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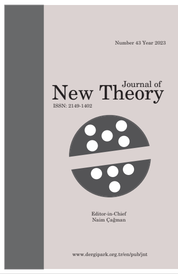

Cofinitely Goldie*-Supplemented Modules

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Research Article

Abstract — One of the generalizations of supplemented modules is the Goldie*-supplemented module, defined by Birkenmeier et al. using β^* relation. In this work, we deal with the concept of the cofinitely Goldie*-supplemented modules as a version of Goldie*-supplemented module. A left R-module M is called a cofinitely Goldie*-supplemented module if there is a supplement submodule S of M with $C\beta^*S$, for each cofinite submodule C of M. Evidently, Goldie*-supplemented are cofinitely Goldie*-supplemented. Further, if M is cofinitely Goldie*-supplemented, then M/C is cofinitely Goldie*-supplemented, for any submodule C of M. If A and B are cofinitely Goldie*-supplemented with $M = A \oplus B$, then M is cofinitely Goldie*-supplemented. Additionally, we investigate some properties of the cofinitely Goldie*-supplemented module and compare this module with supplemented and Goldie*-supplemented modules.

Keywords *Cofinitely supplemented module, Goldie*-supplemented module, cofinitely Goldie*-supplemented module*

Mathematics Subject Classification (2020) 16D10, 16D99

1. Introduction

Cofinitely supplemented modules were introduced by Alizade et al. [1] and Smith [2]. Following these works, various generalizations of cofinitely supplemented modules, such as totally cofinitely supplemented [3], cofinitely weak supplemented [4], an H-cofinitely supplemented [5, 6] and cofinitely weak rad-supplemented [7] were studied. The Goldie*-supplemented modules were introduced and characterized in [8, 9]. A left module M is called a Goldie*-supplemented module (or concisely, $\mathcal{G}^*$s module) if there is a supplement submodule S of M with $C\beta^*S$, for each submodule C of M. Furthermore, the authors [8, 9] stated that Goldie*-supplemented modules ($\mathcal{G}^*$s) are located between amply supplemented and supplemented. Afterward, a new equivalence relation β^{**} was defined, inspired by β^* relation, and the properties of the equivalence relation β^{**} were analyzed in [10]. The relation β^{**} has helped to describe two concepts, namely Goldie-rad-supplemented and amply (weakly) Goldie-rad-supplemented modules. After presenting the relation β^{**}, Talebi et al. [10] characterized Goldie-rad-supplemented modules as a perspective of H-supplemented modules. This module corresponds to rad-H-supplemented modules. Meanwhile, another version of the Goldie-rad-supplemented modules, called amply (weakly) Goldie-rad-supplemented modules, were developed based on the relation β^{**} [11]. It was shown that an amply (weakly) Goldie-rad-supplemented module is a (weakly) Goldie-rad-supplemented [11]. Inspired by these works, we concentrate on cofinitely Goldie*-supplemented modules as a generalization of $\mathcal{G}^*$s modules. A module M is called a cofinitely Goldie*-supplemented

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module (or concisely, c$\mathcal{G}^*$s module) if there is a supplement submodule S of M with $C\beta^* S$, for each cofinite submodule C of M, equivalently, $C+S/C$ is small in M/C, and $C+S/S$ is small in M/S. This definition is closely related to the concept of H-cofinitely supplemented. A module M is called H-cofinitely supplemented if, for each cofinite submodule C of M, there exists a direct summand D of M such that $C+D/C$ is small in M/C, and $C+D/D$ is small in M/D. Clearly, H-cofinitely supplemented is c$\mathcal{G}^*$s. We provide an example to show that the converse implication does not hold. However, if M is refinable, then H-cofinitely supplemented and c$\mathcal{G}^*$s coincide. Therefore, c$\mathcal{G}^*$s modules are situated between H-cofinitely supplemented and cofinitely weak supplemented. Moreover, we observe that if M is c$\mathcal{G}^*$s, then M/C is c$\mathcal{G}^*$s, for any submodule C of M. In addition, we provide that the cofinite direct summand of c$\mathcal{G}^*$s is c$\mathcal{G}^*$s. We investigate the relations between c$\mathcal{G}^*$s, $\mathcal{G}^*$s, and cofinitely supplemented modules under some restrictions.

Section 2 of the handled study presents some basic definitions and properties. Section 3 studies cofinitely Goldie*-supplemented modules. Final section discusses the need for further research.

2. Preliminaries

This section provides some essential definitions to be needed for the following sections. Throughout this paper, let M be an unital left module over an associative unital ring R and $Rad(M)$ be a Jacobson radical of M.

Definition 2.1. [12] Let A be a submodule of M. If $A + B \neq M$, for every proper submodule B of M, A is called superfluous (or small) in M and denoted by $A \ll M$.

Lemma 2.2. [13] Let A, B be submodules of M.

i. If $A \subseteq B \subseteq M$, then $B \ll M$ if and only if $A \ll M$ and $B/A \ll M/A$.

ii. If $A \subseteq B \subseteq M$ and $A \ll B$, then $A \ll M$. Moreover, if B is a direct summand in M and $A \ll M$, then $A \ll B$.

iii. For $A \ll M$, if $f : M \to N$, then $f(A) \ll N$. If f is a small epimorphism, the converse is also true.

Definition 2.3. [13] A submodule A of M is called a (weak) supplement of B in M if $A + B = M$ and $A \cap B \ll A$ ($A \cap B \ll M$), for some submodule B of M. If every submodule of M has a (weak) supplement in M, then M is (weak) supplemented.

It is clear that the supplemented module is weak supplemented.

Lemma 2.4. [14] If $f : M \to N$ is a small epimorphism with a small kernel, and A is a supplement of B in M, then $f(A)$ is a supplement of $f(B)$ in N.

Definition 2.5. [13] A submodule C of M is called a cofinite submodule in M if M/C is finitely generated. A module M is said to be cofinitely weak supplemented (briefly, cws) if every cofinite submodule of M has a weak supplement in M.

Definition 2.6. [13] If every cofinite submodule of M has a supplement in M, M is called a cofinitely supplemented module (briefly, cs).

Indeed, if M is supplemented module, then M is cofinitely supplemented, and cofinitely weak supplemented. For the converse, finitely generated property is needed. Namely, finitely generated cofinitely supplemented is supplemented.

Proposition 2.7. [4] An arbitrary sum of cws-modules is a cws-module.

Theorem 2.8. [4] Let M be an R-module such that $Rad(A) = A \cap Rad(M)$, for every finitely generated submodule A of M. Then, M is cws if and only if M is cs.

Theorem 2.9. [4] Let M be a module with a small radical. Then, the following statements are equivalent:

i. M is a cws-module.

ii. $M/Rad(M)$ is a cws-module.

iii. Every cofinite submodule of $M/Rad(M)$ is a direct summand.

Definition 2.10. [13] Let $M = X + Y$, for submodules X and Y of M. Then, M is called a refinable module if there is a direct summand A of M so that $A \subseteq X$ and $M = A + Y$.

Definition 2.11. [13] Any submodule A of M has ample supplements in M if $A + B = M$, for every submodule B of M, there is a supplement A' of A with $A' \subseteq B$. Then, M is called an amply supplemented if all submodules have ample supplements in M.

Evidently, if M is an amply supplemented module, then M is supplemented. Supplemented modules over a non-local Dedekind domain provided in [2] are amply supplemented. Additionally, if R is semiperfect ring, then every finitely generated left R-module is amply supplemented.

Definition 2.12. [8] Let A and B be submodules of M. Then, $A\beta^*B$ if $A + B/B$ is small in M/B, and $A + B/A$ is small in M/A.

In [8], it is shown that β^* is an equivalence relation, and if A is small in M, then $0\beta^*A$.

Definition 2.13. [8] If there is a supplement submodule B of M with $A\beta^*B$, for each submodule A of M, then M is called a Goldie*-supplemented module ($\mathcal{G}^*$s).

Every linearly compact and semisimple module is $\mathcal{G}^*$s. Moreover, if M is amply supplemented, then M is $\mathcal{G}^*$s. In addition, if M is $\mathcal{G}^*$s, then M is supplemented [8].

Theorem 2.14. [8] Let A, B be submodules of M such that $A\beta^*B$. Then, A has a (weak) supplement C in M if and only if C is a (weak) supplement for B in M.

Corollary 2.15. [8] Let A, B be submodules of M such that $A \subseteq B$, and A has a weak supplement C in M. Then, $A\beta^*B$ if and only if $B \cap C \ll M$.

Proposition 2.16. [8] Let $f : M \to N$ be an epimorphism.

i. If A and B are two submodules of M such that $A\beta^*B$, then $f(A)\beta^*f(B)$.

ii. If A and B are two submodules of N such that $A\beta^*B$, then $f^{-1}(A)\beta^*f^{-1}(B)$.

Corollary 2.17. [8] Let A, B, and C be submodules of M such that $C \ll M$. Then, $A\beta^*B$ if and only if $A\beta^*(B + C)$.

Definition 2.18. [5] A module M is called an H-cofinitely supplemented if, for each cofinite submodule C of M, there exists a direct summand D of M such that $C + D/C$ is small in M/C, and $C + D/D$ is small in M/D. It is obvious that H-cofinitely supplemented is c$\mathcal{G}^*$s.

Definition 2.19. [15] A ring R is called a left V-ring if every simple left R-module is injective.

Theorem 2.20. [15] For any ring R, the following are equivalent:

i. R is a left V-ring.

ii. Any left ideal A of R is an intersection of maximal left ideals.

iii. For any left R-module M, $Rad(M) = 0$.

3. Cofinitely Goldie*-Supplemented Modules

Definition 3.1. A module M is called a cofinitely Goldie*-supplemented (c$\mathcal{G}$*s) if there is a supplement submodule S of M with $C\beta^*S$, for each cofinite submodule C of M. It is obvious that every $\mathcal{G}$*s is c$\mathcal{G}$*s.

Example 3.2. Every semisimple and local module is c$\mathcal{G}$*s. Let M be a semisimple. In other words, M is $\mathcal{G}$*s. Therefore, M is c$\mathcal{G}$*s. Let us take a submodule C as a cofinite in M. Because M is local, C is small in M, that is, $C\beta^*0$. Thereby, M is c$\mathcal{G}$*s.

Proposition 3.3. Every c$\mathcal{G}$*s module is cws.

Proof.
To prove this, consider the cofinite submodule C of M. Then, from the hypothesis, we get $C\beta^*S$ where $M = S + K$ and $K \cap S \ll S$, for some submodule K of M, that is, S is a supplement in M. Besides, $K \cap S$ is also small in M from Lemma 2.2. Thus, S has a weak supplement K by Definition 2.3. Moreover, from Theorem 2.14, C has a weak supplement K in M. Consequently, M is cws. □

Proposition 3.4. If M is a refinable cws-module, then M is c$\mathcal{G}$*s.

Proof.
Assume that C is cofinite in M. Then, C has a weak supplement S in M as M is cws. In other words, $M = C + S$ and $C \cap S$ is small in M. Using the refinable property, we observe that there exists a direct summand A of M, such that $A \subseteq C$ and $M = A + S$. Thus, $A \cap S \subseteq C \cap S \ll M$ implies from Lemma 2.2 *i* that $A \cap S \ll M$. Thus, A has a weak supplement S in M. Hence, $A\beta^*C$ from Corollary 2.15. □

Theorem 3.5. Let M be a module and consider the following conditions:

i. M is amply supplemented.

ii. M is $\mathcal{G}$*s.

iii. M is c$\mathcal{G}$*s.

Then, $i \Rightarrow ii$ and $ii \Rightarrow iii$. Moreover, if M is finitely generated, then $iii \Rightarrow ii$, and if R is a non-local domain, then $ii \Rightarrow i$.

Proof.
$i \Rightarrow ii$ Clear.

$ii \Rightarrow iii$ Clear.

$iii \Rightarrow ii$ Let M be a c$\mathcal{G}$*s module. If M is finitely generated, then every submodule of M is cofinite. Hence, M is $\mathcal{G}$*s.

$ii \Rightarrow i$ M is supplemented since every $\mathcal{G}$*s is supplemented. Hence, M is amply supplemented because R is a non-local domain. □

The following example shows that every H-cofinitely supplemented module need not be c$\mathcal{G}$*s.

Example 3.6. [5] Let $R = F[[x, y]]$ be the ring of formal power series over a field F in the indeterminates x and y. Then, R is a commutative noetherian local domain with maximal ideal $J = Rx + Ry$. Therefore, the ring R is semiperfect, and the ideal J is finitely generated. Since R is a domain, J_R is a uniform module. Thus, J_R is not a direct sum of cylic modules. Then, J_R is not H-cofinitely supplemented. Since R is semiperfect, J_R is amply supplemented. Hence, J_R is c$\mathcal{G}$*s by Theorem 3.5.

The relationships between c$\mathcal{G}$*s and cs modules under some conditions are as follows:

Proposition 3.7. If M is c$\mathcal{G}^*$s with zero radical, then M is cs.

Proof.
Let C be a cofinite submodule of M. From the hypothesis, there exists a supplement submodule S of M such that $C\beta^*S$. We observe that $M = S + K$, and $K \cap S$ is small in S, for some submodule K of M. When the radical is zero, $K \cap S = 0$. This means $M = S \oplus K$. In particular, K is also a supplement of C in M because of Theorem 2.14. Therefore, M is cs. □

Proposition 3.8. If M is refinable c$\mathcal{G}^*$s, then M is cs.

Proof.
Take a cofinite submodule C of M. As M is c$\mathcal{G}^*$s, $C\beta^*S$ where S is a supplement submodule of M. Therefore, $M = S + S'$, and $S' \cap S$ is small in S, for submodule S' of M. According to Lemma 2.2, $S' \cap S$ is small in M. More precisely, S and S' are weak supplements of each other. In addition, based on Theorem 2.14, we realize that C also has a weak supplement S' in M. Then, we mean $M = C + S'$ and $C \cap S'$ is small in M. The refinable property admits a direct summand A of M so that $A \subseteq C$ and $M = S' + A$. Taking a submodule A' of M, we write as $M = A \oplus A'$. In these circumstances, A' is a supplement of A. By the modular property, we see that $C = A + (C \cap S')$. Moreover, $A \cap S'$ is small in M. Here, we emphasize that A is a weak supplement of S' in M. Corollary 2.15 shows that $C\beta^*A$. We conclude from Theorem 2.14 that A' is a supplement of C in M. □

Proposition 3.9. Let M be c$\mathcal{G}^*$s with $Rad(A) = A \cap Rad(M)$, for finitely generated submodule A of M. Therefore, M is cs.

Proof.
Based on Proposition 3.3, we have that M is cws. We provide from Theorem 2.8 that M is cs. □

Proposition 3.10. If M is c$\mathcal{G}^*$s, then M/A is c$\mathcal{G}^*$s, for every small submodule A of M.

Proof.
Take a submodule C of M containing A, and let C/A be a cofinite submodule in M/A. Then, C is a cofinite submodule in M, as $(M/A)/(C/A) \cong M/C$ is finitely generated. From the hypothesis, $C\beta^*S$ with a supplement S in M. If $g : M \to M/A$ is a canonical epimorphism, following Proposition 2.16, we get $g(C)\beta^*g(S)$, that is, $(C/A)\beta^*(S + A/A)$. Taking into account Lemma 2.4, we have that $S + A/A$ is a supplement in M/A. As a consequence, M/A is c$\mathcal{G}^*$s. □

Proposition 3.11. If M/A is refinable c$\mathcal{G}^*$s with $A \ll M$, M is c$\mathcal{G}^*$s.

Proof.
If C is a cofinite submodule in M, then $C + A/A$ is a cofinite in M/A. Since M/A is c$\mathcal{G}^*$s,

$$(C + A/A)\beta^*(S + A/A)$$

where $S+A/A$ is a supplement in M/A. Observe that $M/A = (S+A/A)+(B/A)$ and $(S+A/A)\cap(B/A)$ is small in $S+A/A$, for submodule B of M containing A, equivalently, $M = S+B$, $(S\cap B)+A/A$ is small in $S + A/A$. Furthermore, $(S \cap B) + A/A$ is small in M/A. If $f : M \to M/A$ is a small epimorphism, we obtain $f^{-1}(C + A/A)\beta^*f^{-1}(S + A/A)$ from Proposition 2.16, that is, $(C + A)\beta^*(S + A)$. We can see from Corollary 2.17 that $C\beta^*S$. By Lemma 2.2, $S \cap B$ is small in M. Since $M = S + B$, S has a weak supplement B in M. In fact, following Theorem 2.14, we get $M = C + B$, and $C \cap B$ is small in M. Since M is refinable, $M = C' \oplus C''$ for some submodules C' and C'' of M with $C' \subseteq C$, and $M = C' + B$. If C' is contained in C, by Lemma 2.2, $C' \cap B$ is also small in M. This implies that C' has a weak supplement B in M. Using Corollary 2.15, we have $C\beta^*C'$. Finally, M is c$\mathcal{G}^*$s. □

Proposition 3.12. Let M be a c$\mathcal{G}$*s with a small radical. Then, every cofinite submodule of $M/Rad(M)$ is a direct summand.

PROOF.
We deduce from Proposition 3.3 that M is cws. Then, Theorem 2.9 shows the result. □

Proposition 3.13. Let M be refinable c$\mathcal{G}$*s, and C be a cofinite direct summand of M. Thus, C is c$\mathcal{G}$*s.

PROOF.
Assume that $M = C \oplus B$, for some submodule B of M. Here, B is finitely generated. Consider a cofinite submodule A of C. Then, C/A is finitely generated. Further, A is a cofinite in M because $M/A = (C \oplus B)/A$. Since M is c$\mathcal{G}$*s, there exists a supplement S in M such that $A\beta^*S$. Thus, for submodule S' of M, $M = S + S'$, and $S \cap S'$ is small in S. Note that $S \cap S'$ is small in M from Lemma 2.2. Moreover, S has a weak supplement S' in M. Following Theorem 2.14, $M = A + S'$ and $A \cap S'$ is small in M. Because M is refinable, then $M = X \oplus X'$, for some submodules X and X' of M with $X \subseteq A$ and $M = X + S'$. Since X is contained in A, then $X \cap S' \subseteq A \cap S'$, and $A \cap S' \ll M$ implies that $X \cap S' \ll M$ from Lemma 2.2. Hence, S' is a weak supplement of X in M. Applying Corollary 2.15, we get $X\beta^*A$. From the modular law, $C = X \oplus (C \cap X')$. Obviously, X is a supplement submodule in C. □

Proposition 3.14. Let M be refinable. If $M = A \oplus B$ where A and B are c$\mathcal{G}$*s, then M is c$\mathcal{G}$*s.

PROOF.
A and B are cws by Proposition 3.3. Furthermore, M is cws by Proposition 2.7. Thus, M is c$\mathcal{G}$*s because of Proposition 3.4. □

Proposition 3.15. Let C be a cofinite submodule in M such that $C = S + A$, for some supplement submodule S and small submodule A of M. Then, M is c$\mathcal{G}$*s.

PROOF.
Because β^* is an equivalence relation, $C\beta^*C$. Thus, $C\beta^*(S + A)$. By Corollary 2.17, $C\beta^*S$. □

In addition,the converse of Proposition 3.15 under refinable conditions is as follows:

Proposition 3.16. If M is refinable and c$\mathcal{G}$*s, then $C = S + A$, for every cofinite submodule C of M, such that S is a supplement in M and A is small in M.

PROOF.
From the hypothesis, there is a supplement S in M such that $C\beta^*S$. In this situation, $M = S + S'$ and $S' \cap S \ll S$, for some submodule S' of M. In other words, S' has a weak supplement S in M as $S' \cap S \ll M$ by Lemma 2.2 *ii*. According to Theorem 2.14, we can write as $M = C + S'$ and $S' \cap C$ is small in M. As M is refinable, for the direct summand submodule C' of M, $C' \subseteq C$, and $M = C' + S'$. From modularity, $C = C' + (S' \cap C)$. □

Proposition 3.17. Let M be c$\mathcal{G}$*s module over a commutative V-ring and C be a cofinite submodule in M. Then, C is a direct summand in M.

PROOF.
From the assumption, $C\beta^*S$, for supplement submodule S of M. Thus, $M = S + S'$ and $S' \cap S \ll S$, for some submodule S' of M, and based on Lemma 2.2, we have $S' \cap S \ll M$. Moreover, from Theorem 2.14, $M = S' + C$ and $S' \cap C \ll M$. Then, $S' \cap C \subseteq Rad(M) = 0$ by Theorem 2.20. Consequently, $S' \cap C = 0$ and thus $M = S' \oplus C$. □

Corollary 3.18. If M is c$\mathcal{G}$*s over a commutative V-ring, then M is cs.

Theorem 3.19. If M is a torsion module and R is a Dedekind domain, then $M/Rad(M)$ is c$\mathcal{G}$*s.

Proof.
From assumption, $M/Rad(M)$ is semisimple. Hence, $M/Rad(M)$ is $\mathcal{G}$*s. Therefore, $M/Rad(M)$ is c$\mathcal{G}$*s. □

4. Conclusion

In this study, we discussed some results of cofinitely Goldie*-supplemented modules using β^* relation. We proved that any factor module of cofinitely Goldie*-supplemented is cofinitely Goldie*-supplemented. In addition, the finite sum of cofinitely Goldie*-supplemented is cofinitely Goldie*-supplemented. For future studies, modules for which every submodule is cofinitely Goldie*-supplemented may be an interesting subject. Moreover, one can investigate the rings whose modules are cofinitely Goldie*-supplemented.

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

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