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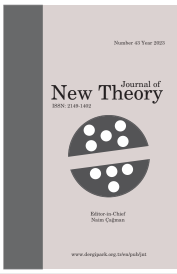

Homoderivations in Prime Rings

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Abstract — The study consists of two parts. The first part shows that if $h_1(x)h_2(y) = h_3(x)h_4(y)$, for all $x, y \in R$, then $h_1 = h_3$ and $h_2 = h_4$. Here, h_1, h_2, h_3, and h_4 are zero-power valued non-zero homoderivations of a prime ring R. Moreover, this study provide an explanation related to h_1 and h_2 satisfying the condition $ah_1 + h_2b = 0$. The second part shows that $L \subseteq Z$ if one of the following conditions is satisfied: $i.\, h(L) = (0)$, $ii.\, h(L) \subseteq Z$, $iii.\, h(xy) = xy$, for all $x, y \in L$, $iv.\, h(xy) = yx$, for all $x, y \in L$, or $v.\, h([x, y]) = 0$, and for all $x, y \in L$. Here, R is a prime ring with a characteristic other than 2, h is a homoderivation of R, and L is a non-zero square closed Lie ideal of R.

Keywords *Prime rings, Lie ideals, homoderivations*

Mathematics Subject Classification (2020) 16N60, 16W25

1. Introduction

Throughout this article, unless otherwise specified, R denotes an associative prime ring, i.e., for all $a, b \in R$, $aRb = 0$ implies $a = 0$ or $b = 0$, with the maximal left ring of quotients $Q = Q_{ml}(R)$. It is well known that R is a subring of Q, Q is a prime ring, and the center C of Q is a field and called the extended centroid of R [1]. Z denotes the center of R, and the notation $\mathrm{Char}(R)$ represents the characteristic of R. For all $a, b \in R$, let $[a, b] := ab - ba$, the Lie commutator of a and b. For a subset A of R, $C_R(A)$ means the centralizer of A and defined by $C_R(A) = \{x \in R \mid [x, a] = 0, \text{ for all } a \in A\}$. If L is an additive subgroup of R and $[x, r] \in L$, for all $x \in L$ and $r \in R$, then L is referred to as a Lie ideal of R. If L is a Lie ideal of R and $x^2 \in L$, for all $x \in L$, then L is called a square closed Lie ideal. Since $(x + y)^2 \in L$ and $[x, y] \in L$, for all $x, y \in L$, then $2xy \in L$. Let $\varnothing \neq S \subseteq R$. A mapping $f : R \to R$ is called zero-power valued on S, if $f(S) \subseteq S$, and, for all $s \in S$, there exists a positive integer $n(s) > 1$ such that $f^{n(s)}(s) = 0$. An additive map $d : R \to R$ is called derivation if $d(xy) = d(x)y + xd(y)$, for all $x, y \in R$. Especially, I_a, defined by $I_a(x) := [a, x]$, for all $x \in R$, is an inner derivation induced by an element $a \in R$.

In [2], El Sofy Aly has introduced a new mapping created by combining the concepts of homomorphisms and derivations on rings. An additive mapping $h : R \to R$ is called a homoderivation if

$$h(xy) = h(x)h(y) + h(x)y + xh(y)$$

for all $x, y \in R$. The only additive mapping, both a derivation and a homoderivation on a prime ring, is the zero map. Some examples of homoderivations are as follows:

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Example 1.1. [2] Let R be a ring and f be an endomorphism of R. Then, the mapping $h : R \to R$ defined by $h(x) = f(x) - x$ is a homoderivation of R.

Example 1.2. [2] Let R be a ring. Then, the additive mapping $h : R \to R$ defined by $h(x) = -x$ is a homoderivation of R.

Example 1.3. [2] Let $R = \mathbb{Z}\left(\sqrt{2}\right)$, a ring of all the real numbers of the form $m + n\sqrt{2}$ such that $m, n \in \mathbb{Z}$, the set of all the integers, under the usual addition and multiplication of real numbers. Then, the map $d : R \to R$ defined by $d(m + n\sqrt{2}) = -2n\sqrt{2}$ is a homoderivation of R.

In 2016, Melaibari et al. [3] have proved the commutativity of a prime ring R admitting a non-zero homoderivation h that satisfies any one of the conditions: *i.* $[x, y] = [h(x), h(y)]$, for all $x, y \in U$, a non-zero ideal of R, *ii.* $h([x, y]) = 0$, for all $x, y \in U$, a non-zero ideal of R, or *iii.* $h([x, y]) \in Z$, for all $x, y \in R$. Alharfie et al. [4] have shown that the commutativity of a prime ring R if any of the following conditions is satisfied: for all $x, y \in I$, *i.* $xh(y) \pm xy \in Z(R)$, *ii.* $xh(y) \pm yx \in Z(R)$, or *iii.* $xh(y) \pm [x, y] \in Z(R)$. Here, I is a non-zero left ideal of R, and h is a homoderivation of R. In 2019, Al Harfien et al. [5] and Rehman et al. [6] have studied the commutativity of a semiprime (prime) ring admitting a homoderivation satisfying some identities on a ring. Researchers [7–14] have executed many noteworthy works concerning various properties of homoderivations during the last decades.

In Theorem 1.4, Bresar [16] has indicated that derivations d, f, g, and h of a prime ring R satisfying the condition $d(x)g(y) = h(x)f(y)$, for all $x, y \in R$, are $C-$dependent. In other words, g and f and h and d are $C-$dependent. In Teorem 1.5, the author has indicated that derivations g and h of a prime ring R satisfying the condition $ag(x) + h(x)b = 0$, for all $x, y \in R$, are $C-$dependent. That is, g and I_b and h and I_a are $C-$dependent. Motivated by the results of Bresar, we create Section 3 of this study. In the section, we research the results of Bresar by homoderivations. We show that homoderivations h_1, h_2, h_3, and h_4 of a prime ring R satisfying the condition $h_1(x)h_2(y) = h_3(x)h_4(y)$, for all $x, y \in R$, are $1-$dependent such that $1 \in C$. That is, $h_1 = h_3$ and $h_2 = h_4$ where $h_{1|_Z} \neq 0$ or $h_{2|_Z} \neq 0$ such that $h_{1|_Z}, h_{2|_Z} : Z \to \mathbb{R}$ are two mapping defined by $h_{1|_Z}(x) := h_1(x)$ and $h_{2|_Z}(x) := h_2(x)$, respectively. In addition, we prove that $a = -b \in Z$, for homoderivations h_1 and h_2 of a prime ring R satisfying the condition $ah_1(x) + h_2(x)b = 0$, for all $x \in R$.

Theorem 1.4. [16] Let R be a prime ring, and d, f, g, and h be derivations of R. Suppose that $d(x)g(y) = h(x)f(y)$, for all $x, y \in R$. If $d \neq 0$ and $f \neq 0$, then there exists a $\lambda \in C$ such that $g(x) = \lambda f(x)$ and $h(x) = \lambda d(x)$, for all $x \in R$

Theorem 1.5. [16] Let R be a prime ring, and g and h be derivations of R. Suppose that there exist $a, b \in R$ such that $ag(x) + h(x)b = 0$, for all $x \in R$. If $a \notin Z$ and $b \notin Z$, then there exists a $\lambda \in C$ such that $g(x) = [\lambda b, x]$ and $h(x) = [\lambda a, x]$, for all $x \in R$. Moreover, if $g \neq 0$, then $ab \in Z$.

The purpose of Section 3 is to prove the following two results:

• Let R be a prime ring and h_1, h_2, h_3, and h_4 be zero-power valued non-zero homoderivations on R. Suppose that $h_1(x)h_2(y) = h_3(x)h_4(y)$, for all $x, y \in R$. If $h_{1|_Z} \neq 0$, then $h_1 = h_3$ and $h_2 = h_4$. Moreover, $h_{1|_Z} = 0$ if and only if (iff) $h_{3|_Z} = 0$. Similarly, If $h_{2|_Z} \neq 0$, then $h_1 = h_3$ and $h_2 = h_4$. Moreover, $h_{2|_Z} = 0$ iff $h_{4|_Z} = 0$.

• Let R be a prime ring and h_1 and h_2 be zero-power valued non-zero homoderivations on R. Suppose that there are $a, b \in R$ such that $ah_1(x) + h_2(x)b = 0$, for all $x \in R$. Then, $a = -b \in Z$ or $h_{1|_Z} = h_{2|_Z} = 0$.

In Lemma 5 and 6 provided in [15], Bergen et al. have showed that a Lie ideal U of a prime ring R such that $\text{Char}(R) \neq 2$ with derivation d satisfying the condition $d(U) = 0$ or $d(U) \subseteq Z$ is central.

One of our motivations for Section 4 is this result. In this paper, we investigate the hypothesis of this result using homoderivations and provide similar results. Another purpose of Section 4 is to generalize some of the well-known results above using square closed Lie ideals of a prime ring.

The purpose of Section 4 is to prove $L \subseteq Z$ if one of the following conditions is satisfied:

i. $h(L) = (0)$,

ii. $h(L) \subseteq Z$,

iii. $h(xy) = xy$, for all $x, y \in L$,

iv. $h(xy) = yx$, for all $x, y \in L$, or

v. $h([x, y]) = 0$, for all $x, y \in L$

Here, R is a prime ring with a $\mathrm{Char}(R) \neq 2$, h is a homoderivation of R and L is a non-zero square closed Lie ideal of R:

Section 2 of the present study provides some properties on commutativity of prime rings. Section 3 investigates the identity $ah_1(x) + h_2(x)b = 0$ on prime rings such that h_1 and h_2 are two homoderivations on R. Section 4 studies commutativity of a prime ring by square closed Lie ideals and homoderivations. Final section discusses the need for further research.

2. Preliminary

This section uses the following basic identities: $[xy, z] = x[y, z] + [x, z]y$ and $[x, yz] = y[x, z] + [x, y]z$, for any $x, y, z \in R$.

Theorem 2.1. [17] Let R be a prime ring whose characteristic is not 2 and d_1 and d_2 derivations of R such that the iterate d_1d_2 is also a derivation, then at least one of d_1 and d_2 is zero.

Lemma 2.2. [15] Let R be a prime ring whose characteristic is not 2. If $U \not\subseteq Z$ is a Lie ideal of R, then $C_R(U) = Z$.

Lemma 2.3. [15] Let R be a prime ring whose characteristic is not 2. If $U \not\subseteq Z$ is a Lie ideal of R and $aUb = 0$, then $a = 0$ or $b = 0$.

Lemma 2.4. [18] If a prime ring R contains a commutative non-zero right ideal I, then R is commutative.

Lemma 2.5. [18] Let b and ab be in the center of a prime ring R. If b is not zero, then $a \in Z$.

Lemma 2.6. [3] Let R be a ring and h be a zero-power valued homoderivation on R. Then, h preserves Z.

Lemma 2.7. Let R be a prime ring. If h is a zero-power valued non-zero homoderivation on R such that $h(x) \in Z$, for all $x \in R$, then R is commutative or

$$h_{|Z} = 0 \quad \text{and} \quad h(xz) = h(x)z \ \ (h(zx) = zh(x)), \text{ for all } x \in R \text{ and } z \in Z$$

PROOF.
Let R be a prime ring and h be a zero-power valued non-zero homoderivation on R such that $h(R) \subseteq Z$. By hypothesis, $h(x_1x_2) \in Z$, for all $x_1, x_2 \in R$. Since Z is a subring of R and h is homoderivation of R, then

$$h(x_1)x_2 + x_1h(x_2) \in Z \tag{1}$$

Replacing x_2 by x_2z such that $z \in Z$, then the following expression is obtained by Expression 1,

$$x_1(h(x_2) + x_2)h(z) \in Z \tag{2}$$

Since h is zero-power valued on R, there exists an integer $n(x_2) > 1$ such that $h^{n(x_2)}(x_2) = 0$, for all $x_2 \in R$. Replacing x_2 by $x_2 - h(x_2) + h^2(x_2) + \cdots + (-1)^{n(x_2)-1}h^{n(x_2)-1}(x_2)$ in Expression 2, for all $x_1, x_2 \in R$ and $z \in Z$,

$$x_1x_2h(z) \in Z$$

In view of Lemma 2.5, we have $x_1x_2 \in Z$ or $h(z) = 0$, for all $x_1, x_2 \in R$ and $z \in Z$. Here, there are two cases:

Case 1: If $x_1x_2 \in Z$, for all $x_1, x_2 \in R$, then $(x_1x_2)x_3 \in Z$, for all $x_3 \in R$. Hence, $[(x_1x_2)x_3, x_4] = 0$, for all $x_4 \in R$. That is, $[(x_1x_2), x_4]\, x_3 + x_1x_2\, [x_3, x_4] = 0$ and thus

$$x_1x_2\, [x_3, x_4] = 0, \text{ for all } x_1, x_2, x_3, x_4 \in R$$

It follows from the fact that R is a prime ring that R is commutative.

Case 2: If $h(z) = 0$, for all $z \in Z$, then $h_{|_Z} = 0$. In this case, for all $x_1 \in R$ and $z \in Z$,

$$h(x_1z) = h(x_1)z \quad (h(zx_1) = zh(x_1))$$

is obtained. □

3. The Identity $ah_1(x) + h_2(x)b = 0$

In this section, unless stated otherwise, let R be a prime ring.

Theorem 3.1. Let h_1, h_2, h_3, and h_4 be zero-power valued non-zero homoderivations on R. Suppose that

$$h_1(x_1)h_2(x_2) = h_3(x_1)h_4(x_2), \text{ for all } x_1, x_2 \in R \tag{3}$$

i. If $h_{1_{|Z}} \neq 0$, then $h_1 = h_3$ and $h_2 = h_4$.

ii. $h_{1_{|Z}} = 0$ iff $h_{3_{|Z}} = 0$

iii. If $h_{2_{|Z}} \neq 0$, then $h_1 = h_3$ and $h_2 = h_4$.

iv. $h_{2_{|Z}} = 0$ iff $h_{4_{|Z}} = 0$

Proof.
Let h_1, h_2, h_3, and h_4 be zero-power valued non-zero homoderivations on R. Suppose that

$$h_1(x_1)h_2(x_2) = h_3(x_1)h_4(x_2), \text{ for all } x_1, x_2 \in R$$

i. Let $h_{1_{|Z}} \neq 0$. There is at least $0 \neq z \in Z$ such that $h_1(z) \neq 0$. By Lemma 2.6, it is clear that $h_1(z) \in Z$. In Expression 3, by replacing x_1 by x_1z, for $x_1 \in R$,

$$h_1(x_1z)h_2(x_2) = h_3(x_1z)h_4(x_2)$$

Thus,

$$h_1(x_1)h_1(z)h_2(x_2) + h_1(x_1)zh_2(x_2) + x_1h_1(z)h_2(x_2) = h_3(x_1)h_3(z)h_4(x_2) + h_3(x_1)zh_4(x_2) + x_1h_3(z)h_4(x_2)$$

From the last equation,for all $x_1, x_2 \in R$, the equation

$$h_1(x_1)h_1(z)h_2(x_2) = h_3(x_1)h_3(z)h_4(x_2)$$

is obtained. In view of hypothesis, for all $x_2 \in R$,

$$h_1(z)h_2(x_2) = h_3(z)h_4(x_2)$$

Using the last equation in the equation $h_1(x_1)h_1(z)h_2(x_2) = h_3(x_1)h_3(z)h_4(x_2)$,

$$(h_1(x_1) - h_3(x_1))h_1(z)h_2(x_2) = 0, \text{ for all } x_1, x_2 \in R$$

The primeness of R and $0 \neq h_1(z) \in Z$ imply that

$$(h_1(x_1) - h_3(x_1))h_2(x_2) = 0, \text{ for all } x_1, x_2 \in R \tag{4}$$

In Expression 4, replacing x_2 by x_2x_3 such that $x_3 \in R$ and using Expression 4,

$$(h_1(x_1) - h_3(x_1))x_2h_2(x_3) = 0$$

for all $x_1, x_2, x_3 \in R$. Since R is a prime ring and h_2 is a non-zero homoderivation of R, then $h_1(x_1) = h_3(x_1)$, for all $x_1 \in R$. In that case, by hypothesis, $h_1(x_1)h_2(x_2) = h_1(x_1)h_4(x_2)$ for all $x_1, x_2 \in R$. That is,

$$h_1(x_1)(h_2(x_2) - h_4(x_2)) = 0, \text{ for all } x_1, x_2 \in R \tag{5}$$

In Expression 5, replacing x_1 by x_1x_3, $x_3 \in R$, and using Expression 5,

$$h_1(x_1)x_3(h_2(x_2) - h_4(x_2)) = 0, \text{ for all } x_1, x_2, x_3 \in R$$

Since R is a prime ring and h_1 is a non-zero homoderivation of R, then $h_2(x_2) = h_4(x_2)$, for all $x_2 \in R$.

ii. ($\Rightarrow$): Let $h_{1|_Z} = 0$. In Expression 3, replacing x_1 by $z \in Z$ for and using $h_1(z) = 0$,

$$h_3(z)h_4(x_2) = 0, \text{ for all } x_2 \in R$$

In this equation, replacing x_2 by x_3x_2 for $x_3 \in R$ and using the hypothesis,

$$h_3(z)x_3h_4(x_2) = 0, \text{ for all } x_2, x_3 \in R$$

The primeness of R implies $h_{3|_Z} = 0$. Thus, if $h_{1|_Z} = 0$, then $h_{3|_Z} = 0$.

($\Leftarrow$): Let $h_{3|_Z} = 0$. With similar steps above, $h_{1|_Z} = 0$ is obtained. Hence, if $h_{3|_Z} = 0$, then $h_{1|_Z} = 0$.

The proofs of *iii.* and *iv.* are similar to *i.* and *ii.*, respectively. □

Example 3.2. Let $\Re$ be a ring with the unit and no zero divisors. For the subring

$$\wp = \{r_{11}e_{11} + r_{12}e_{12} + r_{22}e_{22} : r_{11}, r_{12}, r_{22} \in \Re\}$$

of $M_2(\Re)$, the ring of 2×2 matrices over $\Re$, it is easy to validate that $\wp$ is not a prime ring. Here,

$$e_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad e_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad e_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Moreover, $Z_\wp = \{ze_{11} + ze_{22} : z \in Z_\Re\}$ is the center of ring $\wp$. Let

$$\begin{array}{rccl} h_1 : & \wp & \to & \wp \\ & r_{11}e_{11} + r_{12}e_{12} + r_{22}e_{22} & \to & -r_{11}e_{11} - r_{12}e_{12} \end{array}$$

and

$$\begin{array}{rccl} h_2 : & \wp & \to & \wp \\ & r_{11}e_{11} + r_{12}e_{12} + r_{22}e_{22} & \to & -r_{12}e_{12} - r_{22}e_{22} \end{array}$$

Then, it is easy to check that h_1 and h_2 are homoderivations of $\wp$. Let $\Im = \wp \times \wp$. It is easy to validate that $\Im$ is not a prime ring. Besides,

$$Z_\Im = \{(z_{11}e_{11} + z_{22}e_{22}, \alpha_{11}e_{11} + \alpha_{22}e_{22}) : z_{11}, z_{22}, \alpha_{11}, \alpha_{22} \in Z_\Re\}$$

is the center of ring $\Im$. Let $X = (r_{11}e_{11} + r_{12}e_{12} + r_{22}e_{22}, s_{11}e_{11} + s_{12}e_{12} + s_{22}e_{22}) \in \Im$ and $Y = (x_{11}e_{11} + x_{12}e_{12} + x_{22}e_{22}, y_{11}e_{11} + y_{12}e_{12} + y_{22}e_{22}) \in \Im$. Define the maps $H_1, H_2, H_3, H_4 : \Im \to \Im$ as follows:

$$\begin{array}{rcl} H_1 : \Im & \to & \Im \\ X & \to & (h_1(r_{11}e_{11} + r_{12}e_{12} + r_{22}e_{22}), 0_\Im) = (-r_{11}e_{11} - r_{12}e_{12}, 0_\Im) \end{array}$$

$$\begin{aligned} H_2 \ : \ \Im &\to \Im \\ X &\to (0_\Im, h_1(s_{11}e_{11} + s_{12}e_{12} + s_{22}e_{22})) = (0_\Im, -s_{11}e_{11} - s_{12}e_{12}) \end{aligned}$$

$$\begin{aligned} H_3 \ : \ \Im &\to \Im \\ X &\to (h_2(r_{11}e_{11} + r_{12}e_{12} + r_{22}e_{22}), 0_\Im) = (-r_{12}e_{12} - r_{22}e_{22}, 0_\Im) \end{aligned}$$

and

$$\begin{aligned} H_4 \ : \ \Im &\to \Im \\ X &\to (0_\Im, h_2(s_{11}e_{11} + s_{12}e_{12} + s_{22}e_{22})) = (0_\Im, -s_{12}e_{12} - s_{22}e_{22}) \end{aligned}$$

Then, it is easy to check that H_1, H_2, H_3, and H_4 are homoderivations of $\Im$. For any two elements $X, Y \in \Im$,

$$H_1(X)H_2(Y) = H_3(X)H_4(Y)$$

However, neither

$$H_1 = H_3 \quad \text{and } H_2 = H_4$$

nor

$$H_{1|_{Z_\Im}} = 0, \quad H_{2|_{Z_\Im}} = 0, \quad H_{3|_{Z_\Im}} = 0, \quad \text{and} \quad H_{4|_{Z_\Im}} = 0$$

Hence, this example shows that it is crucial that the considered ring is a prime ring and the selected homoderivations are zero-power valued, as stated in Theorem 3.1.

Note 3.3. From Theorem 3.1, it can be observed that the statements "If $h_{3|_Z} \neq 0$, then $h_1 = h_3$ and $h_2 = h_4$" and "If $h_{4|_Z} \neq 0$, then $h_1 = h_3$ and $h_2 = h_4$" are valid.

From Theorem 3.1, the following corollaries are obtained.

Corollary 3.4. Let h_1 and h_2 be zero-power valued non-zero homoderivations on R satisfying the condition

$$h_1(x)h_1(y) = h_2(x)h_2(y), \text{ for all } x, y \in R$$

Then, $h_1 = h_2$ or $h_{1|_Z} = h_{2|_Z} = 0$.

Corollary 3.5. Let h_1 and h_2 be zero-power valued non-zero homoderivations on R. Suppose that

$$h_1(x)h_2(y) = h_2(x)h_1(y), \text{ for all } x, y \in R$$

Then, $h_1 = h_2$ or $h_{1|_Z} = h_{2|_Z} = 0$.

Theorem 3.6. Let h_1 and h_2 be zero-power valued non-zero homoderivations on R. Suppose that there are $a, b \in R$ such that

$$ah_1(x) + h_2(x)b = 0, \text{ for all } x \in R \tag{6}$$

Then, $a = -b \in Z$ or $h_{1|_Z} = h_{2|_Z} = 0$.

PROOF.
Let h_1 and h_2 be zero-power valued non-zero homoderivations on R. Suppose that there are $a, b \in R$ such that

$$ah_1(x) + h_2(x)b = 0, \text{ for all } x \in R$$

If $a = b = 0$, then the proof is clear. From now on, $a \neq 0$ and $b \neq 0$. Suppose that $h_{1|_Z} = 0$. In Expression 6, replacing x by z for $z \in Z$,

$$ah_1(z) + h_2(z)b = 0$$

Since $h_{1|_Z} = 0$, then $h_2(z)b = 0$. This means that $h_2(z) = 0$, for all $z \in Z$, by the primeness of R. With the same arguments above, it can be shown that if $h_{2|_Z} = 0$, then $h_{1|_Z} = 0$. Assume that

$h_{1|_Z} \neq 0$. In light of Lemma 2.6 and $h_{1|_Z} \neq 0$, there is at least $0 \neq z_1 \in Z$ such that $0 \neq h_1(z_1) \in Z$ and $h_2(z_1) \in Z$. Replacing x by xz_1 in Expression 6,

$$0 = ah_1(x)h_1(z_1) + ah_1(x)z_1 + axh_1(z_1) + h_2(x)h_2(z_1)b + h_2(x)z_1b + xh_2(z_1)b$$

Using $z_1, h_1(z_1), h_2(z_1) \in Z$ and Expression 6 in the last equation,

$$(a(h_1(x) + x) - (h_2(x) + x)a)h_1(z_1) = 0$$

Since R is a prime ring and $h_1(z_1) \neq 0$, for all $x \in R$,

$$a(h_1(x) + x) - (h_2(x) + x)a = 0 \tag{7}$$

Since $h_{2|_Z} \neq 0$, there is at least $0 \neq z_2 \in Z$ such that $0 \neq h_2(z_2) \in Z$. In Expression 7, replacing x by z_2,

$$ah_1(z_2) + h_2(z_2)(-a) = 0$$

Combining the last equations and Expression 6,

$$h_2(z_2)(b + a) = 0$$

The primeness of R and $h_2(z_2) \neq 0$ implies $a = -b$. In that case, for any $x \in R$,

$$ah_1(x) - h_2(x)a = 0 \tag{8}$$

In Expression 8, replacing x by xz_1,

$$0 = ah_1(x)h_1(z_1) + axh_1(z_1) - h_2(x)h_2(z_1)a - xh_2(z_1)a$$

According to the last equation and Expression 8,

$$ah_1(x)h_1(z_1) + axh_1(z_1) - ah_1(x)h_1(z_1) - xah_1(z_1) = 0$$

This implies $[a, x]\, h_1(z_1) = 0$, for all $x \in R$. The primeness of R and $h_1(z_1) \neq 0$ implies $a \in Z$. $\square$

Example 3.7. Consider the ring $\wp$ provided in Example 3.2. Let

$$\begin{array}{rccl} h_1 & : & \wp & \rightarrow \ \wp \\ & & r_{11}e_{11} + r_{12}e_{12} + r_{22}e_{22} & \rightarrow \ -r_{11}e_{11} - r_{12}e_{12} - r_{22_{e_{22}}} \end{array}$$

and

$$\begin{array}{rccl} h_2 & : & \wp & \rightarrow \ \wp \\ & & r_{11}e_{11} + r_{12}e_{12} + r_{22}e_{22} & \rightarrow \ -r_{11}e_{12} - r_{12}e_{12} \end{array}$$

Then, it is easy to check that h_1 and h_2 are homoderivations of $\wp$. Let $\alpha = -1_{\Re}e_{11}$ and $\beta = 1_{\Re}e_{11} + 1_{\Re}e_{22}$ be fixed elements. For any element $X = r_{11}e_{11} + r_{12}e_{12} + r_{22}e_{22} \in \wp$,

$$\alpha h_1(X) + h_2(X)\beta = 0_{\wp}$$

However, neither $\alpha = -\beta$ nor $h_{1|_{Z_\wp}} = h_{2|_{Z_\wp}} = 0$. Hence, this examples show that it is crucial that the considered ring is a prime ring and the selected homoderivations are zero-power valued, as stated in Theorem 3.6.

4. Central Lie Ideals of Prime Rings with Homoderivations

In this section, unless stated otherwise, R is a prime ring with $\mathrm{Char}(R) \neq 2$.

Lemma 4.1. Let L be a non-zero Lie ideal of R and h be a non-zero homoderivation of R such that $h(x) = 0$, for all $x \in L$. Then, $L \subseteq Z$.

Proof.
Let L be a non-zero Lie ideal of R and h be a non-zero homoderivation of R such that $h(x) = 0$, for all $x \in L$. Since h is a homoderivations of R,

$$h([x_1, r_1]) = [h(x_1), h(r_1)] + [h(x_1), r_1] + [x_1, h(r_1)], \text{ for all } x_1 \in L, r_1 \in R$$

By hypothesis, $[x_1, h(r_1)] = 0$, for all $x_1 \in L$ and $r_1 \in R$. By taking $r_1 = r_1 x_2$, for any $x_2 \in L$, in the last equation,

$$h(r_1)[x_1, x_2] = 0, \text{ for all } x_1, x_2 \in L, r_1 \in R \tag{9}$$

In Expression 9, replacing r_1 by $r_1 r_2$, $r_2 \in R$,

$$h(r_1) r_2 [x_1, x_2] = 0$$

Hence, $[x_1, x_2] = 0$, for all $x_1, x_2 \in L$, by the primeness of R. By replacing x_2 by $[x_2, r_1]$ in the last equation,

$$[x_1, [x_2, r_1]] = 0, \text{ for all } x_1, x_2 \in L, r_1 \in R \tag{10}$$

Consider two inner derivations of R, $I_{x_1} : R \to R$ and $I_{x_2} : R \to R$ defined by $I_{x_1}(s) = [x_1, s]$ and $I_{x_2}(s) = [x_2, s]$, respectively. Thus, $I_{x_1} I_{x_2}(r_1) = 0$, for all $r_1 \in R$, by Expression 10. In view of Theorem 2.1, $I_{x_1} = 0$ or $I_{x_2} = 0$. That is, $x_1 \in Z$ or $x_2 \in Z$. This prove that $L \subseteq Z$. □

Lemma 4.2. Let L be a non-zero square closed Lie ideal of R and h be a non-zero homoderivation of R such that $h(x) \in Z$, for all $x \in L$. Then, $L \subseteq Z$.

Proof.
Let L be a non-zero square closed Lie ideal of R and h be a non-zero homoderivation of R such that $h(x) \in Z$, for all $x \in L$. By hypothesis for all $x_1 \in L$ and $r_1 \in R$,

$$[h(x_1), r_1] = 0 \tag{11}$$

In Expression 11, by replacing x_1 by x_1^2, $[h(x_1^2), r_1] = 0$. From the last equation, since $h(x_1) \in Z$ and using $\text{Char}(R) \neq 2$,

$$h(x_1)[x_1, r_1] = 0 \tag{12}$$

In Expression 12, substituting $r_1 r_2$ instead of r_1, $r_2 \in R$,

$$h(x_1) r_1 [x_1, r_2] = 0$$

The primeness of R implies $h(x_1) = 0$ or $x_1 \in Z$, for all $x_1 \in L$. Define

$$A = \{x \in L : h(x) = 0\}$$

and

$$B = \{x \in L : x \in Z\}$$

Note that both are additive subgroups of L, and their union equals L. Thus, either $A = L$ or $B = L$. Suppose first that $A = L$. Then, $h(L) = 0$. In view of Lemma 4.1, $L \subseteq Z$. In other case, $x_1 \in Z$, for all $x_1 \in L$. That is $L \subseteq Z$. □

The following example shows that the above result is not true in the types of some other rings. In the example, it is emphasized that the hypothesis primeness of the result provided above is all-important.

Example 4.3. Let R_1 be a non-commutative ring with the unit, no zero divisors, and $\text{Char}(R_1) \neq 2$, and R_2 be a non-commutative ring with the unit, no zero divisors, and $\text{Char}(R_2) \neq 2$. For a fixed $(1_{R_1}, 0_{R_2}), (0_{R_1}, 1_{R_2}) \neq (0_{R_1}, 0_{R_2}) \in R^* = R_1 \times R_2$, it holds that $(1_{R_1}, 0_{R_2}) R^* (0_{R_1}, 1_{R_2}) = (0_{R_1}, 0_{R_2})$. Thus, R^* is not a prime ring. Let $L = Z_{R_1} \times R_2$ such that Z_{R_1} is a the center of R_1. It is easy to verify that L is a subgroup of R^*. For $(z,s_1) \in L$ and $(r,s_2) \in R^*$,

$$[(z,s_1),(r,s_2)] = (zr - rz, s_1s_2 - s_2s_1) \overset{z\in Z_{R_1}}{=} (0_{R_1}, s_1s_2 - s_2s_1) \in L$$

and

$$(z,s_1)(z,s_1) = \left(z^2, s_1s_2\right) \in L$$

Thus, L is a square closed Lie ideal of R^* and $L \not\subseteq Z_{R^*}$. Let

$$\begin{array}{cccc} h & : & R^* & \to & R^* \\ & & (r,s) & \to & (-r, 0_{R_2}) \end{array}$$

Then, it is easy to check that h is a homoderivation of R^*. For any element $(z,s_1) \in L$, $h(z,s_1) \in Z_{R^*}$. However, L is not a central square closed Lie ideal of R^*.

Theorem 4.4. Let L be a non-zero square closed Lie ideal of R and h be a non-zero homoderivation of R such that

$$h(xy) = xy \text{ (or } h(xy) = yx\text{), for all } x,y \in L$$

Then, $L \subseteq Z$.

Proof.
Let L be a non-zero square closed Lie ideal of R and h be a non-zero homoderivation of R such that

$$h(xy) = xy, \text{ for all } x,y \in L$$

Suppose that $L \not\subseteq Z$. Since h is homoderivation of R, for all $x_1,x_2,x_3 \in L$,

$$\begin{aligned} x_1 2(x_2x_3) &= h(x_1 2(x_2x_3)) = 2h(x_1(x_2x_3)) \\ &= 2(h(x_1)h(x_2x_3) + h(x_1)x_2x_3 + x_1h(x_2x_3)) \\ &= 2(h(x_1)x_2x_3 + h(x_1)x_2x_3 + x_1x_2x_3) \end{aligned}$$

This implies $4h(x_1)x_2x_3 = 0$. Since $\text{Char}(R) \neq 2$,

$$h(x_1)x_2x_3 = 0, \text{ for all } x_1,x_2,x_3 \in L \tag{13}$$

In Expression 13, replacing x_2 by $2x_4x_2$ such that $x_4 \in L$ and using $\text{Char}(R) \neq 2$,

$$h(x_1)x_4x_2x_3 = 0 \tag{14}$$

Multiplying Expression 13 by x_4 from the left,

$$x_4h(x_1)x_2x_3 = 0 \tag{15}$$

Combining Expression 14 and Expression 15,

$$[h(x_1), x_4]\, x_2x_3 = 0$$

for all $x_1,x_2,x_3,x_4 \in L$. In view of Lemma 2.3 and $L \neq (0)$, for all $x_1,x_4 \in L$,

$$[h(x_1), x_4] = 0$$

We have proved $h(L) \subseteq C_R(L)$. In this case, $h(L) \subseteq Z$ by Lemma 2.2. In view of Lemma 4.2, $L \subseteq Z$. This is a contradiction. That proves that $L \subseteq Z$.

For the condition $h(xy) = yx$, for all $x,y \in L$, the proof is similar. $\square$

Since every ideal is a square closed Lie ideal, an ideal can be considered instead of a square closed Lie ideal in Theorem 4.4. Thus, Corollary 4.5 is obtained by Lemma 2.4.

Corollary 4.5. Let R be a prime ring with $\mathrm{Char}(R) \neq 2$, I be a non-zero ideal of R, and h be a non-zero homoderivation of R. If one of the following conditions is satisfied, for all $x, y \in I$,

i. $h(xy) = xy$

ii. $h(xy) = yx$

then R is commutative.

Here, it can be observed that Corollary 4.5 without hypothesis "zero-power valued homoderivation on the ideal" is a more general version of Theorem 3 provided in [4].

Theorem 4.6. Let L be a non-zero square closed Lie ideal of R and h be a non-zero homoderivation of R such that

$$h([x, y]) = 0, \text{ for all } x, y \in L \tag{16}$$

Then, $L \subseteq Z$.

Proof.
Let L be a non-zero square closed Lie ideal of R and h be a non-zero homoderivation of R such that

$$h([x, y]) = 0, \text{ for all } x, y \in L$$

Suppose that $L \not\subseteq Z$. Let $x_1, x_2 \in L$. By taking $x = 2x_2x_1$ and $y = x_2$ in Expression 16 and using $\mathrm{Char}(R) \neq 2$,

$$h(x_2)\,[x_1, x_2] = 0 \tag{17}$$

and then replacing x_1 with $2x_1x_3$ such that $x_3 \in L$ in Expression 17 and using $\mathrm{Char}(R) \neq 2$,

$$h(x_2)x_1\,[x_3, x_2] = 0 \tag{18}$$

Let $x_4 \in L$. In Expression 18, replacing x_1 by $2x_4x_1$ and using $\mathrm{Char}(R) \neq 2$,

$$h(x_2)x_4x_1\,[x_3, x_2] = 0 \tag{19}$$

Multiplying Expression 18 by x_4 from the left,

$$x_4h(x_2)x_1\,[x_3, x_2] = 0 \tag{20}$$

By comparing Expression 19 and Expression 20,

$$[h(x_2), x_4]\,x_1\,[x_3, x_2] = 0, \text{ for all } x_1, x_2, x_3, x_4 \in L$$

In view of Lemma 2.3,

$$[h(x_2), x_4] = 0 \quad \text{ or } \quad [x_3, x_2] = 0$$

for all $x_2, x_3, x_4 \in L$. This proves that $h(x_2) \in C_R(L)$ or $[x_3, x_2] = 0$, for all $x_2, x_3 \in L$. Define

$$A = \{x \in L : h(x) \in C_R(L)\}$$

and

$$B = \{x \in L : [y, x] = 0, \text{ for all } y \in L\}$$

Note that both are additive subgroups of L and their union equals L. Thus either $A = L$ or $B = L$. Suppose first that $A = L$. Then, $h(x_2) \in C_R(L)$, for all $x_2 \in L$. Moreover, by Lemma 2.2, $h(x_2) \in Z$, for all $x_2 \in L$. In view of Lemma 4.2, $L \subseteq Z$, a contradiction. Suppose that $B = L$. Then, $[x_3, x_2] = 0$, for all $x_2, x_3 \in L$. Let $r \in R$ and fix $x_2, x_3 \in L$. By replacing x_2 by $[x_2, r]$ in $[x_3, x_2] = 0$,

$$[x_3, [x_2, r]] = 0$$

Using similar techniques after Expression 10, $L \subseteq Z$, a contradiction. That proves that $L \subseteq Z$. $\square$

5. Conclusion

In this paper, Section 3 discussed algebraic identities including homoderivations on a prime ring. Section 4 also investigated algebraic identities involving homoderivations on a square closed Lie ideal of a prime ring. It proved that a square closed Lie ideal, satisfying the identities discussed in the section, is contained in the center of a prime ring. The obtained results extended several well-known results in the literature. In future studies, the hypotheses in this study can be studied using a semiprime ring and an ideal of a semiprime ring.

Author Contributions

All the authors equally contributed to this work. This paper is derived from the first author's doctoral dissertation supervised by the second author. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

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