) = \operatorname{ber}(T)$, we have

$$\operatorname{ber}(T) = \max\{\operatorname{ber}(P+R), \operatorname{ber}(P-R)\}.$$

For $P = 0$, we get

$$\operatorname{ber}\left(\begin{bmatrix} 0 & R \\ R & 0 \end{bmatrix}\right) = \operatorname{ber}(R).$$

□

Theorem 2.7. *Let $\mathscr{H} = \mathscr{H}(\Upsilon)$ be a FHS and $P,R,S,T \in \mathbb{L}(\mathscr{H})$. Then*

$$\begin{aligned} \operatorname{ber}^2\left(\begin{bmatrix} P & R \\ S & T \end{bmatrix}\right) &= \max\left\{\operatorname{ber}^2(P), \operatorname{ber}^2(T)\right\} + \operatorname{ber}^2\left(\begin{bmatrix} 0 & R \\ S & 0 \end{bmatrix}\right) \\ &\quad + \max\{\|P\|_{\text{Ber}}, \|T\|_{\text{Ber}}\}\max\{\|R\|_{\text{Ber}}, \|S\|_{\text{Ber}}\} + \operatorname{ber}\left(\begin{bmatrix} 0 & S^*T \\ R^*P & 0 \end{bmatrix}\right). \end{aligned} \tag{2.9}$$

Proof. For every $(\omega_1,\omega_2),(\varkappa_1,\varkappa_2)\in\Upsilon_1\times\Upsilon_2$, let $\widehat{k}_w=\widehat{k}_{(\omega_1,\omega_2)}=\begin{bmatrix}k_{\omega_1}\\ k_{\omega_2}\end{bmatrix}$, $\widehat{k}_\varkappa=\widehat{k}_{(\varkappa_1,\varkappa_2)}=\begin{bmatrix}k_{\varkappa_1}\\ k_{\varkappa_2}\end{bmatrix}$ be a normalized reproducing kernel in $\mathscr{H}(\Upsilon_1)\oplus\mathscr{H}(\Upsilon_2)$. Then we have

$$\begin{aligned}
\left|\left\langle\begin{bmatrix}P & R\\ S & T\end{bmatrix}\widehat{k}_w,\widehat{k}_\varkappa\right\rangle\right|^2 &= \left|\left\langle\left(\begin{bmatrix}P & 0\\ 0 & T\end{bmatrix}+\begin{bmatrix}0 & R\\ S & 0\end{bmatrix}\right)\widehat{k}_w,\widehat{k}_\varkappa\right\rangle\right|^2\\
&\le \left|\left\langle\begin{bmatrix}P & 0\\ 0 & T\end{bmatrix}\widehat{k}_w,\widehat{k}_\varkappa\right\rangle+\left\langle\begin{bmatrix}0 & R\\ S & 0\end{bmatrix}\widehat{k}_w,\widehat{k}_\varkappa\right\rangle\right|^2\\
&\le \left(\left|\left\langle\begin{bmatrix}P & 0\\ 0 & T\end{bmatrix}\widehat{k}_w,\widehat{k}_\varkappa\right\rangle\right|+\left|\left\langle\begin{bmatrix}0 & R\\ S & 0\end{bmatrix}\widehat{k}_w,\widehat{k}_\varkappa\right\rangle\right|\right)^2\\
&= \left|\left\langle\begin{bmatrix}P & 0\\ 0 & T\end{bmatrix}\widehat{k}_w,\widehat{k}_\varkappa\right\rangle\right|^2+\left|\left\langle\begin{bmatrix}0 & R\\ S & 0\end{bmatrix}\widehat{k}_w,\widehat{k}_\varkappa\right\rangle\right|^2+2\left|\left\langle\begin{bmatrix}P & 0\\ 0 & T\end{bmatrix}\widehat{k}_w,\widehat{k}_\varkappa\right\rangle\right|\left|\left\langle\begin{bmatrix}0 & R\\ S & 0\end{bmatrix}\widehat{k}_w,\widehat{k}_\varkappa\right\rangle\right|\\
&= \left|\left\langle\begin{bmatrix}P & 0\\ 0 & T\end{bmatrix}\widehat{k}_w,\widehat{k}_w\right\rangle\right|^2+\left|\left\langle\begin{bmatrix}0 & R\\ S & 0\end{bmatrix}\widehat{k}_w,\widehat{k}_w\right\rangle\right|^2+2\left|\left\langle\begin{bmatrix}P & 0\\ 0 & T\end{bmatrix}\widehat{k}_w,\widehat{k}_\varkappa\right\rangle\right|\left|\left\langle\widehat{k}_\varkappa,\begin{bmatrix}0 & R\\ S & 0\end{bmatrix}\widehat{k}_w\right\rangle\right|\\
&\le \left|\left\langle\begin{bmatrix}P & 0\\ 0 & T\end{bmatrix}\widehat{k}_w,\widehat{k}_\varkappa\right\rangle\right|^2+\left|\left\langle\begin{bmatrix}0 & R\\ S & 0\end{bmatrix}\widehat{k}_w,\widehat{k}_\varkappa\right\rangle\right|^2+\left\|\begin{bmatrix}P & 0\\ 0 & T\end{bmatrix}\widehat{k}_w\right\|\left\|\begin{bmatrix}0 & R\\ S & 0\end{bmatrix}\widehat{k}_w\right\|\\
&\quad+\left|\left\langle\begin{bmatrix}0 & S^*T\\ R^*P & 0\end{bmatrix}\widehat{k}_w,\widehat{k}_w\right\rangle\right|.
\end{aligned}$$

where the sixth inequality follows from Buzano's inequality (see [27]), i.e., if $u,v,e\in\mathscr{H}$ and $\|e\|=1$, then

$$|\langle u,e\rangle\langle e,v\rangle|\le\frac{1}{2}\left(\|u\|\,\|v\|+|\langle u,v\rangle|\right). \tag{2.10}$$

So, we get

$$\begin{aligned}
\left|\left\langle\begin{bmatrix}P & R\\ S & T\end{bmatrix}\widehat{k}_w,\widehat{k}_\varkappa\right\rangle\right|^2 &\le \left|\left\langle\begin{bmatrix}P & 0\\ 0 & T\end{bmatrix}\widehat{k}_w,\widehat{k}_\varkappa\right\rangle\right|^2+\left|\left\langle\begin{bmatrix}0 & R\\ S & 0\end{bmatrix}\widehat{k}_w,\widehat{k}_\varkappa\right\rangle\right|^2+\left\|\begin{bmatrix}P & 0\\ 0 & T\end{bmatrix}\widehat{k}_w\right\|\left\|\begin{bmatrix}0 & R\\ S & 0\end{bmatrix}\widehat{k}_w\right\|\\
&\quad+\left|\left\langle\begin{bmatrix}0 & S^*T\\ R^*P & 0\end{bmatrix}\widehat{k}_w,\widehat{k}_w\right\rangle\right|.
\end{aligned}$$

Now, taking the supremum over $(\omega_1,\omega_2),(\varkappa_1,\varkappa_2)\in\Upsilon_1\times\Upsilon_2$ with $(\omega_1,\omega_2)=(\varkappa_1,\varkappa_2)$ and then applying inequality (2.1) in the above inequality, we have

$$\begin{aligned}
\operatorname{ber}^2\left(\begin{bmatrix}P & R\\ S & T\end{bmatrix}\right) &= \max\left\{\operatorname{ber}^2(P),\operatorname{ber}^2(T)\right\}+\operatorname{ber}^2\left(\begin{bmatrix}0 & R\\ S & 0\end{bmatrix}\right)\\
&\quad+\max\left\{\|P\|_{\text{Ber}},\|T\|_{\text{Ber}}\right\}\max\left\{\|R\|_{\text{Ber}},\|S\|_{\text{Ber}}\right\}+\operatorname{ber}\left(\begin{bmatrix}0 & S^*T\\ R^*P & 0\end{bmatrix}\right).
\end{aligned}$$

The proof is now complete. □

Several inequalities for Berezin radius inequalities of operator matrices are included in the preceding theorem. The following corollaries show these inequities.

Corollary 2.8. *If $(\omega_1,\omega_2)\neq(\varkappa_1,\varkappa_2)$, then we have*

$$\begin{aligned}
\left\|\begin{bmatrix}P & R\\ S & T\end{bmatrix}\right\|^2 &= \max\left\{\|P\|^2,\|T\|^2\right\}+\max\left\{\|R\|^2,\|S\|^2\right\}\\
&\quad+\max\left\{\|P\|_{\text{Ber}},\|T\|_{\text{Ber}}\right\}\max\left\{\|R\|_{\text{Ber}},\|S\|_{\text{Ber}}\right\}+\operatorname{ber}\left(\begin{bmatrix}0 & S^*T\\ R^*P & 0\end{bmatrix}\right).
\end{aligned}$$

Corollary 2.9. *If we apply the inequality (2.2) to inequality (2.9), we have*

$$\begin{aligned}
\operatorname{ber}^2\left(\begin{bmatrix}P & R\\ S & T\end{bmatrix}\right) &= \max\left\{\operatorname{ber}^2(P),\operatorname{ber}^2(T)\right\}+\frac{1}{4}\left(\|R\|+\|S\|\right)^2\\
&\quad+\max\left\{\|P\|_{\text{Ber}},\|T\|_{\text{Ber}}\right\}\max\left\{\|R\|_{\text{Ber}},\|S_{\text{Ber}}\|\right\}+\frac{1}{4}\left(\|R^*P\|+\|S^*T\|\right).
\end{aligned}$$

The following corollary follows easily from the inequality (2.8) in Theorem 2.6.

Corollary 2.10. *If we take $P=T$ and $R=S$ in the Theorem 2.7 and Corollary 2.8, we get*

(i) $\mathrm{ber}^2\left(\begin{bmatrix} P & R \\ R & P \end{bmatrix}\right) = \max\left\{\mathrm{ber}^2(P+R), \mathrm{ber}^2(P-R)\right\} \leq \mathrm{ber}^2(P) + \mathrm{ber}^2(R) + \|P\|\|R\| + \mathrm{ber}(R^*P)$,

(ii) $\left\|\begin{bmatrix} P & R \\ R & P \end{bmatrix}\right\|^2 = \max\left\{\|P+R\|^2, \|P-R\|^2\right\} \leq \|P\|^2 + \|R\|^2 + \|P\|\|R\| + \mathrm{ber}(R^*P)$.

After that, it is simple that to prove Corollary 2.10 may be used to provide a required condition for the equality case in the triangle inequality for Berezin radius.

Proposition 2.11. *Let $\mathscr{H} = \mathscr{H}(\Upsilon)$ be a FHS and $P, R \in \mathbb{L}(\mathscr{H})$.*

(i) If $\mathrm{ber}(P+R) = \mathrm{ber}(P) + \mathrm{ber}(R)$, then we have $2\mathrm{ber}(P)\,\mathrm{ber}(R) - \mathrm{ber}(R^*P) = \|P\|_{\mathrm{Ber}}\|R\|_{\mathrm{Ber}}$.
(ii) If $\|P+R\| = \|P\| + \|R\|$, then we have $\mathrm{ber}(R^*P) = \|P\|_{\mathrm{Ber}}\|R\|_{\mathrm{Ber}}$.

Proof. By using $\mathrm{ber}(P+R) = \mathrm{ber}(P) + \mathrm{ber}(R)$ and $\|P+R\| = \|P\| + \|R\|$ in Corollary 2.10, respectively, we get the required inequalities. □

Theorem 2.12. *Let $P, R, S, T \in \mathbb{L}(\mathscr{H})$. Then*

$$\begin{aligned} \mathrm{ber}^2\left(\begin{bmatrix} P & R \\ S & T \end{bmatrix}\right) &= \max\left\{\mathrm{ber}^2(P), \mathrm{ber}^2(T)\right\} + \mathrm{ber}^2\left(\begin{bmatrix} 0 & R \\ S & 0 \end{bmatrix}\right) \\ &\quad + \frac{1}{2}\max\left\{\left\||P|^2 + |R^*|^2\right\|_{\mathrm{ber}}, \left\||T|^2 + |S^*|^2\right\|_{\mathrm{ber}}\right\} + \mathrm{ber}\left(\begin{bmatrix} 0 & RT \\ SP & 0 \end{bmatrix}\right). \end{aligned} \tag{2.11}$$

Proof. For every $(\omega_1, \omega_2) \in \Upsilon_1 \times \Upsilon_2$, let $\widehat{k}_w = \widehat{k}_{(\omega_1,\omega_2)} = \begin{bmatrix} k_{\omega_1} \\ k_{\omega_2} \end{bmatrix}$ be a normalized reproducing kernel in $\mathscr{H}(\Upsilon_1) \oplus \mathscr{H}(\Upsilon_2)$. Using the AM-GM inequality and (2.10), we establish the following inequality using the same approach as in Theorem 2.7:

$$\begin{aligned} \left|\left\langle \begin{bmatrix} P & R \\ S & T \end{bmatrix} \widehat{k}_w, \widehat{k}_w \right\rangle\right|^2 &= \left|\left\langle \left(\begin{bmatrix} P & 0 \\ 0 & T \end{bmatrix} + \begin{bmatrix} 0 & R \\ S & 0 \end{bmatrix}\right) \widehat{k}_w, \widehat{k}_w \right\rangle\right|^2 \\ &\leq \left|\left\langle \begin{bmatrix} P & 0 \\ 0 & T \end{bmatrix} \widehat{k}_w, \widehat{k}_w \right\rangle + \left\langle \begin{bmatrix} 0 & R \\ S & 0 \end{bmatrix} \widehat{k}_w, \widehat{k}_w \right\rangle\right|^2 \\ &\leq \left(\left|\left\langle \begin{bmatrix} P & 0 \\ 0 & T \end{bmatrix} \widehat{k}_w, \widehat{k}_w \right\rangle\right| + \left|\left\langle \begin{bmatrix} 0 & R \\ S & 0 \end{bmatrix} \widehat{k}_w, \widehat{k}_w \right\rangle\right|\right)^2 \\ &= \left|\left\langle \begin{bmatrix} P & 0 \\ 0 & T \end{bmatrix} \widehat{k}_w, \widehat{k}_w \right\rangle\right|^2 + \left|\left\langle \begin{bmatrix} 0 & R \\ S & 0 \end{bmatrix} \widehat{k}_w, \widehat{k}_w \right\rangle\right|^2 + 2\left|\left\langle \begin{bmatrix} P & 0 \\ 0 & T \end{bmatrix} \widehat{k}_w, \widehat{k}_w \right\rangle\right|\left|\left\langle \begin{bmatrix} 0 & R \\ S & 0 \end{bmatrix} \widehat{k}_w, \widehat{k}_w \right\rangle\right| \\ &= \left|\left\langle \begin{bmatrix} P & 0 \\ 0 & T \end{bmatrix} \widehat{k}_w, \widehat{k}_w \right\rangle\right|^2 + \left|\left\langle \begin{bmatrix} 0 & R \\ S & 0 \end{bmatrix} \widehat{k}_w, \widehat{k}_w \right\rangle\right|^2 + 2\left|\left\langle \begin{bmatrix} P & 0 \\ 0 & T \end{bmatrix} \widehat{k}_w, \widehat{k}_w \right\rangle\right|\left|\left\langle \widehat{k}_w, \begin{bmatrix} 0 & R \\ S & 0 \end{bmatrix} \widehat{k}_w \right\rangle\right| \\ &\leq \left|\left\langle \begin{bmatrix} P & 0 \\ 0 & T \end{bmatrix} \widehat{k}_w, \widehat{k}_w \right\rangle\right|^2 + \left|\left\langle \begin{bmatrix} 0 & R \\ S & 0 \end{bmatrix} \widehat{k}_w, \widehat{k}_w \right\rangle\right|^2 + \left\|\begin{bmatrix} P & 0 \\ 0 & T \end{bmatrix} \widehat{k}_w\right\|\left\|\begin{bmatrix} 0 & R \\ S & 0 \end{bmatrix} \widehat{k}_w\right\| \\ &\quad + \left|\left\langle \begin{bmatrix} P & 0 \\ 0 & T \end{bmatrix} \widehat{k}_w, \begin{bmatrix} 0 & S^* \\ R^* & 0 \end{bmatrix} \widehat{k}_w \right\rangle\right| \\ &\leq \left|\left\langle \begin{bmatrix} P & 0 \\ 0 & T \end{bmatrix} \widehat{k}_w, \widehat{k}_w \right\rangle\right|^2 + \left|\left\langle \begin{bmatrix} 0 & R \\ S & 0 \end{bmatrix} \widehat{k}_w, \widehat{k}_w \right\rangle\right|^2 + \sqrt{\left\langle \left|\begin{bmatrix} P & 0 \\ 0 & T \end{bmatrix}\right|^2 \widehat{k}_w, \widehat{k}_w \right\rangle \left\langle \left|\begin{bmatrix} 0 & S^* \\ R^* & 0 \end{bmatrix}\right|^2 \widehat{k}_w, \widehat{k}_w \right\rangle} \\ &\quad + \left|\left\langle \begin{bmatrix} 0 & RT \\ SP & 0 \end{bmatrix} \widehat{k}_w, \widehat{k}_w \right\rangle\right| \\ &\leq \left|\left\langle \begin{bmatrix} P & 0 \\ 0 & T \end{bmatrix} \widehat{k}_w, \widehat{k}_w \right\rangle\right|^2 + \left|\left\langle \begin{bmatrix} 0 & R \\ S & 0 \end{bmatrix} \widehat{k}_w, \widehat{k}_w \right\rangle\right|^2 + \frac{1}{2}\left\langle \begin{bmatrix} |P|^2 + |R^*|^2 & 0 \\ 0 & |T|^2 + |S^*|^2 \end{bmatrix} \widehat{k}_w, \widehat{k}_w \right\rangle \\ &\quad + \left|\left\langle \begin{bmatrix} 0 & RT \\ SP & 0 \end{bmatrix} \widehat{k}_w, \widehat{k}_w \right\rangle\right|. \end{aligned}$$

By taking the supremum over $(\omega_1, \omega_2) \in \Upsilon_1 \times \Upsilon_2$ and applying the inequality (2.1) in the above inequality, we have

$$\begin{aligned} \mathrm{ber}^2\left(\begin{bmatrix} P & R \\ S & T \end{bmatrix}\right) &= \max\left\{\mathrm{ber}^2(P), \mathrm{ber}^2(T)\right\} + \mathrm{ber}^2\left(\begin{bmatrix} 0 & R \\ S & 0 \end{bmatrix}\right) \\ &\quad + \frac{1}{2}\max\left\{\left\||P|^2 + |R^*|^2\right\|_{\mathrm{ber}}, \left\||T|^2 + |S^*|^2\right\|_{\mathrm{ber}}\right\} + \mathrm{ber}\left(\begin{bmatrix} 0 & RT \\ SP & 0 \end{bmatrix}\right). \end{aligned}$$

We have desired result. □

Several Berezin radius inequalities for operator matrices are contained in Theorem 2.12. The following corollaries demonstrate some of these inequalities.

Corollary 2.13. *If we apply the inequality (2.2) to inequality (2.11), then we have*

$$\begin{aligned} \mathrm{ber}^2\left(\begin{bmatrix} P & R \\ S & T \end{bmatrix}\right) &= \max\left\{\mathrm{ber}^2(P), \mathrm{ber}^2(T)\right\} + \frac{1}{4}(\|R\|+\|S\|)^2 \\ &\quad + \frac{1}{2}\max\left\{\left\| |P|^2+|R^*|^2 \right\|_{\mathrm{ber}}, \left\| |T|^2+|S^*|^2 \right\|_{\mathrm{ber}}\right\} + \frac{1}{2}(\|RT\|+\|SP\|). \end{aligned}$$

The inequality (2.8) in Theorem 2.6 simply leads to the following consequence.

Corollary 2.14. *If we take $P=T$ and $R=S$ in Theorem 2.12, then we get*

$$\mathrm{ber}^2\left(\begin{bmatrix} P & R \\ R & P \end{bmatrix}\right) = \max\left\{\mathrm{ber}^2(P+R), \mathrm{ber}^2(P-R)\right\} \le \mathrm{ber}^2(P) + \mathrm{ber}^2(R) + \frac{1}{2}\left\| |P|^2+|R^*|^2 \right\|_{\mathrm{ber}} + \mathrm{ber}(RP).$$

Corollary 2.15. *If we take $P=R$ in Corollary 2.14, then we have*

$$\mathrm{ber}^2(P) \le \frac{1}{4}\left\| |P|^2+|P^*|^2 \right\|_{\mathrm{ber}} + \frac{1}{2}\mathrm{ber}\left(P^2\right),$$

(see, [28, Corollary 2.14]).

Theorem 2.16. *Let $P,R,S,T \in \mathbb{L}(\mathscr{H})$. Then*

$$\begin{aligned} \mathrm{ber}^2\left(\begin{bmatrix} P & R \\ S & T \end{bmatrix}\right) &= \max\left\{\mathrm{ber}^2(P), \mathrm{ber}^2(T)\right\} + \frac{1}{4}\max\left\{\left\| |S|^2+|R^*|^2 \right\|_{\mathrm{ber}}, \left\| |R|^2+|S^*|^2 \right\|_{\mathrm{ber}}\right\} \\ &\quad + \max\{\mathrm{ber}(P), \mathrm{ber}(T)\}\max\left\{\| |S|+|R^*| \|_{\mathrm{ber}}, \| |R|+|S^*| \|_{\mathrm{ber}}\right\} + \frac{1}{2}\max\{\mathrm{ber}(SR), \mathrm{ber}(RS)\}. \end{aligned}$$

Proof. For every $(\omega_1,\omega_2) \in \Upsilon_1 \times \Upsilon_2$, let $\widehat{k}_w = \widehat{k}_{(\omega_1,\omega_2)} = \begin{bmatrix} k_{\omega_1} \\ k_{\omega_2} \end{bmatrix}$ be a normalized reproducing kernel in $\mathscr{H}(\Upsilon_1) \oplus \mathscr{H}(\Upsilon_2)$. Following the same procedure as in Theorem 2.7 and by using the AM-GM inequality, (2.6) and (2.10), the following inequality is obtained:

$$\begin{aligned} \left|\left\langle \begin{bmatrix} P & R \\ S & T \end{bmatrix} \widehat{k}_w, \widehat{k}_w \right\rangle\right|^2 &= \left|\left\langle \left(\begin{bmatrix} P & 0 \\ 0 & T \end{bmatrix} + \begin{bmatrix} 0 & R \\ S & 0 \end{bmatrix}\right) \widehat{k}_w, \widehat{k}_w \right\rangle\right|^2 \\ &\le \left(\left|\left\langle \begin{bmatrix} P & 0 \\ 0 & T \end{bmatrix} \widehat{k}_w, \widehat{k}_w \right\rangle + \left\langle \begin{bmatrix} 0 & R \\ S & 0 \end{bmatrix} \widehat{k}_w, \widehat{k}_w \right\rangle\right|\right)^2 \\ &\le \left(\left|\left\langle \begin{bmatrix} P & 0 \\ 0 & T \end{bmatrix} \widehat{k}_w, \widehat{k}_w \right\rangle\right| + \left|\left\langle \begin{bmatrix} 0 & R \\ S & 0 \end{bmatrix} \widehat{k}_w, \widehat{k}_w \right\rangle\right|\right)^2 \\ &= \left|\left\langle \begin{bmatrix} P & 0 \\ 0 & T \end{bmatrix} \widehat{k}_w, \widehat{k}_w \right\rangle\right|^2 + 2\left|\left\langle \begin{bmatrix} P & 0 \\ 0 & T \end{bmatrix} \widehat{k}_w, \widehat{k}_w \right\rangle\right| \left|\left\langle \begin{bmatrix} 0 & R \\ S & 0 \end{bmatrix} \widehat{k}_w, \widehat{k}_w \right\rangle\right| + \left|\left\langle \begin{bmatrix} 0 & R \\ S & 0 \end{bmatrix} \widehat{k}_w, \widehat{k}_w \right\rangle\right|^2 \\ &\le \left|\left\langle \begin{bmatrix} P & 0 \\ 0 & T \end{bmatrix} \widehat{k}_w, \widehat{k}_w \right\rangle\right|^2 + 2\left|\left\langle \begin{bmatrix} P & 0 \\ 0 & T \end{bmatrix} \widehat{k}_w, \widehat{k}_w \right\rangle\right| \sqrt{\left\langle \left|\begin{bmatrix} 0 & R \\ S & 0 \end{bmatrix}\right| \widehat{k}_w, \widehat{k}_w \right\rangle \left\langle \left|\begin{bmatrix} 0 & S^* \\ R^* & 0 \end{bmatrix}\right| \widehat{k}_w, \widehat{k}_w \right\rangle} \\ &\quad + \frac{1}{2}\left(\sqrt{\left\langle \left|\begin{bmatrix} 0 & R \\ S & 0 \end{bmatrix}\right|^2 \widehat{k}_w, \widehat{k}_w \right\rangle \left\langle \left|\begin{bmatrix} 0 & S^* \\ R^* & 0 \end{bmatrix}\right|^2 \widehat{k}_w, \widehat{k}_w \right\rangle} + \left|\left\langle \begin{bmatrix} 0 & R \\ S & 0 \end{bmatrix} \widehat{k}_w, \begin{bmatrix} 0 & S^* \\ R^* & 0 \end{bmatrix} \widehat{k}_w \right\rangle\right|\right) \\ &\le \left|\left\langle \begin{bmatrix} P & 0 \\ 0 & T \end{bmatrix} \widehat{k}_w, \widehat{k}_w \right\rangle\right|^2 + \left|\left\langle \begin{bmatrix} P & 0 \\ 0 & T \end{bmatrix} \widehat{k}_w, \widehat{k}_w \right\rangle\right| \left(\left\langle \left|\begin{bmatrix} 0 & R \\ S & 0 \end{bmatrix}\right| \widehat{k}_w, \widehat{k}_w \right\rangle + \left\langle \left|\begin{bmatrix} 0 & S^* \\ R^* & 0 \end{bmatrix}\right| \widehat{k}_w, \widehat{k}_w \right\rangle\right) \\ &\quad + \frac{1}{4}\left(\left\langle \left|\begin{bmatrix} 0 & R \\ S & 0 \end{bmatrix}\right|^2 \widehat{k}_w, \widehat{k}_w \right\rangle + \left\langle \left|\begin{bmatrix} 0 & S^* \\ R^* & 0 \end{bmatrix}\right|^2 \widehat{k}_w, \widehat{k}_w \right\rangle\right) + \frac{1}{2}\left|\left\langle \begin{bmatrix} RS & 0 \\ 0 & SR \end{bmatrix} \widehat{k}_w, \widehat{k}_w \right\rangle\right| \\ &\le \left|\left\langle \begin{bmatrix} P & 0 \\ 0 & T \end{bmatrix} \widehat{k}_w, \widehat{k}_w \right\rangle\right|^2 + \left|\left\langle \begin{bmatrix} P & 0 \\ 0 & T \end{bmatrix} \widehat{k}_w, \widehat{k}_w \right\rangle\right| \left(\left\langle \begin{bmatrix} |S|+|R^*| & 0 \\ 0 & |R|+|S^*| \end{bmatrix} \widehat{k}_w, \widehat{k}_w \right\rangle\right) \\ &\quad + \frac{1}{4}\left(\left\langle \left|\begin{bmatrix} |S|^2+|R^*|^2 & 0 \\ 0 & |R|^2+|S^*|^2 \end{bmatrix}\right|^2 \widehat{k}_w, \widehat{k}_w \right\rangle\right) + \frac{1}{2}\left|\left\langle \begin{bmatrix} RS & 0 \\ 0 & SR \end{bmatrix} \widehat{k}_w, \widehat{k}_w \right\rangle\right|, \end{aligned}$$

and, so

$$\left|\left\langle\begin{bmatrix}P & R\\ S & T\end{bmatrix}\widehat{k}_w,\widehat{k}_w\right\rangle\right|^2 \leq \left|\left\langle\begin{bmatrix}P & 0\\ 0 & T\end{bmatrix}\widehat{k}_w,\widehat{k}_w\right\rangle\right|^2+\left|\left\langle\begin{bmatrix}P & 0\\ 0 & T\end{bmatrix}\widehat{k}_w,\widehat{k}_w\right\rangle\right|\left(\left\langle\begin{bmatrix}|S|+|R^*| & 0\\ 0 & |R|+|S^*|\end{bmatrix}\widehat{k}_w,\widehat{k}_w\right\rangle\right)$$
$$+\frac{1}{4}\left(\left\langle\left|\begin{bmatrix}|S|^2+|R^*|^2 & 0\\ 0 & |R|^2+|S^*|^2\end{bmatrix}\right|^2\widehat{k}_w,\widehat{k}_w\right\rangle\right)+\frac{1}{2}\left|\left\langle\begin{bmatrix}RS & 0\\ 0 & SR\end{bmatrix}\widehat{k}_w,\widehat{k}_w\right\rangle\right|.$$

By taking the supremum over $(\omega_1,\omega_2)\in\Upsilon_1\times\Upsilon_2$ and using inequality (2.1) in the above inequality, we have

$$\begin{aligned}\operatorname{ber}^2\left(\begin{bmatrix}P & R\\ S & T\end{bmatrix}\right) &= \max\left\{\operatorname{ber}^2(P),\operatorname{ber}^2(T)\right\}+\frac{1}{4}\max\left\{\left\||S|^2+|R^*|^2\right\|_{\operatorname{ber}},\left\||R|^2+|S^*|^2\right\|_{\operatorname{ber}}\right\}\\ &+\max\left\{\operatorname{ber}(P),\operatorname{ber}(T)\right\}\max\left\{\||S|+|R^*|\|_{\operatorname{ber}},\||R|+|S^*|\|_{\operatorname{ber}}\right\}\\ &+\frac{1}{2}\max\left\{\operatorname{ber}(RS),\operatorname{ber}(SR)\right\}.\end{aligned}$$

□

The inequality (2.8) in Theorem 2.6 and a particular application of Theorem 2.16 lead to the following conclusion, which is straightforward to deduce.

Corollary 2.17. *If $P=T$ and $R=S$, then it follows from Theorem 2.16 that*

$$\operatorname{ber}^2\left(\begin{bmatrix}P & R\\ R & P\end{bmatrix}\right)=\max\left\{\operatorname{ber}^2(P+R),\operatorname{ber}^2(P-R)\right\}\leq\operatorname{ber}^2(P)+\frac{1}{4}\left\||R|^2+|R^*|^2\right\|_{\operatorname{ber}}+\operatorname{ber}(P)\||R|+|R^*|\|_{\operatorname{ber}}+\frac{1}{2}\operatorname{ber}\left(R^2\right).$$

Corollary 2.18. *If we take $P=R$ in Corollary 2.17, then we get the inequality (1.2)*

$$\operatorname{ber}^2(P)\leq\frac{1}{2}\left\||P|^2+|P^*|^2\right\|_{\operatorname{ber}}$$

(see, [18, Theorem 3.2]).

Proof. From Corollary 2.17, we reach

$$4\operatorname{ber}^2(P)\leq\operatorname{ber}^2(P)+\frac{1}{4}\left\||P|^2+|P^*|^2\right\|_{\operatorname{ber}}+\operatorname{ber}(P)\||P|+|P^*|\|_{\operatorname{ber}}+\frac{1}{2}\operatorname{ber}\left(P^2\right).$$

Also, by using the inequality (1.1), we have

$$3\operatorname{ber}^2(P)\leq\frac{1}{4}\left\||P|^2+|P^*|^2\right\|_{\operatorname{ber}}+\operatorname{ber}(P)\||P|+|P^*|\|_{\operatorname{ber}}+\frac{1}{2}\operatorname{ber}^2(P).$$

Then using the inequalities (1.1) and (2.5), we get

$$\begin{aligned}\frac{5}{2}\operatorname{ber}^2(P) &\leq \frac{1}{4}\left\||P|^2+|P^*|^2\right\|_{\operatorname{ber}}+\frac{1}{2}\||P|+|P^*|\|^2_{\operatorname{ber}}\\ &= \frac{1}{4}\left\||P|^2+|P^*|^2\right\|_{\operatorname{ber}}+\left\|\frac{|P|+|P^*|}{2}\right\|^2_{\operatorname{ber}}\\ &\leq \frac{1}{4}\left\||P|^2+|P^*|^2\right\|_{\operatorname{ber}}+\left\|\frac{|P|^2+|P^*|^2}{2}\right\|_{\operatorname{ber}}\\ &\leq \frac{1}{4}\left\||P|^2+|P^*|^2\right\|_{\operatorname{ber}}+\left\||P|^2+|P^*|^2\right\|_{\operatorname{ber}}\\ &\leq \frac{5}{4}\left\||P|^2+|P^*|^2\right\|_{\operatorname{ber}}\end{aligned}$$

and

$$\operatorname{ber}^2(P)\leq\frac{1}{2}\left\||P|^2+|P^*|^2\right\|_{\operatorname{ber}}$$

as desired. □

Taking $P=0$ and $P=T=0$ in Corollary 2.17 and Theorem 2.16, respectively, we obtain the following corollary.

Remark 2.19. *$\mathscr{H}=\mathscr{H}(\Upsilon)$ be a FHS and $R,S\in\mathbb{L}(\mathscr{H})$. Then*

$$\mathrm{ber}^2(R)\le\frac{1}{4}\left\||R|^2+|R^*|^2\right\|_{\mathrm{ber}}+\frac{1}{2}\mathrm{ber}\left(R^2\right)$$

(see, [28, Corollary 2.14]) and

$$\mathrm{ber}^2\left(\begin{bmatrix}0&R\\S&0\end{bmatrix}\right)=\frac{1}{4}\max\left\{\left\||S|^2+|R^*|^2\right\|_{\mathrm{ber}},\left\||R|^2+|S^*|^2\right\|_{\mathrm{ber}}\right\}+\frac{1}{2}\max\left\{\mathrm{ber}(RS),\mathrm{ber}(SR)\right\}.$$

Theorem 2.20. *Let $P,R,S,T\in\mathbb{L}(\mathscr{H})$. Then*

$$\begin{aligned}\mathrm{ber}^4\left(\begin{bmatrix}P&R\\S&T\end{bmatrix}\right)\ &\le\ 8\max\left\{\mathrm{ber}^4(P),\mathrm{ber}^4(T)\right\}+3\max\left\{\left\||S|^4+|R^*|^4\right\|_{\mathrm{ber}},\left\||R|^4+|S^*|^4\right\|_{\mathrm{ber}}\right\}\\&\quad+\max\left\{\mathrm{ber}(SR),\mathrm{ber}(RS)\right\}\max\left\{\left\||S|^2+|R^*|^2\right\|_{\mathrm{ber}},\left\||R|^2+|S^*|^2\right\|_{\mathrm{ber}}\right\}.\end{aligned}$$

Proof. For every $(\omega_1,\omega_2)\in\Upsilon_1\times\Upsilon_2$, let $\widehat{k}_w=\widehat{k}_{(\omega_1,\omega_2)}=\begin{bmatrix}k_{\omega_1}\\k_{\omega_2}\end{bmatrix}$ be a normalized reproducing kernel in $\mathscr{H}(\Upsilon_1)\oplus\mathscr{H}(\Upsilon_2)$. Then

$$\begin{aligned}\left|\left\langle\begin{bmatrix}P&R\\S&T\end{bmatrix}\widehat{k}_w,\widehat{k}_w\right\rangle\right|^4&=\left|\left\langle\left(\begin{bmatrix}P&0\\0&T\end{bmatrix}+\begin{bmatrix}0&R\\S&0\end{bmatrix}\right)\widehat{k}_w,\widehat{k}_w\right\rangle\right|^4\\&\le\left(\left|\left\langle\begin{bmatrix}P&0\\0&T\end{bmatrix}\widehat{k}_w,\widehat{k}_w\right\rangle+\left\langle\begin{bmatrix}0&R\\S&0\end{bmatrix}\widehat{k}_w,\widehat{k}_w\right\rangle\right|\right)^4\\&\le\left(\left|\left\langle\begin{bmatrix}P&0\\0&T\end{bmatrix}\widehat{k}_w,\widehat{k}_w\right\rangle\right|+\left|\left\langle\begin{bmatrix}0&R\\S&0\end{bmatrix}\widehat{k}_w,\widehat{k}_w\right\rangle\right|\right)^4\\&=\left(\frac{2\left|\left\langle\begin{bmatrix}P&0\\0&T\end{bmatrix}\widehat{k}_w,\widehat{k}_w\right\rangle\right|+2\left|\left\langle\begin{bmatrix}0&R\\S&0\end{bmatrix}\widehat{k}_w,\widehat{k}_w\right\rangle\right|}{2}\right)^4.\end{aligned}$$

By convexity of $f(t)=t^4$, the inequality (2.7) with $M=\begin{bmatrix}0&R\\S&0\end{bmatrix}$, the AM-GM inequality and inequality (2.3), we get

$$\begin{aligned}\left|\left\langle\begin{bmatrix}P&R\\S&T\end{bmatrix}\widehat{k}_w,\widehat{k}_w\right\rangle\right|^4&\le 8\left(\left|\left\langle\begin{bmatrix}P&0\\0&T\end{bmatrix}\widehat{k}_w,\widehat{k}_w\right\rangle\right|^4+\left|\left\langle\begin{bmatrix}0&R\\S&0\end{bmatrix}\widehat{k}_w,\widehat{k}_w\right\rangle\right|^4\right)\\&\le 8\left|\left\langle\begin{bmatrix}P&0\\0&T\end{bmatrix}\widehat{k}_w,\widehat{k}_w\right\rangle\right|^4\\&\quad+2\left(3\left\|M\widehat{k}_w\right\|^2\left\|M^*\widehat{k}_w\right\|^2+\left\|M\widehat{k}_w\right\|\left\|M^*\widehat{k}_w\right\|\left|\left\langle M\widehat{k}_w,M^*\widehat{k}_w\right\rangle\right|\right)\\&=8\left|\left\langle\begin{bmatrix}P&0\\0&T\end{bmatrix}\widehat{k}_w,\widehat{k}_w\right\rangle\right|^4+2\left(3\left\langle|M|^2\widehat{k}_w,\widehat{k}_w\right\rangle\left\langle|M^*|^2\widehat{k}_w,\widehat{k}_w\right\rangle\right.\\&\quad\left.+\sqrt{\left\langle|M|^2\widehat{k}_w,\widehat{k}_w\right\rangle\left\langle|M^*|^2\widehat{k}_w,\widehat{k}_w\right\rangle}\left|\left\langle M\widehat{k}_w,M^*\widehat{k}_w\right\rangle\right|\right)\\&\le 8\left|\left\langle\begin{bmatrix}P&0\\0&T\end{bmatrix}\widehat{k}_w,\widehat{k}_w\right\rangle\right|^4+3\left\langle\left(|M|^2+|M^*|^2\right)\widehat{k}_w,\widehat{k}_w\right\rangle\\&\quad+\left\langle\left(|M|^2+|M^*|^2\right)\widehat{k}_w,\widehat{k}_w\right\rangle\left|\left\langle M^2\widehat{k}_w,\widehat{k}_w\right\rangle\right|\\&=8\left|\left\langle\begin{bmatrix}P&0\\0&T\end{bmatrix}\widehat{k}_w,\widehat{k}_w\right\rangle\right|^4+3\left\langle\begin{bmatrix}|S|^4+|R^*|^4&0\\0&|R|^4+|S^*|^4\end{bmatrix}\widehat{k}_w,\widehat{k}_w\right\rangle\\&\quad+\left\langle\begin{bmatrix}|S|^2+|R^*|^2&0\\0&|R|^2+|S^*|^2\end{bmatrix}\widehat{k}_w,\widehat{k}_w\right\rangle\left|\left\langle\begin{bmatrix}RS&0\\0&SR\end{bmatrix}\widehat{k}_w,\widehat{k}_w\right\rangle\right|.\end{aligned}$$

and, so

$$\begin{aligned}\left|\left\langle\begin{bmatrix}P&R\\S&T\end{bmatrix}\widehat{k}_w,\widehat{k}_w\right\rangle\right|^4&\le 8\left|\left\langle\begin{bmatrix}P&0\\0&T\end{bmatrix}\widehat{k}_w,\widehat{k}_w\right\rangle\right|^4+3\left\langle\begin{bmatrix}|S|^4+|R^*|^4&0\\0&|R|^4+|S^*|^4\end{bmatrix}\widehat{k}_w,\widehat{k}_w\right\rangle\\&\quad+\left\langle\begin{bmatrix}|S|^2+|R^*|^2&0\\0&|R|^2+|S^*|^2\end{bmatrix}\widehat{k}_w,\widehat{k}_w\right\rangle\left|\left\langle\begin{bmatrix}RS&0\\0&SR\end{bmatrix}\widehat{k}_w,\widehat{k}_w\right\rangle\right|.\end{aligned}$$

Taking the supremum over $(\omega_1,\omega_2)\in\Upsilon_1\times\Upsilon_2$ and then applying the inequality (2.1) in the above inequality, we have

$$\begin{aligned}\operatorname{ber}^4\left(\begin{bmatrix}P & R\\ S & T\end{bmatrix}\right) &\leq 8\max\left\{\operatorname{ber}^4(P),\operatorname{ber}^4(T)\right\}+3\max\left\{\left\||S|^4+|R^*|^4\right\|_{\text{ber}},\left\||R|^4+|S^*|^4\right\|_{\text{ber}}\right\}\\ &\quad+\max\left\{\operatorname{ber}(SR),\operatorname{ber}(RS)\right\}\max\left\{\left\||S|^2+|R^*|^2\right\|_{\text{ber}},\left\||R|^2+|S^*|^2\right\|_{\text{ber}}\right\}.\end{aligned}$$

This completes the proof. □

Now, considering $P=T$ and $R=S$ in Theorem 2.20, we obtain the following case.

Corollary 2.21. *Let $\mathscr{H}=\mathscr{H}(\Upsilon)$ be a FHS and $P,R\in\mathbb{L}(\mathscr{H})$. Then we have*

$$\operatorname{ber}^4\left(\begin{bmatrix}P & R\\ R & P\end{bmatrix}\right)\leq\max\left\{\operatorname{ber}^4(P+R),\operatorname{ber}^4(P-R)\right\}\leq 8\operatorname{ber}^4(P)+3\left\||R|^4+|R^*|^4\right\|_{\text{ber}}+\left\||R|^2+|R^*|^2\right\|_{\text{ber}}\operatorname{ber}\left(R^2\right).$$

It should be noted that the case $P=R$ of Corollary 2.21 is same as the inequality

$$\operatorname{ber}^4(P)\leq\frac{3}{8}\left\||P|^4+|P^*|^4\right\|_{\text{ber}}+\frac{1}{8}\left\||P|^2+|P^*|^2\right\|_{\text{ber}}\operatorname{ber}\left(P^2\right)$$

obtained in [28, Theorem 2.5].
We give the following example which show that $\operatorname{ber}(T)=\max_{1\leq j\leq n}\left|t_{jj}\right|$ for any complex $n\times n$ matrix $T=\left(t_{jk}\right)_{j,k=1}^{n}$ (see, [17, Example 2.17]). Let's think about a situation with finite dimensions. $T=\left(t_{jk}\right)_{j,k=1}^{n}$ be a $n\times n$ matrix. Let $\omega=(\omega_1,\ldots,\omega_n)\in\mathbb{C}^n$ and $Y=\{1,\ldots,n\}$. We can consider $\mathbb{C}^n$ as the set of all functions mapping $Y\to\mathbb{C}$ by $\omega(j)=\omega_j$. Letting z_j be the jth standard basis vector for $\mathbb{C}^n$ under the standard inner product, we can view $\mathbb{C}^n$ as a FHS with kernel

$$k(i,j)=\left\langle z_j,z_i\right\rangle.$$

Note that $k_j=\widehat{k}_j$ for each $j=1,\ldots,n$. We get $t_{jj}=\left\langle Tz_j,z_j\right\rangle$. So, the Berezin set of T is simply

$$\operatorname{Ber}(T)=\left\{t_{jj}:j=1,\ldots,n\right\},$$

which is just the collection of diagonal elements of T. Hence $\operatorname{ber}(T)=\max_{1\leq j\leq n}\left|t_{jj}\right|$.

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