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Approximate Solutions for A Fractional Shallow Water Flow Model

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Conformable Calculus; Residual Power Series Method, Drinfeld-Sokolov-Wilson Equation, Shallow water waves.

Abstract — This paper presents the solutions of fractional Drinfeld-Sokolov-Wilson (DSW) equations that occur in shallow water flow models using the residual power series method. The fractional derivatives and integrals are considered in the conformable sense. In addition, surface plots of the solutions are given. The solutions and results show that the present method is very efficient and effective due to the lack of a need for complex calculations and that the method also has a wide range of practicability in the resolution of partial differential fractional equations.

Subject Classification (2020): 34KXX, 39AXX.

1. Introduction

The use of fractional differential operators and integral operators in mathematical models has become increasingly popular in recent decades. Fractional calculus has, therefore, found numerous applications in different technical and scientific fields, such as fluid mechanics [1], signal processing [2], thermodynamics [3], biology [4], economics [5], viscoelasticity [6], control [7] and many other physical mechanisms.

In conjunction with these efforts in research, fractional differential equations (FDEs) have also been proposed and implemented in modeling several physical and engineering problems. As a result, an active consulting firm has been involved in discovering reliable and effective methods for resolving FDEs. Since, it is not easy to find the exact solutions of most FDEs, some approximate and numerical schemes must be produced. Some of the numerical methods used to solve FDEs are differential transform method [8] for fractional partial differential equation from finance, Adams-Bashforth method [9] for chaotic differential equations and Fisher's equation, homotopy analysis method [10] for Nizhnik-Novikov-Veselov system, q-homotopy analysis method [11] for seventh-order time-fractional Lax's Korteweg-de Vries and Sawada-Kotera equations, Shehu transform method [12] for Burgers-Fisher, backward Klomogorov and Klein-Gordon

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equations and some other systems, perturbation-iteration algorithm [13] for fuzzy partial differential equations and Adomian decomposition method [14] for Burger-Huxley's equation.

Besides, as analytical methods, the functional variable method [15] for the Zakharov-Kuznetsov equation, the Benjamin-Bona-Mahony equation and the Korteweg-de Vries equation, Sine-Gordon expansion method [16] for RLW-class equations, modified Khater method and sech-tanh functions expansion method [17] for emerging telecommunication model, $Exp(-\phi(\xi))$-expansion method [18] for nematicons, new extended direct algebraic method [19] for Konno-Ono equation and Kudryashov's method [20] for nonlinear schrödinger equation are worth mentioning.

In this piece of research, the residual power series method [21–23] is used to obtain new approximate solutions for below mentioned time-fractional Drinfeld-Sokolov-Wilson equation that arise in shallow water flow models. We successfully solved differential equations with this method before [26–29]. Also Jaradat et. al.[30] used RPSM for solving DSW equation where the fractional derivatives are in Caputo sense.

Consider the following nonlinear conformable time-fractional DSW equation, as

$$\begin{aligned} &\frac{\partial^\alpha w(x,t)}{\partial t^\alpha} + \mu v(x,t)\frac{\partial v(x,t)}{\partial x} = 0, \\ &\frac{\partial^\alpha v(x,t)}{\partial t^\alpha} + \eta\frac{\partial^3 v(x,t)}{\partial x^3} + \gamma w(x,t)\frac{\partial v(x,t)}{\partial x} + \xi v(x,t)\frac{\partial w(x,t)}{\partial x} = 0, \quad 0<\alpha<1, \end{aligned} \tag{1.1}$$

subject to the initial conditions

$$\begin{aligned} w(x,0) &= \hbar(x), \\ v(x,0) &= \aleph(x). \end{aligned} \tag{1.2}$$

The purpose of this study is to construct a power series solution for Eqs. (1.1) and (1.2) by its power series expansion among its truncated residual function.The major improvement of the RPSM is that by choosing suitable initial conditions, it can be applied directly to the problem without perturbation, linearization or discretization, in other words, without any adjustments. Furthermore, present method is capable of obtaining results without complicated calculations.

The remainder of the study is carried out as follows: In Section 2, we present essential definitions and results for RPSM. Within Section 3, general procedure of the RPSM is summarized In Sections 4, Implementation of RPSM for Drinfeld-Sokolov-Wilson system is presented. In Section 5, numerical results illustrated. Finally, Section 5 is reserved for conclusion.

2. Essential Definitions and Results for RPSM

Suppose that f is an infinitely α-differentiable function, for some $\alpha \in (0,1]$ at a neighborhood of a point $t = t_0$ then f has the following conformable fractional power series expansion [24, 25]:

$$f(t) = \sum_{p=0}^{\infty} \frac{(T_\alpha^{t_0} f)^{(p)}(t-t_0)^{p\alpha}}{\alpha^p p!}, \quad t_0 < t < t_0 + R^{1/\alpha},\ R>0. \tag{2.1}$$

where $(T_\alpha^{t_0} f)^{(p)}$ is the application of the fractional derivative p times. [23, 25] A power series of the form $\sum_{p=0}^{\infty} g_p(x)(t)^{p\alpha}$ for $0 \le m-1 < \alpha \le m$ is called multiple fractional power series about $t_0 = 0$, where g_p's are functions of x called the coefficients of the series. [25] Suppose that $u(x,t)$ has the following multiple fractional power series representation at $t_0 = 0$:

$$u(x,t) = \sum_{p=0}^{\infty} g_p(x) t^{p\alpha}, \quad 0 < \alpha \leq 1, x \in I, 0 \leq t \leq R^{1/\alpha}. \tag{2.2}$$

If $u_t^{(p\alpha)}(x,t)$ are continuous on $I \times (0, R^{1/\alpha}),\ k = 0,1,2,\ldots,$ then $g_p(x) = \frac{u_t^{(p\alpha)}(x,0)}{\alpha^p p!}$.

3. General Procedure of the RPSM

The main steps of this procedure are described as follows:

Step 1. Suppose that the solution of Eq. (1.1) and Eq. (1.2) is expressed in the form of fractional power series expansion about the initial point $t = 0$, as

$$\begin{aligned} w(x,t) &= \sum_{p=0}^{\infty} h_p(x) \frac{t^{p\alpha}}{\alpha^p p!}, \\ v(x,t) &= \sum_{p=0}^{\infty} z_p(x) \frac{t^{p\alpha}}{\alpha^p p!}, \quad 0 < \alpha \leq 1,\ x \in I,\ 0 \leq t < R^{\frac{1}{\alpha}}. \end{aligned} \tag{3.1}$$

The RPSM guarantees that the analytical approximate solution for Eq. (1.1) and Eq. (1.2) are in the form of an infinite fractional power series. To obtain the numerical values from these series, let $w_k(x,t)$ and $v_k(x,t)$ denotes the k-th truncated series of $w(x,t)$ and $v(x,t)$, respectively. *i.e.*,

$$\begin{aligned} w_k(x,t) &= \sum_{p=0}^{k} h_p(x) \frac{t^{p\alpha}}{\alpha^p p!}, \\ v_k(x,t) &= \sum_{p=0}^{k} z_p(x) \frac{t^{p\alpha}}{\alpha^p p!}, \quad 0 < \alpha \leq 1,\ x \in I,\ 0 \leq t < R^{\frac{1}{\alpha}}. \end{aligned} \tag{3.2}$$

Take $k = 0$ and by the initial condition, the 0-th residual power series approximate solution of $w(x,t)$ and $v(x,t)$ can be written in the following form, as

$$\begin{aligned} w_0(x,t) &= h_0(x) = w(x,0) = \hbar(x), \\ v_0(x,t) &= z_0(x) = v(x,0) = \aleph(x). \end{aligned} \tag{3.3}$$

The Eq. (3.2) can be rewritten, as

$$\begin{aligned} w_k(x,t) &= \hbar(x) + \sum_{p=1}^{k} h_p(x) \frac{t^{p\alpha}}{\alpha^p p!}, \\ v_k(x,t) &= \aleph(x) + \sum_{p=1}^{k} z_p(x) \frac{t^{p\alpha}}{\alpha^p p!}, \quad 0 < \alpha \leq 1,\ x \in I,\ 0 \leq t, \end{aligned} \tag{3.4}$$

where $k = 1,2,3,\ldots$. By viewing the representations of $w_k(x,t)$ and $v_k(x,t)$, the k-th residual power series approximate solutions will be obtained after $h_p(x), z_p(x),\ p = 1,2,3,\ldots,k$, are available.

Step 2. Define the residual function, for Eq. (1.1) and Eq. (1.2), as

$$\begin{aligned} Res_w(x,t) &= \frac{\partial^\alpha w(x,t)}{\partial t^\alpha} + \mu v(x,t) \frac{\partial v(x,t)}{\partial x}, \\ Res_v(x,t) &= \frac{\partial^\alpha v(x,t)}{\partial t^\alpha} + \eta \frac{\partial^3 v(x,t)}{\partial x^3} + \gamma w(x,t) \frac{\partial v(x,t)}{\partial x} + \xi v(x,t) \frac{\partial w(x,t)}{\partial x} \end{aligned} \tag{3.5}$$

and the k-th residual functions, $k = 1,2,3,\ldots$, can be expressed, as

$$\begin{aligned} Res_{w,k}(x,t) &= \frac{\partial^\alpha w_k(x,t)}{\partial t^\alpha} + \mu v_k(x,t)\frac{\partial v_k(x,t)}{\partial x}, \\ Res_{v,k}(x,t) &= \frac{\partial^\alpha v_k(x,t)}{\partial t^\alpha} + \eta\frac{\partial^3 v_k(x,t)}{\partial x^3} + \gamma w_k(x,t)\frac{\partial v_k(x,t)}{\partial x} + \xi v_k(x,t)\frac{\partial w_k(x,t)}{\partial x}. \end{aligned} \tag{3.6}$$

From [25], some useful results for $Res_{w,k}(x,t)$ and $Res_{v,k}(x,t)$ which are essential in the residual power series solution for $j = 0,1,2,...,k$ are stated as follows:

$$\begin{aligned} &(i)\ Res_w(x,t) = 0, Res_v(x,t) = 0, \\ &(ii)\ \lim_{k\to\infty} Res_{w,k}(x,t) = Res_w(x,t),\ \lim_{k\to\infty} Res_{v,k}(x,t) = Res_v(x,t),\quad for\ each\ x \in I\ and\ t \geq 0, \\ &(iii)\ \frac{\partial^{j\alpha}}{\partial t^{j\alpha}}Res_w(x,0) = \frac{\partial^{j\alpha}}{\partial t^{j\alpha}}Res_{w,k}(x,0) = 0, \frac{\partial^{j\alpha}}{\partial t^{j\alpha}}Res_v(x,0) = \frac{\partial^{j\alpha}}{\partial t^{j\alpha}}Res_{v,k}(x,0) = 0. \end{aligned} \tag{3.7}$$

Step 3. Substitute the k-th truncated series of $w(x,t)$ and $v(x,t)$ into Eq. (3.6) and calculate the fractional derivative $\frac{\partial^{(k-1)\alpha}}{\partial t^{(k-1)\alpha}}$ of $Res_{w,k}(x,t)$ and $Res_{v,k}(x,t)$, $k = 1,2,3,...$ at $t = 0$, together with Eq. (3.7), the following algebraic systems are obtained:

$$\begin{aligned} &\frac{\partial^{(k-1)\alpha}}{\partial t^{(k-1)\alpha}}Res_{w,k}(x,0) = 0, \\ &\frac{\partial^{(k-1)\alpha}}{\partial t^{(k-1)\alpha}}Res_{v,k}(x,0) = 0, \quad 0 < \alpha \leq 1,\ k = 1,2,3,\ldots. \end{aligned} \tag{3.8}$$

Step 4. After solving the systems (3.8), the values of the coefficients $h_p(x), z_p(x),\ p = 1,2,3,...,k$ are obtained. Thus, the k-th residual power series approximate solutions is derived.

In the next discussion, the 1st, 2nd, 3rd and 4th residual power series approximate solutions are determined in detail by following the above steps.

4. Implementation of RPSM

For $\boldsymbol{k} = \mathbf{1}$, the 1st-residual power series solutions can be written, as

$$\begin{aligned} w_1(x,t) &= \hbar(x) + h_1(x)\frac{t^\alpha}{\alpha}, \\ v_1(x,t) &= \aleph(x) + z_1(x)\frac{t^\alpha}{\alpha}. \end{aligned} \tag{4.1}$$

The 1st-residual functions can be written, as

$$\begin{aligned} Res_{w,1}(x,t) &= \frac{\partial^\alpha w_1(x,t)}{\partial t^\alpha} + \mu v_1(x,t)\frac{\partial v_1(x,t)}{\partial x}, \\ Res_{v,1}(x,t) &= \frac{\partial^\alpha v_1(x,t)}{\partial t^\alpha} + \eta\frac{\partial^3 v_1(x,t)}{\partial x^3} + \gamma w_1(x,t)\frac{\partial v_1(x,t)}{\partial x} + \xi v_1(x,t)\frac{\partial w_1(x,t)}{\partial x}. \end{aligned} \tag{4.2}$$

Substitute the 1st truncated series, $w_1(x,t)$ and $v_1(x,t)$ into the 1st residual functions, $Res_{w,1}(x,t)$ and $Res_{v,1}(x,t)$, respectively. *i.e.*,

$$\begin{aligned}
Res_{w,1}(x,t) &= h_1(x) + \mu\Big(\aleph(x) + z_1(x)\frac{t^\alpha}{\alpha}\Big)\Big(\aleph^{(1)}(x) + z_1^{(1)}(x)\frac{t^\alpha}{\alpha}\Big),\\
Res_{v,1}(x,t) &= z_1(x) + \eta\Big(\aleph^{(3)}(x) + z_1^{(3)}(x)\frac{t^\alpha}{\alpha}\Big) + \gamma\Big(\hbar(x) + h_1(x)\frac{t^\alpha}{\alpha}\Big)\Big(\aleph^{(1)}(x) + z_1^{(1)}(x)\frac{t^\alpha}{\alpha}\Big)\\
&\quad + \xi\Big(\aleph(x) + z_1(x)\frac{t^\alpha}{\alpha}\Big)\Big(\hbar^{(1)}(x) + h_1^{(1)}(x)\frac{t^\alpha}{\alpha}\Big)
\end{aligned} \tag{4.3}$$

From Eq. (3.8) and Eq. (4.3), it can be written, as

$$\begin{aligned}
h_1(x) &= -\big(\mu\aleph(x)\aleph^{(1)}(x)\big),\\
z_1(x) &= -\big(\xi\aleph(x)\hbar^{(1)}(x) + \gamma\hbar(x)\aleph^{(1)}(x) + \eta\aleph^{(3)}(x)\big).
\end{aligned} \tag{4.4}$$

The 1st RPS approximate solutions can be written in the following form, as

$$\begin{aligned}
w_1(x,t) &= \hbar(x) - \frac{t^\alpha}{\alpha}\Big(\mu\aleph(x)\aleph^{(1)}(x)\Big),\\
v_1(x,t) &= \aleph(x) - \frac{t^\alpha}{\alpha}\Big(\xi\aleph(x)\hbar^{(1)}(x) + \gamma\hbar(x)\aleph^{(1)}(x) + \eta\aleph^{(3)}(x)\Big).
\end{aligned} \tag{4.5}$$

For $\boldsymbol{k} = \mathbf{2}$, the 2nd-residual power series solution can be written, as

$$\begin{aligned}
w_1(x,t) &= \hbar(x) + h_1(x)\frac{t^\alpha}{\alpha} + h_2(x)\frac{t^{2\alpha}}{2\alpha^2},\\
v_1(x,t) &= \aleph(x) + z_1(x)\frac{t^\alpha}{\alpha} + z_2(x)\frac{t^{2\alpha}}{2\alpha^2}.
\end{aligned} \tag{4.6}$$

Substitute the 2nd truncated series $u_2(x,t)$ into the 2nd residual function $Res_2(x,t)$, *i.e.*,

$$\begin{aligned}
Res_{w,2}(x,t) &= h_1(x) + h_2(x)\frac{t^\alpha}{\alpha} + \mu\Big(\aleph(x) + z_1(x)\frac{t^\alpha}{\alpha} + z_2(x)\frac{t^{2\alpha}}{2\alpha^2}\Big)\Big(\aleph^{(1)}(x) + z_1^{(1)}(x)\frac{t^\alpha}{\alpha} + z_2^{(1)}(x)\frac{t^{2\alpha}}{2\alpha^2}\Big),\\
Res_{v,2}(x,t) &= z_1(x) + z_2(x)\frac{t^\alpha}{\alpha} + \eta\Big(\aleph^{(3)}(x) + z_1^{(3)}(x)\frac{t^\alpha}{\alpha} + z_2^{(3)}(x)\frac{t^{2\alpha}}{2\alpha^2}\Big) + \gamma\Big(\hbar(x) + h_1(x)\frac{t^\alpha}{\alpha}\\
&\quad + h_2(x)\frac{t^{2\alpha}}{2\alpha^2}\Big)\Big(\aleph^{(1)}(x) + z_1^{(1)}(x)\frac{t^\alpha}{\alpha} + z_2^{(1)}(x)\frac{t^{2\alpha}}{2\alpha^2}\Big) + \xi\Big(\aleph(x) + z_1(x)\frac{t^\alpha}{\alpha}\\
&\quad + z_2(x)\frac{t^{2\alpha}}{2\alpha^2}\Big)\Big(\hbar^{(1)}(x) + h_1^{(1)}(x)\frac{t^\alpha}{\alpha} + h_2^{(1)}(x)\frac{t^{2\alpha}}{2\alpha^2}\Big)
\end{aligned} \tag{4.7}$$

From Eq.(3.8) and Eq.(4.7), it can be written, as

$$\begin{aligned}
h_2(x) &= -\Big(\mu z_1(x)\aleph^{(1)}(x) + \mu\aleph(x)z_1^{(1)}(x)\Big),\\
z_2(x) &= -\Big(\xi z_1(x)\hbar^{(1)}(x) + \xi\aleph(x)h_1^{(1)}(x) + \gamma h_1(x)\aleph^{(1)}(x)\\
&\quad + \gamma\hbar(x)z_1^{(1)}(x) + \eta z_1^{(3)}(x)\Big)
\end{aligned} \tag{4.8}$$

The 2nd residual power series approximate solutions can be written in the following form, as

$$\begin{aligned} w_2(x,t) &= \hbar(x) - \frac{t^{\alpha}}{\alpha}\Big(\mu\aleph(x)\aleph^{(1)}(x)\Big) - \frac{t^{2\alpha}}{2\alpha^2}\Big(\mu z_1(x)\aleph^{(1)}(x) + \mu\aleph(x)z_1^{(1)}(x)\Big),\\ v_2(x,t) &= \aleph(x) - \frac{t^{\alpha}}{\alpha}\Big(\xi\aleph(x)\hbar^{(1)}(x) + \gamma\hbar(x)\aleph^{(1)}(x) + \eta\aleph^{(3)}(x)\Big) - \frac{t^{2\alpha}}{2\alpha^2}\Big(\xi z_1(x)\hbar^{(1)}(x)\\ &\quad + \xi\aleph(x)h_1^{(1)}(x) + \gamma h_1(x)\aleph^{(1)}(x) + \gamma\hbar(x)z_1^{(1)}(x) + \eta z_1^{(3)}(x)\Big). \end{aligned} \tag{4.9}$$

For $\boldsymbol{k} = \mathbf{3}$, substitute the 3rd truncated series,

$$\begin{aligned} w_3(x,t) &= \hbar(x) + h_1(x)\frac{t^{\alpha}}{\alpha} + h_2(x)\frac{t^{2\alpha}}{2\alpha^2} + h_3(x)\frac{t^{3\alpha}}{6\alpha^3},\\ v_3(x,t) &= \aleph(x) + z_1(x)\frac{t^{\alpha}}{\alpha} + z_2(x)\frac{t^{2\alpha}}{2\alpha^2} + z_3(x)\frac{t^{3\alpha}}{6\alpha^3}. \end{aligned} \tag{4.10}$$

of Eq. (1.1) and Eq. (1.2) into the 3rd residual function, $Res_3(x,t)$, of Eq.(3.6), i.e.,

$$\begin{aligned} Res_{w,3}(x,t) &= h_1(x) + h_2(x)\frac{t^{\alpha}}{\alpha} + h_3(x)\frac{t^{2\alpha}}{2\alpha^2} + \mu\Big(\aleph(x) + z_1(x)\frac{t^{\alpha}}{\alpha} + z_2(x)\frac{t^{2\alpha}}{2\alpha^2} + z_3(x)\frac{t^{3\alpha}}{6\alpha^3}\Big)\Big(\aleph^{(1)}(x)\\ &\quad + z_1^{(1)}(x)\frac{t^{\alpha}}{\alpha} + z_2^{(1)}(x)\frac{t^{2\alpha}}{2\alpha^2} + z_3^{(1)}(x)\frac{t^{3\alpha}}{6\alpha^3}\Big),\\ Res_{v,3}(x,t) &= z_1(x) + z_2(x)\frac{t^{\alpha}}{\alpha} + z_3(x)\frac{t^{2\alpha}}{2\alpha^2} + \eta\Big(\aleph^{(3)}(x) + z_1^{(3)}(x)\frac{t^{\alpha}}{\alpha} + z_2^{(3)}(x)\frac{t^{2\alpha}}{2\alpha^2} + z_3^{(3)}(x)\frac{t^{3\alpha}}{6\alpha^3}\Big)\\ &\quad + \gamma\Big(\hbar(x) + h_1(x)\frac{t^{\alpha}}{\alpha} + h_2(x)\frac{t^{2\alpha}}{2\alpha^2} + h_3(x)\frac{t^{3\alpha}}{6\alpha^3}\Big)\Big(\aleph^{(1)}(x) + z_1^{(1)}(x)\frac{t^{\alpha}}{\alpha}\\ &\quad + z_2^{(1)}(x)\frac{t^{2\alpha}}{2\alpha^2} + z_3^{(1)}(x)\frac{t^{3\alpha}}{6\alpha^3}\Big) + \xi\Big(\aleph(x) + z_1(x)\frac{t^{\alpha}}{\alpha} + z_2(x)\frac{t^{2\alpha}}{2\alpha^2} + z_3(x)\frac{t^{3\alpha}}{6\alpha^3}\Big)\Big(\hbar^{(1)}(x)\\ &\quad + h_1^{(1)}(x)\frac{t^{\alpha}}{\alpha} + h_2^{(1)}(x)\frac{t^{2\alpha}}{2\alpha^2} + h_3^{(1)}(x)\frac{t^{3\alpha}}{6\alpha^3}\Big) \end{aligned} \tag{4.11}$$

Now, solving the equation $\frac{\partial^{(k-1)\alpha}}{\partial t^{(k-1)\alpha}}Res_k(x,0) = 0$, for $k = 3$ gives the required value of $g_3(x)$, as

$$\begin{aligned} h_3(x) &= -\Big(\mu z_2(x)\aleph^{(1)}(x) + 2\mu z_1(x)z_1^{(1)}(x) + \mu\aleph(x)z_2^{(1)}(x)\Big),\\ z_3(x) &= -\Big(\xi z_2(x)\hbar^{(1)}(x) + 2\xi z_1(x)h_1^{(1)}(x) + \xi\aleph(x)h_2^{(1)}(x) + \gamma h_2(x)\aleph^{(1)}(x)\\ &\quad + 2\gamma h_1(x)z_1^{(1)}(x) + \gamma\hbar(x)z_2^{(1)}(x) + \eta z_2^{(3)}(x)\Big) \end{aligned} \tag{4.12}$$

Based on the previous results for $g_0(x)$, $g_1(x)$ and $g_2(x)$, the 3rd residual power series approximate solution becomes

$$\begin{aligned} w_3(x,t) &= \hbar(x) - \frac{t^{\alpha}}{\alpha}\Big(\mu\aleph(x)\aleph^{(1)}(x)\Big) - \frac{t^{2\alpha}}{2\alpha^2}\Big(\mu z_1(x)\aleph^{(1)}(x) + \mu\aleph(x)z_1^{(1)}(x)\Big)\\ &\quad - \frac{t^{3\alpha}}{6\alpha^3}\Big(\mu z_2(x)\aleph^{(1)}(x) + 2\mu z_1(x)z_1^{(1)}(x) + \mu\aleph(x)z_2^{(1)}(x)\Big),\\ v_3(x,t) &= \aleph(x) - \frac{t^{\alpha}}{\alpha}\Big(\xi\aleph(x)\hbar^{(1)}(x) + \gamma\hbar(x)\aleph^{(1)}(x) + \eta\aleph^{(3)}(x)\Big) - \frac{t^{2\alpha}}{2\alpha^2}\Big(\xi z_1(x)\hbar^{(1)}(x)\\ &\quad + \xi\aleph(x)h_1^{(1)}(x) + \gamma h_1(x)\aleph^{(1)}(x) + \gamma\hbar(x)z_1^{(1)}(x) + \eta z_1^{(3)}(x)\Big)\\ &\quad - \frac{t^{3\alpha}}{6\alpha^3}\Big(\xi z_2(x)\hbar^{(1)}(x) + 2\xi z_1(x)h_1^{(1)}(x) + \xi\aleph(x)h_2^{(1)}(x) + \gamma h_2(x)\aleph^{(1)}(x)\\ &\quad + 2\gamma h_1(x)z_1^{(1)}(x) + \gamma\hbar(x)z_2^{(1)}(x) + \eta z_2^{(3)}(x)\Big). \end{aligned} \tag{4.13}$$

For $\boldsymbol{k}=\mathbf{4}$, substitute the 4th truncated series,
$w_4(x,t)=\hbar(x)+h_1(x)\frac{t^\alpha}{\alpha}+h_2(x)\frac{t^{2\alpha}}{2\alpha^2}+h_3(x)\frac{t^{3\alpha}}{6\alpha^3}+h_4(x)\frac{t^{4\alpha}}{24\alpha^4}$, $v_4(x,t)=\aleph(x)+z_1(x)\frac{t^\alpha}{\alpha}+z_2(x)\frac{t^{2\alpha}}{2\alpha^2}+z_3(x)\frac{t^{3\alpha}}{6\alpha^3}+z_4(x)\frac{t^{4\alpha}}{24\alpha^4}$ of Eq. (1.1) and Eq. (1.2) into the 4th residual function,$Res_4(x,t)$, of Eq.(3.6), *i.e.*, $Res_4(x,t)$ is equals to

$$\begin{aligned}
Res_{w,4}(x,t) &= h_1(x)+h_2(x)\frac{t^\alpha}{\alpha}+h_3(x)\frac{t^{2\alpha}}{2\alpha^2}+h_4(x)\frac{t^{3\alpha}}{6\alpha^3}+\mu\Big(\aleph(x)+z_1(x)\frac{t^\alpha}{\alpha}+z_2(x)\frac{t^{2\alpha}}{2\alpha^2}\\
&+z_3(x)\frac{t^{3\alpha}}{6\alpha^3}+z_4(x)\frac{t^{4\alpha}}{24\alpha^4}\Big)\Big(\aleph^{(1)}(x)+z_1^{(1)}(x)\frac{t^\alpha}{\alpha}+z_2^{(1)}(x)\frac{t^{2\alpha}}{2\alpha^2}+z_3^{(1)}(x)\frac{t^{3\alpha}}{6\alpha^3}\\
&+z_4^{(1)}(x)\frac{t^{4\alpha}}{24\alpha^4}\Big),\\
Res_{v,4}(x,t) &= z_1(x)+z_2(x)\frac{t^\alpha}{\alpha}+z_3(x)\frac{t^{2\alpha}}{2\alpha^2}+z_4(x)\frac{t^{3\alpha}}{6\alpha^3}+\eta\Big(\aleph^{(3)}(x)+z_1^{(3)}(x)\frac{t^\alpha}{\alpha}\\
&+z_2^{(3)}(x)\frac{t^{2\alpha}}{2\alpha^2}+z_3^{(3)}(x)\frac{t^{3\alpha}}{6\alpha^3}+z_4^{(3)}(x)\frac{t^{4\alpha}}{24\alpha^4}\Big)+\gamma\Big(\hbar(x)+h_1(x)\frac{t^\alpha}{\alpha}+h_2(x)\frac{t^{2\alpha}}{2\alpha^2}\\
&+h_3(x)\frac{t^{3\alpha}}{6\alpha^3}+h_4(x)\frac{t^{4\alpha}}{24\alpha^4}\Big)\Big(\aleph^{(1)}(x)+z_1^{(1)}(x)\frac{t^\alpha}{\alpha}+z_2^{(1)}(x)\frac{t^{2\alpha}}{2\alpha^2}+z_3^{(1)}(x)\frac{t^{3\alpha}}{6\alpha^3}\\
&+z_4^{(1)}(x)\frac{t^{4\alpha}}{24\alpha^4}\Big)+\xi\Big(\aleph(x)+z_1(x)\frac{t^\alpha}{\alpha}+z_2(x)\frac{t^{2\alpha}}{2\alpha^2}+z_3(x)\frac{t^{3\alpha}}{6\alpha^3}\\
&+z_4(x)\frac{t^{4\alpha}}{24\alpha^4}\Big)\Big(\hbar^{(1)}(x)+h_1^{(1)}(x)\frac{t^\alpha}{\alpha}+h_2^{(1)}(x)\frac{t^{2\alpha}}{2\alpha^2}+h_3^{(1)}(x)\frac{t^{3\alpha}}{6\alpha^3}+h_4^{(1)}(x)\frac{t^{4\alpha}}{24\alpha^4}\Big)
\end{aligned} \tag{4.14}$$

From Eq. (3.8) and Eq. (4.14), it can be written, as

$$\begin{aligned}
h_4(x) &= -\Big(\mu z_3(x)\aleph^{(1)}(x)+3\mu z_2(x)z_1^{(1)}(x)+3\mu z_1(x)z_2^{(1)}(x)+\mu\aleph(x)z_3^{(1)}(x)\Big),\\
z_4(x) &= -\Big(\xi z_3(x)\hbar^{(1)}(x)+\gamma h_3(x)\aleph^{(1)}(x)+3\xi z_2(x)h_1^{(1)}(x)+3\xi z_1(x)h_2^{(1)}(x)+\xi\aleph(x)h_3^{(1)}(x)\\
&+2\gamma h_2(x)z_1^{(1)}(x)+3\gamma h_1(x)z_2^{(1)}(x)+\gamma\hbar(x)z_3^{(1)}(x)+\eta z_3^{(3)}(x)\Big)
\end{aligned} \tag{4.15}$$

Based on the previous results for $h_0(x)$, $h_1(x)$, $h_2(x)$ and $h_3(x)$ and $z_0(x)$, $z_1(x)$, $z_2(x)$ and $z_3(x)$, the 4th residual power series approximate solution can be obtained. For the convergence analysis, see [25]

5. Numerical Results

To illustrate the authenticity of the RPSM method to solve the nonlinear conformable time-fractional Drinfeld-Sokolov-Wilson equation, three applications are considered. Consider the following time-fractional Drinfeld-Sokolov-Wilson equation, as

$$\begin{aligned}
&\frac{\partial^\alpha w(x,t)}{\partial t^\alpha}+3v(x,t)\frac{\partial v(x,t)}{\partial x}=0,\\
&\frac{\partial^\alpha v(x,t)}{\partial t^\alpha}+2\frac{\partial^3 v(x,t)}{\partial x^3}+2w(x,t)\frac{\partial v(x,t)}{\partial x}+v(x,t)\frac{\partial w(x,t)}{\partial x}=0, \quad 0<\alpha<1,
\end{aligned} \tag{5.1}$$

subject to the initial conditions

$$\begin{aligned}
w(x,0) &= 3sech^2(x),\\
v(x,0) &= 2sech(x).
\end{aligned} \tag{5.2}$$

The exact solution of this problem for $\alpha = 1$ is given in , as

$$\begin{aligned} w(x,t) &= 3sech^2(x-2t), \\ v(x,t) &= 2sech(x-2t). \end{aligned} \tag{5.3}$$

It can be observed that numerical results are agreement with the exact solution with a high accuracy. Also, it is clear that the adding new terms of the residual power series approximations can make the overall error smaller.

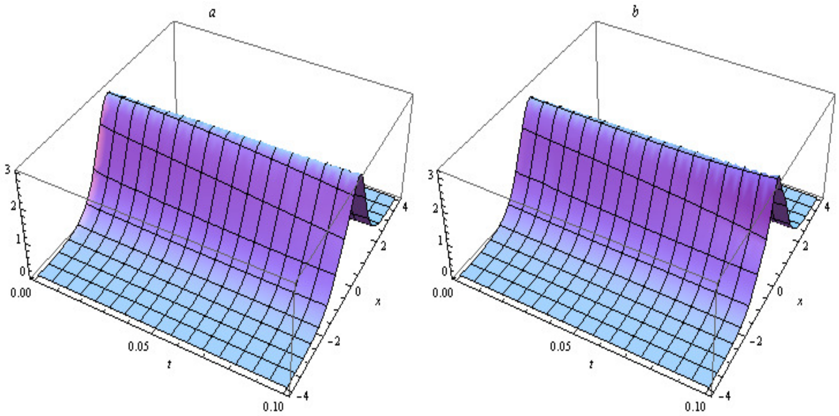

Figure 1. 3D surface plots for the 4th residual power series solution $w_4(x,t)$ with a. $\alpha = 0.8$ and b. $\alpha = 0.9$ for Example 5

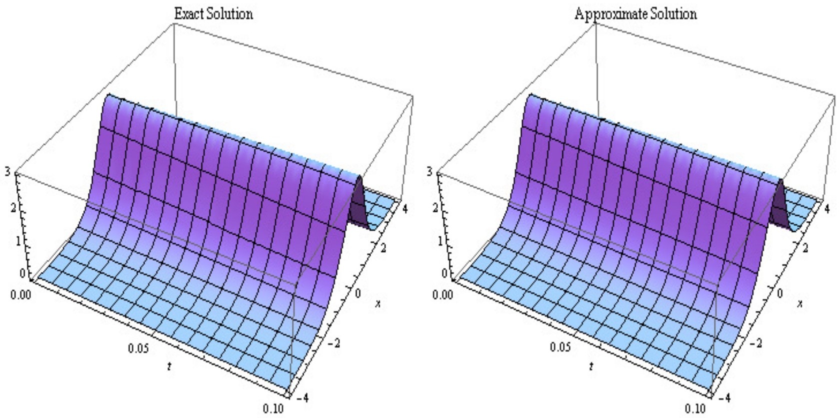

Figure 2. 3D surface plots of exact solution and approximate solution $w_4(x,t)$ at $\alpha = 1$ of Example 5.

6. Conclusion

In this research, we have given an algorithm, namely the Residual Power Series Method (RPSM), for the approximate solution of the fractional Drinfeld - Sokolov - Wilson equation system. The scheme is based on the power series and the solutions are determined in the form of a converging series with simple calculations. The approach offers approximate solutions with a good level of precision. Summing up these results, we can conclude that the residual power series method, in its general form, offers a fair amount of calculations, is an efficient method and simple to apply for nonlinear fractional differential equations in general form.

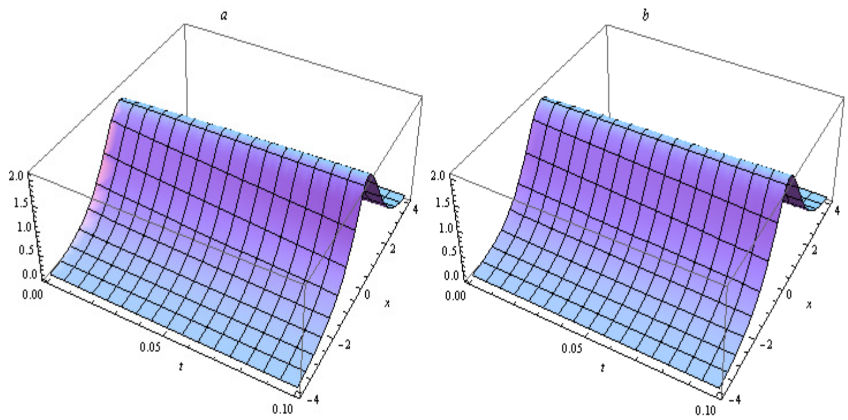

Figure 3. 3D surface plots for the 4th residual power series solution $v_4(x,t)$ with a. $\alpha = 0.8$ and b. $\alpha = 0.9$ for Example 5

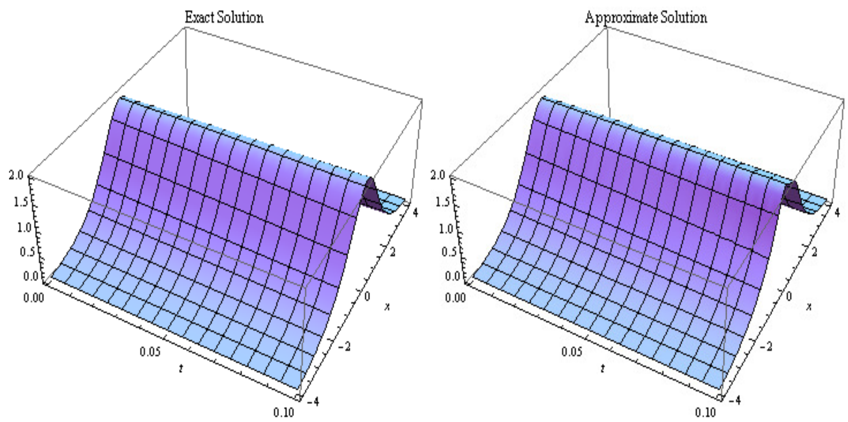

Figure 4. 3D surface plots of exact solution and approximate solution $v_4(x,t)$ at $\alpha = 1$ of Example 5.

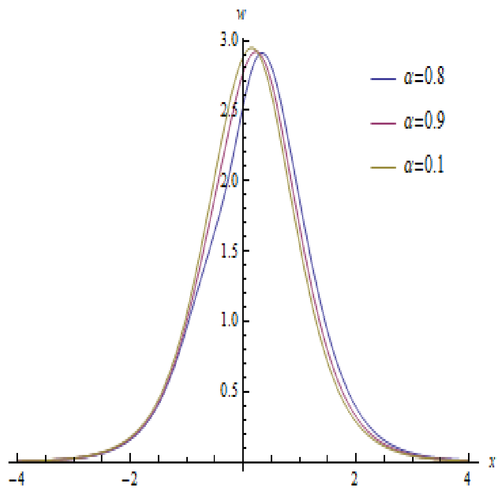

Figure 5. 2D plot of solutions w_3 at $t = 0.1$ for Example 5.

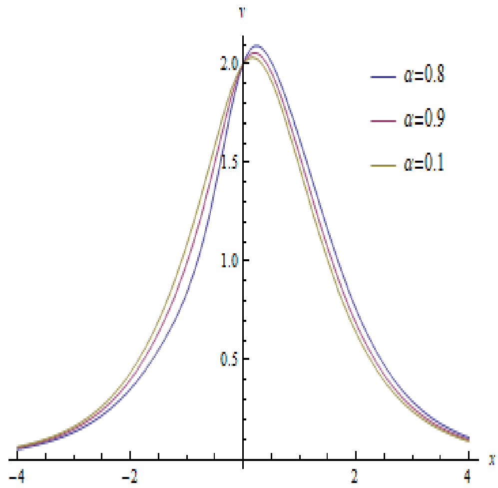

Figure 6. 2D plot of solutions v_2 at $t = 0.1$ for Example 5.

Author Contributions

All authors contributed equally to this work. They all read and approved the final version of the manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

References

[1] Kulish, V. V., Lage, J. L. (2002). Application of fractional calculus to fluid mechanics. J. Fluids Eng., 124(3), 803-806.

[2] Cruz Duarte, J.M., Rosales Garcia, J., Correa Cely, C.R., Garcia Perez, A., Avina Cervantes, J.G. (2018). A closed form expression for the Gaussian-based Caputo-Fabrizio fractional derivative for signal processing applications. Communications in Nonlinear Science and Numerical Simulation, 61, 138-148.

[3] He, J.H., Ji, F.Y. (2019). Two-scale mathematics and fractional calculus for thermodynamics. Thermal Science, 23(4), 2131-2133.

[4] Landman, K. A., Pettet, G. J., Newgreen, D. F. (2003). Mathematical models of cell colonization of uniformly growing domains. Bulletin of mathematical biology, 65(2), 235-262.

[5] Tarasov, V. E. (2019). On history of mathematical economics: Application of fractional calculus. Mathematics, 7(6), 509.

[6] Ionescu, C., Kelly, J.F. (2017). Fractional calculus for respiratory mechanics: Power law impedance, viscoelasticity, and tissue heterogeneity. Chaos, Solitons & Fractals, 102, 433-440.

[7] Tufenkci, S., Senol, B., Alagoz, B. B. (2019, September). Disturbance Rejection Fractional Order PID Controller Design in v-domain by Particle Swarm Optimization. In 2019 International Artificial Intelligence and Data Processing Symposium (IDAP) (pp. 1-6). IEEE.

[8] Yavuz, M., Ozdemir, N., Okur, Y. Y. (2016, July). Generalized differential transform method for fractional partial differential equation from finance. In Proceedings, International Conference on Fractional Dif-

ferentiation and its Applications, Novi Sad, Serbia (pp. 778-785).

[9] Atangana, A., Owolabi, K.M. (2018). New numerical approach for fractional differential equations. Mathematical Modelling of Natural Phenomena, 13(1), 3.

[10] Kurt, A., Tasbozan, O., Baleanu, D. (2017). New solutions for conformable fractional Nizhnik-Novikov-Veselov system via G'/G G'/G expansion method and homotopy analysis methods. Optical and Quantum Electronics, 49, 1-16.

[11] Akinyemi, L. (2019). q-Homotopy analysis method for solving the seventh-order time-fractional Laxâs Korteweg-de Vries and Sawada-Kotera equations. Computational and Applied Mathematics, 38(4), 191.

[12] Akinyemi, L., Iyiola, O. S. (2020). Exact and approximate solutions of time-fractional models arising from physics via Shehu transform. Mathematical Methods in the Applied Sciences, 43(12), 7442-7464.

[13] Senol, M., Atpinar, S., Zararsiz, Z., Salahshour, S., Ahmadian, A. (2019). Approximate solution of time-fractional fuzzy partial differential equations. Computational and Applied Mathematics, 38(1), 1-18.

[14] Hammouch, Z., Mekkaoui, T. (2012). Adomian decomposition method for solving a time-fractional Burger-Huxley's equation. Nonlinear Studies, 19(3).

[15] Rezazadeh, H., Vahidi, J., Zafar, A., Bekir, A. (2019). The Functional Variable Method to Find New Exact Solutions of the Nonlinear Evolution Equations with Dual-Power-Law Nonlinearity. International Journal of Nonlinear Sciences and Numerical Simulation. DOI: https://doi.org/10.1515/ijnsns-2019-0064.

[16] Korkmaz, A., Hepson, O. E., Hosseini, K., Rezazadeh, H., Eslami, M. (2018). Sine-Gordon expansion method for exact solutions to conformable time fractional equations in RLW-class. Journal of King Saud University-Science. 32 (1), 567-574.

[17] Park, C., Khater, M.M., Abdel-Aty, A.H., Attia, R.A., Rezazadeh, H., Zidan, A.M., Mohamed, A.B. (2020). Dynamical analysis of the nonlinear complex fractional emerging telecommunication model with higherâorder dispersive cubic-quintic. Alexandria Engineering Journal, 59(3), 1425-1433.

[18] Raza, N., Afzal, U., Butt, A. R., Rezazadeh, H. (2019). Optical solitons in nematic liquid crystals with Kerr and parabolic law nonlinearities. Optical and Quantum Electronics, 51(4), 107.

[19] Mirhosseini-Alizamini, S. M., Rezazadeh, H., Srinivasa, K., Bekir, A. (2020). New closed form solutions of the new coupled Konno-Oono equation using the new extended direct algebraic method. Pramana, 94, 1-12.

[20] Biswas, A., Rezazadeh, H., Mirzazadeh, M., Eslami, M., Zhou, Q., Moshokoa, S. P., & Belic, M. (2018). Optical solitons having weak non-local nonlinearity by two integration schemes. Optik, 164, 380-384.

[21] Arqub, O. A. (2013). Series solution of fuzzy differential equations under strongly generalized differentiability. Journal of Advanced Research in Applied Mathematics, 5(1), 31-52.

[22] Arqub, O. A., El-Ajou, A., Bataineh, A. S., Hashim, I. (2013, January). A representation of the exact solution of generalized Lane-Emden equations using a new analytical method. In Abstract and Applied Analysis (Vol. 2013). Hindawi.

[23] El-Ajou, A., Arqub, O. A., Zhour, Z. A., Momani, S. (2013). New results on fractional power series: theories and applications. Entropy, 15(12), 5305-5323.

[24] Abdeljawad, T. (2015). On conformable fractional calculus. Journal of computational and Applied Mathematics, 279, 57-66.

[25] Senol, M., Tasbozan, O., Kurt, A. (2019). Numerical solutions of fractional Burgers' type equations with conformable derivative. Chinese Journal of Physics, 58, 75-84.

[26] Tariq, H., Akram, G. (2017). New traveling wave exact and approximate solutions for the nonlinear Cahn-Allen equation: evolution of a nonconserved quantity. Nonlinear Dynamics, 88, 581-594.

[27] Tariq, H., Sadaf, M., Akram, G., Rezazadeh, H., Baili, J., Lv, Y.P., Ahmad, H. (2021). Computational study for the conformable nonlinear Schrödinger equation with cubic-quintic-septic nonlinearities. Results in Physics, 30, 104839.

[28] Tariq, H., Günerhan, H., Rezazadeh, H., Adel, W. (2021). A numerical approach for the nonlinear temporal conformable fractional foam drainage equation. Asian-European Journal of Mathematics, 14(06), 2150089.

[29] Tariq, H., Akram, G. (2017). Residual power series method for solving time-space-fractional Benney-Lin equation arising in falling film problems. Journal of Applied Mathematics and Computing, 55(1), 683-708.

[30] Jaradat, H. M., Al-Shara, S., Khan, Q. J., Alquran, M., Al-Khaled, K. (2016). Analytical solution of time-fractional Drinfeld-Sokolov-Wilson system using residual power series method. IAENG International Journal of Applied Mathematics, 46(1), 64-70.