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Identifying Mathematical Connections in Model-Eliciting Activities: A Case Study with Pre-service Mathematics Teachers as Designers

Model Oluşturma Etkinliklerinde Matematiksel İlişkilendirmelerin Belirlenmesi: Tasarımcı Olarak Matematik Öğretmen Adaylarıyla Bir Durum Çalışması

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ABSTRACT: This paper aimed to explore mathematical connections that pre-service elementary mathematics teachers identified in model-eliciting activities that they designed. The study used a qualitative, in-depth exploratory case study approach conducted with a total of 42 undergraduate students selected using a purposive sampling method. Data were collected from mathematical modeling problems prepared by pre-service mathematics teachers and from written assignments in which pre-service teachers evaluated their problems in terms of mathematical connections. Data analysis utilized open coding to identify and label common patterns thematically. Results showed that most of the problems designed were consistent with the principles of the model-eliciting activities. A cross-case analysis of model-eliciting activities revealed that four types of mathematical connections were identified: (i) connections to real-world situations, (ii) connections between mathematical concepts, (iii) connections between different modes of representation, and (iv) connections to other disciplines. The mathematical connections identified in the modeling activities did not seem to be mutually exclusive, but rather supportive and sometimes overlapping. Pre-service mathematics teachers' identification of mathematical connections in modeling problems reflects both their awareness of mathematical connections and their perception that the skills required for the mathematical modeling process are directly related to mathematical connection skills. Therefore, it is important that pre-service mathematics teachers have a comprehensive understanding of mathematical connections and believe that this is an essential element of the modeling process so that the various dimensions of mathematical connections, along with relevant mathematical, pedagogical, and curricular knowledge, can be effectively integrated into model-eliciting activities.

Keywords: Mathematical connections, model-eliciting activities, mathematical modeling, pre-service mathematics teachers

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ÖZ: Bu araştırmanın amacı, ilköğretim matematik öğretmen adaylarının tasarladıkları model oluşturma etkinliklerinde belirledikleri matematiksel ilişkilendirmeleri incelemektir. Amaçlı örnekleme yöntemine göre seçilen kırk iki lisans öğrencisiyle yürütülen çalışmada nitel araştırma desenlerinden keşfetmeye dayalı durum çalışması kullanılmıştır. Araştırmanın verileri matematik öğretmen adaylarının hazırladıkları matematiksel modelleme problemlerinden ve öğretmen adaylarının problemlerini matematiksel ilişkilendirmeler açısından değerlendirdikleri yazılı ödevlerden toplanmıştır. Çalışmanın veri analizi, ortak yaklaşımları tematik olarak belirlemek ve sınıflandırmak için açık kodlama kullanılarak yapılmıştır. Araştırmanın sonuçları tasarlanan problemlerin çoğunun model oluşturma etkinliklerinin prensipleri ile tutarlı olduğunu göstermiştir. Model oluşturma etkinliklerinin analizinden elde edilen veriler ise dört tür matematiksel ilişkilendirmenin belirlendiğini ortaya çıkarmıştır: (i) gerçek yaşam durumlarıyla ilişkilendirmeler, (ii) matematiksel kavramlar arasındaki ilişkilendirmeler, (iii) farklı temsil biçimleri arasındaki ilişkilendirmeler ve (iv) diğer disiplinlerle ilişkilendirmeler. Modelleme etkinliklerinde belirlenen bu matematiksel ilişkilendirmeler birbirini dışlayan değil aksine birbirini destekleyen ve bazen örtüşen niteliktedir. Matematik öğretmen adaylarının modelleme problemlerindeki matematiksel ilişkilendirmeleri belirlemeleri hem matematiksel ilişkilendirmelere yönelik farkındalıklarını hem de matematiksel modelleme süreci için gerekli olan becerilerin matematiksel ilişkilendirme becerileri ile doğrudan ilişkili olduğuna dair algılarını yansıtmaktadır. Bu nedenle, matematik öğretmen adaylarının matematiksel ilişkilendirmelere ilişkin kapsamlı bir anlayışa sahip olmaları ve bunun modelleme sürecinin temel bir unsuru olduğuna inanmaları matematiksel ilişkilendirmelerin çeşitli boyutlarının ilgili matematik, pedagojik ve müfredat bilgisi ile birlikte model oluşturma etkinliklerine etkili bir şekilde entegre edilebilmesi açısından önemlidir.

Anahtar sözcükler: Matematiksel ilişkilendirmeler, model oluşturma etkinlikleri, matematiksel modelleme, matematik öğretmen adayları

1. INTRODUCTION

Mathematical connections have become increasingly important in mathematics education since they were introduced as an essential mathematical process standard for teaching mathematics by the National Council of Teachers of Mathematics [NCTM] in 2000. In its Principles and Standards for School Mathematics, the NCTM (2000) states that mathematics is not a collection of unrelated facts, rules, or procedures, even though it is often presented that way in the teaching and learning process. Rather, mathematics is viewed as an integrated subject area and a network of strongly interrelated concepts, definitions, theorems, or meanings. The long-term retention of mathematical knowledge and the transfer of that knowledge to new situations are closely related to the development of this network of connections among mathematical ideas, structures, and practices. In addition, the National Research Council (2001), in its report *Adding it up: Helping children learn mathematics*, identified conceptual understanding as one of the five strands that promote the acquisition of mathematical proficiency. The authors of the *"Adding it up"* report believe that a conceptual understanding of mathematics requires a shift in perspective from viewing mathematics as a set of isolated concepts, operations, and relationships. Connections between concepts and procedures make it easier to justify why certain facts are the result of other facts. In other words, since the knowledge gained through understanding is interconnected, it is easier to retain and apply, and it can be reconstructed if forgotten. Therefore, the way students present and relate information is critical to whether they can grasp it thoroughly and use it to solve problems.

The fact that the ability to make connections has been cited as an important skill that learners need to develop in mathematics classrooms has also revealed the importance of how prospective mathematics teachers approach this issue in mathematics education. Although students can make connections on their own, we cannot assume that they are able to do so without teacher intervention (Weinberg, 2001). The teacher plays a critical role in this process by providing opportunities for learners to identify and use connections between mathematical ideas. The implicit responsibility of the teacher is to act in ways that make students explicitly aware of mathematical connections by creating an environment in which opportunities for connections are encouraged (Businskas, 2008; Leikin & Levav-Waynberg, 2007; Mhlolo et al., 2012; NCTM, 2020a). Since teachers play a critical role in making connections, the way they design their lessons can increase or decrease students' chances of making connections (Venkat & Adler, 2012). Teachers need to make connections between students' prior mathematical knowledge and future learning in the classroom. Mathematics instruction should enable students to understand how mathematical ideas are interconnected and build on each other to form a coherent whole, as well as to recognize and apply mathematics in contexts other than mathematics itself (NCTM, 2020b).

Despite the call for teachers to make connections, the literature shows that teachers, especially novice teachers, rarely do so in the classroom or that their connections are implied rather than made explicitly (Bartels, 1995; Eisenhart et al., 1993). If teachers fail to make connections in their classes, then learning mathematics with understanding is likely to be limited or hindered. Thus, most students struggle in school not because their teachers lack subject knowledge, but because they are unable to connect the content of the subject to students' existing patterns of thinking or prior knowledge (Irvine, 2003). Therefore, a teacher who does not view mathematics as an inherently coherent body of knowledge is unlikely to create experiences for students that make these connections explicit. In turn, learners have difficulty seeing mathematics as a holistic structure and are unable to take advantage of opportunities to connect mathematical ideas (Wright, 2018). In addition, based on cross-cultural classroom comparisons, Richland et al. (2012) noted that what distinguished high-performing countries from others were the types of learning opportunities instructors offered for learners, such as making clear connections between mathematical procedures, problems, and concepts in class and finding ways to engage students in the kind of constructive conflict needed to fully understand those connections. However, many teachers typically do not take advantage of opportunities to make connections and therefore do not motivate students to organize their knowledge around mathematical connections.

Mathematical modeling has the potential to provide opportunities for students to connect and experience mathematics in ways that enable them to navigate the realities of their daily lives (Suh et al., 2021). Mathematical modeling can be broadly described as the process of expressing real-world problems through mathematical representations (Barbosa, 2003; Blum & Borromeo Ferri, 2009; Tekin-Dede & Bukova-Güzel, 2018) and involves a cyclical process (Blum & Leiβ, 2007; Borromeo Ferri, 2006). In mathematical modeling, students are presented with a real-world situation and are expected to understand the problems they encounter and make the necessary configurations to solve the problem, just as in the problem-solving process. According to Maab (2006), in order for students to successfully complete a mathematical modeling process, they must first identify the necessary or unnecessary variables for the solution by establishing connections with different disciplines and using additional mathematical information. After evaluating these variables through various reasoning processes, they need to create a mathematical model. The solution is achieved by applying appropriate problem-solving strategies and mathematical information to the model created. They then interpret the results by going back over the thought processes and comparing them to the results that might occur in real life and checking the results they found. In this way, students have the opportunity to compare their results to real life and other disciplines as they interpret and validate the modeling. Since mathematical modeling tasks are expected to be worked on in complex situations that are open-ended and not precise, implicitly involve mathematics, and require generating different ideas and interpretations by making connections between mathematical structures (Borromeo Ferri, 2006; Kaiser et al., 2011; Lesh & Zawojewski, 2007), modeling tasks can be said to be closely related to mathematical process skills and to be an effective tool for acquiring these skills. Hence, it can be assumed that activities that encourage modeling contribute to the development of students' mathematical connection skills. In this sense, it might be critical for pre-service mathematics teachers to design and solve model-eliciting activities to develop their mathematical connection skills. Such activities can help them develop their mathematical thinking skills, relate concepts in different ways, and make connections between mathematics and the real world and other disciplines. Therefore, it is important to understand how prospective teachers identify mathematical connections in the context of modeling problems in order to help them develop a broader understanding of mathematical connections and integrate appropriate practices into their future teaching. Accordingly, this paper aims to discuss the mathematical connections that pre-service mathematics teachers recognized in the model-eliciting activities that they designed.

1.1. Theoretical Background and Literature Review

Hiebert and Carpenter (1992) characterize mathematical connections as essential components of a network designed like a web-like structure. While the nodes in the network represent the bits of information, the threads between them reflect the connections or relationships established between the nodes. According to Eli et al. (2013), the term 'mathematical connection' refers to a bridge that connects mathematical facts, procedures, representations, or meanings. Therefore, an alternative way to describe mathematical connections is to think of them as parts of a schema or as clusters of related schemata in a mental network. Despite the fact that mathematical connections have been expressed or classified in several different ways, the commonalities are usually treated under three headings: (i) connections to real-world applications of mathematics, (ii) connections of mathematics with other fields, and (iii) connections between mathematical concepts or ideas (Bingölbali & Coşkun, 2016; Blum et al., 2007; Özgen 2016; Yavuz-Mumcu, 2018).

Businskas (2008) argues that most of the papers in the literature focus on connections in real-life contexts rather than mathematical connections or interconnectedness within mathematics, and she proposes a framework with five categories that can be used as signs for mathematical relationships: (i) multiple representations as a form of mathematical connections, (ii) part-whole relationships (hierarchical nature of concepts), (iii) connections where A implies B (logical reasoning), (iv) connections showing A is a procedure to do B (algorithms), and (v) instructional-oriented connections (building on students prior knowledge). While Businskas (2008) refers to connections between mathematical ideas or concepts as interconnections in mathematics, García-García and Dolores-Flores (2018) call them intra-mathematical connections as opposed to extra-mathematical connections, which establish a relationship between a mathematical concept and a problem in a context outside of mathematics. They assume that mathematical connections are the cognitive process by which a person relates two or more ideas, notions, definitions, arguments, processes, representations, and implications to one another, to other subjects, or to the real world. García-García and Dolores-Flores (2018) also introduced a framework that includes several types of mathematical connections: (i) procedure, (ii) different representations, (iii) features, (iv) reversibility, and (v) meaning. Finally, based on the model of Businskas (2008) with the models developed by Eli et al. (2011), García-García and Dolores-Flores (2018, 2019, 2020) and Rodríguez-Nieto et al. (2020), Rodríguez-Nieto et al. (2021) consider nine types of intra-mathematical connections: (i) instruction-oriented connection, (ii) different representations, (iii) procedural, (iv) implication, (v) part-whole, (vi) meaning, (vii) feature, (viii) reversibility, and (ix) metaphorical.

Research supports that connecting in mathematics provides meaningful learning experiences for students, and contributes to deeper and more permanent learning (e.g., Ball et al., 2005; Barmby et al., 2009; Carlson & Rasmussen, 2008; García-García & Dolores-Flores, 2018, 2019, 2020; Özgen, 2013, 2016; Rodríguez-Nieto et al., 2020; Star & Stylianides, 2013; Van de Walle et al., 2019). Classes in which the interrelatedness of mathematical ideas is emphasized not only help students learn mathematics but also allow them to see mathematics as a whole and to better understand its function in modern society (Eli et al., 2011, 2013). Making connections between mathematical facts, procedures, and meanings is part of learning mathematics with understanding (Bay-Williams & Stokes Levine, 2017; NCTM, 2014; Stylianides & Stylianides, 2008) as recognizing the connections between mathematical concepts, their properties, and representations is an important hallmark of mathematical understanding (Dolores-Flores et al., 2019; Hiebert & Carpenter 1992; Silver et al., 2009; Skemp, 1987). This implies that while learning mathematics with understanding can be the process of creating connections in mathematics, mathematical connections themselves can also be the product of meaningful learning in mathematics (Cai & Ding, 2015). Sawyer (2008) maintains that having the ability to understand and work with numbers depends on students' capacity to recognize the connections between mathematics and other fields of knowledge, and between mathematics and their personal experiences. Therefore, developing students' mathematical connection skills is important as it deepens their understanding of mathematics, which ultimately increases their mathematical achievement and prepares them for a constantly evolving and increasingly competitive life arena.

In this context, Dede et al. (2020) emphasize that mathematical modeling is closely related to mathematical process skills, including making connections, and can be an effective method for acquiring these skills. Students can develop important conceptual structures in mathematics when working on modeling activities (Gravemeijer, 2007; Hennig, 2010; Lesh & Doerr, 2003; Lesh & English, 2005; Lesh & Zawojewski, 2007). Encouraging students to develop their own models through modeling activities supports both their active participation and the construction of related mathematical concepts based on students' mental actions (Gravemeijer, 1999). Various approaches with different theoretical perspectives have been proposed for the use of modeling in mathematics education, and there is no unified view among educators. Kaiser and Sriraman's (2006) classification scheme for representing modeling approaches can be considered a guiding classification. This scheme classifies modeling perspectives as (i) realistic or applied modeling, (ii) contextual modeling, (iii) educational modeling, (iv) socio-critical modeling, (v) epistemological or theoretical modeling, and (vi) cognitive modeling. In general, modeling is also distinguished according to its purpose in mathematics education, such as (i) modeling as a purpose of teaching mathematics or (ii) modeling as a means for teaching mathematics (Galbraith, 2012; Gravemeijer, 2002; Julie & Mudaly, 2007; Niss et al., 2007). One of the most well-known approaches to the contextual modeling perspective is the model and modeling perspective (MMP) developed by Lesh and Doerr (2003). In the MMP, which views mathematical modeling as a means for teaching mathematics (Galbraith, 2012), mathematical modeling activities are referred to as model-eliciting activities (MEAs) because they involve both models and modeling concepts (Lesh & Doerr, 2003). The salient feature of the activities is that they involve real-world scenarios that are meaningful to students. In that regard, the MEAs are contextualized activities in which students can develop important mathematical and conceptual structures as they engage in them (Abassian et al., 2020). In other words, MEAs require students to explore and understand the nature of mathematical concepts, processes, or relationships. Students attempt to build models based on their own knowledge and experience. Thus, MEAs allow students to self-monitor or self-regulate their own cognitive processes (Lesh & Zawojewski, 2007).

The MEAs also require complex and non-algorithmic thinking skills. They are designed to expose students' thought processes. Therefore, the model-eliciting activities provide teachers with opportunities to identify students' thinking styles and possible conceptualization steps (Lesh & English, 2005). Teachers are able to understand their students' conceptual strengths and weaknesses during modeling activities in order to prepare effective instruction based on these insights (Lesh et al., 2000). Therefore, modeling activities can serve as a valuable guide for teachers to help them capture their students' mathematical comprehension skills and abilities that are generally not captured when using traditional word problem-solving activities (Erbaş et al., 2014). Moreover, for MEAs to serve their purpose, they must be designed according to the modeling principles established by Lesh et al. (2000). In the modeling activity designed according to these principles, students have the opportunity to reveal their thinking through meaningful, shareable, testable, and assessable models to mathematically solve a complex real-world situation (Lesh & Doerr, 2003). Lesh et al. (2000) listed the principles that should be included in MEAs as follows: model construction principle, reality principle, self-assessment principle, construct documentation principle, construct shareability and reusability principle, and effective prototype principle. Thus, if teachers are to use MEAs effectively to develop students' mathematical connection skills, they must see themselves as teachers who practice effective ways of embedding these principles in model-eliciting activities. In other words, recognizing connections is important for developing an identity as a teacher who embeds mathematical connections in model-eliciting activities. Therefore, how pre-service mathematics teachers identify mathematical connections in modeling problems is worth exploring as it plays a critical role in designing model-eliciting activities.

2. METHOD

2.1. Research Design

In an exploratory manner, qualitative research is often preferred to describe, analyze, and assess the complex nature of the problem, as it provides a comprehensive understanding of the issue under study when "how" and "why" questions are asked (Yin, 2018). Therefore, this study used a qualitative, in-depth exploratory case study approach to investigate pre-service mathematics teachers' awareness of mathematical connections in the model-eliciting activities they design and to reveal and report the elusive and multifaceted nature of their perspectives on this matter (Cohen et al., 2017).

2.2. Participants

The study was conducted with a total of forty-two undergraduate students (twenty-seven females and fifteen males) who were selected on a voluntary basis from among forty-nine students studying in the fourth year of the elementary mathematics teacher education program at a state university and enrolled in the Modeling in Mathematics Education course. No pressure was placed on those who did not want to take part in the study. The participants of the study were selected via the criterion sampling method, which is one of the purposive (or purposeful) sampling methods used in qualitative research (Patton, 2014). Having previously taken the Making Connections in Mathematics Education course was taken as a criterion for selecting the study group for this research. Participants were also coded from P1 to P42 to protect their personal rights and privacy and to allow for easy referencing.

2.3. Data Collection

The data for the study came from the mathematical modeling problems prepared by the pre-service mathematics teachers and the written assignments in which they evaluated their problems in terms of mathematical connections (see list of problems in Appendix 1). First of all, the researcher informed the prospective teachers about the model, modeling, mathematical modeling, perspectives, processes and competencies of mathematical modeling, principles of model-eliciting activities, and application and assessment of modeling activities in the classroom for nine weeks in the Modeling in Mathematics Education course, which lasted two hours per week. Thus, the pre-service teachers had the opportunity to learn the theoretical background of mathematical modeling as well as how to solve and develop modeling problems. In addition, a total of eight groups, consisting of six groups of five and two groups of six, were formed so that the prospective teachers could work in groups in the classroom for five weeks. Each group was asked to form a mathematical modeling activity that incorporated the principles of model-eliciting activity design and then to present the modeling activity to the other groups and show possible solutions. In this sense, the prospective teachers also had the opportunity to experience the model-eliciting activities by exploring, thinking about, and evaluating different examples of mathematical modeling tasks. At the end of the fourteen weeks, the essential knowledge and skills in mathematical modeling had been substantially revised and the pre-service mathematics teachers who volunteered to participate in the study were asked to individually design a mathematical modeling problem on any topic from the middle school mathematics curriculum within two weeks and to write a report on what kind of mathematical connections associated with that problem. In addition, the prospective teachers were informed that the modeling problems they designed and the reports they wrote about the mathematical connections in those problems would be kept strictly confidential, would not be used for any purpose other than research, and would not be graded.

2.4. Data Analysis

Data were analyzed using thematic analysis to identify, compare, and label patterns across qualitative data from participants. NVivo (QSR International, 2012), a qualitative analysis software program, allowed researchers to organize the data and search for common themes. First, the mathematical modeling problems written by the pre-service teachers were analyzed according to the principles of model-eliciting activities highlighted by Lesh et al. (2000), and then the reports in which the prospective teachers evaluated their problems in terms of the mathematical connections they referred to were analyzed. The reports were analyzed thematically using emerging codes. After assigning an initial code list to all the data, the researcher analyzed them systematically to verify the codes extracted from the data and performed a content analysis in this way to find common patterns. An initial coding scheme was developed to reflect key elements of the literature (e.g., Bingölbali & Coşkun, 2016; Businskas, 2008; Eli et al., 2011; Özgen 2016; Yavuz-Mumcu, 2018). Relevant information that emerged from the reports was also added to this scheme as new codes. The original coding list of emerging patterns was then revised to match the original data. This method allowed us to check the reliability and internal consistency of the coding scheme by grouping similar or related codes if they appeared redundant, or by extracting codes from passages that were not critical to the study (Patton, 2014). Therefore, similar codes were grouped into meaningful categories in accordance with the research purpose. For example, the category 'Connections between different modes of representation' was created by combining the codes from pre-service teachers' assessments such as 'use of tabular representations', 'use of verbal representations', 'use of algebraic representations', and 'use of graphical representations'. Similarly, the category 'Connections to real-world situations' was formed by grouping codes such as 'related to knowledge and life experience', 'conducive to generating different real ideas', 'non-routine', and 'interesting'. In presenting the results, an attempt was made to directly cite the most salient mathematical connections that participants made in the modeling problems, and the results were interpreted based on these connections. Throughout all of these stages, the content of the data was constantly reviewed with the purpose of the research in mind, and care was taken to ensure that the mathematical connections assigned to the same category were as consistent and meaningful as possible (Merriam & Tisdell, 2015). In addition, to ensure that the research findings accurately reflected the mathematical connections made by prospective teachers, a researcher in the field of mathematics education was asked to serve as an external rater. An external researcher familiar with the same issue verified that the assigned codes were appropriate and reasonable, and thus matched those that the third-party researcher could label or create. In the final phase of analysis, the codes in all model-eliciting activities were double-checked by the authors and an external rater and re-examined for emergent themes. A comparison of two coding schemes yielded an inter-coder reliability of 0.91, indicating high reliability (Miles & Huberman, 1994). The raters resolved all inconsistencies and refined the code definitions until all conflicts were completely settled to ensure consistency in the coding process.

2.5. Ethical Issues

The approval of the Human Research Ethics Committee of Zonguldak Bülent Ecevit University dated 14.02.2023 with the number 275340 was obtained to conduct this study. In addition, the consent forms of the pre-service teachers were obtained before starting the research. Accordingly, they were informed that participation in the study was not compulsory, that they could withdraw at any time after participation, that the data obtained from the study would be used only for scientific research, that the existing data would not be included in the study and would remain confidential if the participant wished to withdraw from the study or was removed by the researcher, and that the identity information of the participants included in the study would be kept confidential when the research results were published.

3. FINDINGS

The findings showed that most of the problems (*n*=31) designed by the prospective teachers conformed to the principles of model-eliciting activities including the model construction principle, reality principle, self-assessment principle, construct documentation principle, model generalization principle, and effective prototype principle. As seen from Table 1, thirty-one problems out of forty-two were found to satisfy the six design principles that made them count as modeling problems, while eleven problems did not satisfy at least one of these principles. Therefore, only these 31 modeling problems were analyzed in this study in terms of the mathematical connections to which they refer.

Table 1: *Analysis of the Problems Based on the Principles of Model-Eliciting Activities*

		Principles of Model-Eliciting Activities					
Pre-service Teacher	**List of Mathematical Modeling Problems**	**Model Construction**	**Reality**	**Self-assessment**	**Construct Documentation**	**Model Generalization Principle**	**Effective Prototype**
*P1	MP election	✓	✓	✓	✓	✓	✓
*P2	Unity neighborhood event	✓	✓	✓	✓	✓	✓
*P3	Surprise heritage	✓	✓	✓	✓	✓	✓
*P4	The housing problem	✓	✓	✓	✓	✓	✓
*P5	A takeover of a restaurant	✓	✓	✓	✓	✓	✓
*P6	What to do to reduce global warming?	✓	✓	✓	✓	✓	✓
*P7	Preference for the city in graduate education	✓	✓	✓	✓	✓	✓
P8	Win or lose	✓	✓	✓	✓	✓	×
P9	Material for hospital indoor wall cladding	✓	✓	×	×	×	×
*P10	What to do with these tomatoes?	✓	✓	✓	✓	✓	✓
*P11	Door design	✓	✓	✓	✓	✓	✓
*P12	The cheapest but funniest birthday party	✓	✓	✓	✓	✓	✓
*P13	Fishing net order	✓	✓	✓	✓	✓	✓
*P14	Tutoring preferences	✓	✓	✓	✓	✓	✓
*P15	Production of free-range geese	✓	✓	✓	✓	✓	✓
*P16	Job application	✓	✓	✓	✓	✓	✓

***Appropriate model-eliciting activities**

Table 1: *(Continued)*

Pre-service Teacher	List of Mathematical Modeling Problems	Principles of Model-Eliciting Activities					
		Model Construction	Reality	Self-assessment	Construct Documentation	Model Generalization Principle	Effective Prototype
P17	Home selection problem	✓	✓	✓	✗	✓	✓
*P18	Livestock breeding and farming	✓	✓	✓	✓	✓	✓
P19	Bodybuilding nutrition program	✓	✓	✗	✗	✗	✗
P20	KPSS exam preparation course preference	✓	✓	✓	✗	✓	✓
*P21	The problem of inheritance	✓	✓	✓	✓	✓	✓
*P22	Dangerous roads for water supply	✓	✓	✓	✓	✓	✓
P23	Bus ticket at the best price	✓	✓	✓	✗	✓	✓
*P24	Buying the best diapers for two babies	✓	✓	✓	✓	✓	✓
P25	The best smart TV to buy	✓	✓	✓	✗	✓	✗
*P26	The amount of oxygen in the classroom	✓	✓	✓	✓	✓	✓
P27	Deciding where to work	✓	✓	✗	✗	✗	✗
*P28	Buying a summer house	✓	✓	✓	✓	✓	✓
P29	Online sales in a boutique	✓	✓	✓	✗	✓	✓
*P30	Productivity of workers	✓	✓	✓	✓	✓	✓
*P31	Buying the most suitable car	✓	✓	✓	✓	✓	✓
*P32	Which cell phone plan?	✓	✓	✓	✓	✓	✓
*P33	Buying hay bales and feed	✓	✓	✓	✓	✓	✓
*P34	Natural dairy products	✓	✓	✓	✓	✓	✓
*P35	Animal shelters	✓	✓	✓	✓	✓	✓
*P36	Banks and interest system	✓	✓	✓	✓	✓	✓
P37	Movie theaters	✓	✓	✗	✗	✓	✓
*P38	Summer vacation work	✓	✓	✓	✓	✓	✓
*P39	Production of cardboard boxes	✓	✓	✓	✓	✓	✓
*P40	Let us dig a well	✓	✓	✓	✓	✓	✓
*P41	Iftar menu	✓	✓	✓	✓	✓	✓
P42	Making salad at the best price	✓	✓	✓	✗	✓	✓

***Appropriate model-eliciting activities**

A cross-case analysis of 31 pre-service teachers' model-eliciting activities revealed four themes of mathematical connections that these pre-service mathematics teachers identified in the activities they designed: (i) connections to real-world situations, (ii) connections between mathematical concepts, (iii) connections between different modes of representation, and (iv) connections to other disciplines. Figure 1 summarizes the results in a single graph using the frequencies of the emergent themes to visualize the data set. The numbers in parentheses next to each theme indicate the number of pre-service teachers who mentioned that theme. Thus, as shown in Figure 1, pre-service teachers were found to place the most importance on making connections to real life, while making connections to different disciplines was less emphasized than other types of mathematical connections. All themes are presented in detail in the following sections.

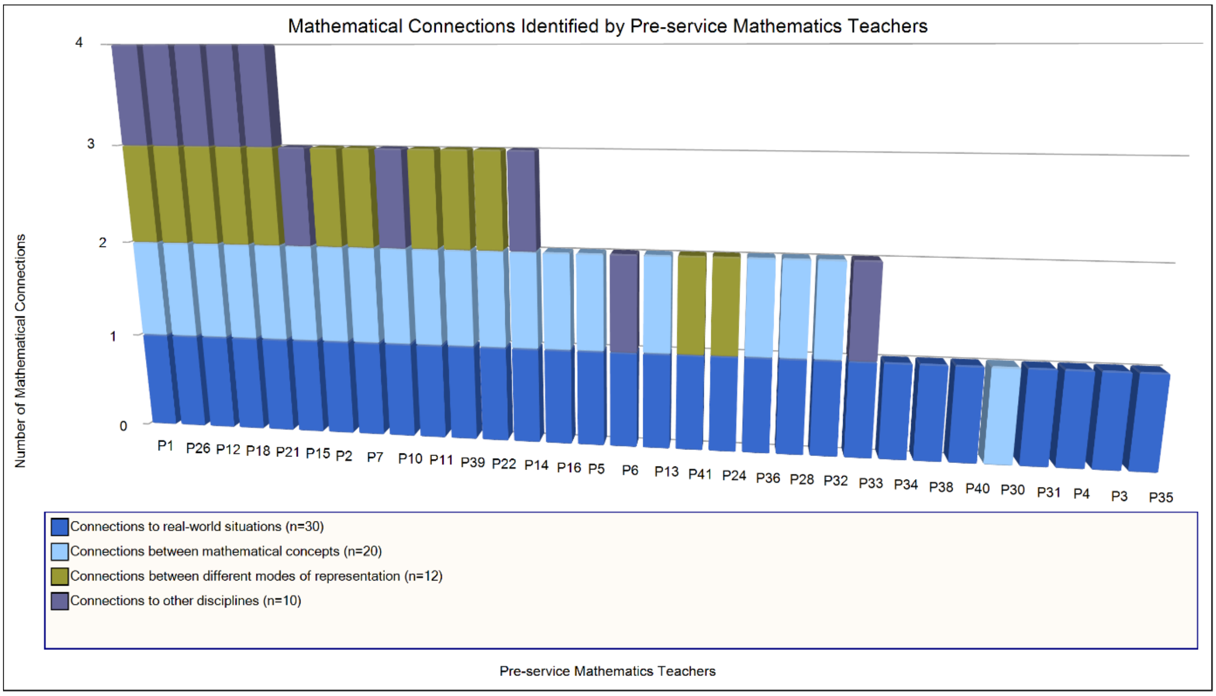

Figure 1: *Graphical Representation of Mathematical Connections Identified by Pre-Service Mathematics Teachers in Model-Eliciting Activities They Designed*

3.1. Connections to Real-World Situations

Most of the pre-service mathematics teachers (*n*=30) stated that mathematical modeling problems are not routine and correspond to the kinds of problems we encounter in real life. They explained that mathematical modeling is a process of using mathematics to solve, explain, and interpret problems that occur in daily life. It is the process of expressing a problem situation in everyday life mathematically and explaining it using mathematical models. Through this process, individuals not only gain the ability to create models that they can use in real life but the mathematics they learn gains meaning through reference to these models. Therefore, mathematical modeling provides a unique opportunity to explore mathematics in context by relating it to real life. For example, one of the pre-service teachers (P34) expressed that when modeling problems, students are usually expected to first understand the everyday situation given in the task and then find an appropriate way to effectively apply their mathematical skills. In this way, students have to create models by using their mathematical knowledge. Then, they are supposed to solve the problem using their mathematical thinking, reasoning, and interpretations. Finally, they check whether the results obtained have any meaning in real life. Moreover, the need to design mathematical modeling activities in accordance with the reality principle is an indication that they have connections to real life. Therefore, when designing mathematical modeling tasks, care is usually taken to include situations from everyday life. For instance, the modeling problem titled “*Natural Dairy Products*” written by participant P34 is as follows:

Oray, one of the entrepreneurs of the 21st century, wants to create a source of income in the field of organic food, as people’s demand for natural (organic) food has increased *recently. Oray wants to enter the organic food sector by producing milk, but he also wants to build his business at the lowest possible cost while making a high profit. For this reason, he talked to members of a union that deals with cattle breeding about the milk productivity of cattle and decided to keep 20 Simmental cattle on the dairy farm he is going to establish. However, thinking about the Turkish market and production costs, Oray is still undecided about the region in which he would like to establish his company. In which region of Turkey do you think the dairy that Oray wants to start can become a high-margin business? Explain this in detail and make sure that your solutions are based on convincing, solid foundations so that Oray can make a decision. Also, decide with justifications whether the model you propose is an effective way to go and whether the solution you have found can be used in other similar situations. (P34)*

Participant P34 expressed the following regarding the mathematical connection included in his problem:

In the modeling problem, a possible real-life situation is presented. The first phase of the company's creation, as mentioned in the problem, starts with a detailed consideration of the real data. For this purpose, the characteristics of the provinces within the borders of Turkey are studied and possible solutions are compared. The obtained result is interpreted, verified, and defended based on real-life experiences and knowledge. In this way, a connection between mathematics and real life is established in the problem. (P34)

3.2. Connections Between Mathematical Concepts

Many of the pre-service teachers (*n*=20) emphasized that mathematical modeling activities can help students develop mathematical knowledge, understanding, and skills and raise awareness of the multiple connections among mathematical concepts. They explained that modeling activities typically have high expectations for all students. Students are presented with complex mathematical problems and are expected to look for patterns, find relationships, generalize, verify, and share and critique the work of others as they investigate these problems. According to pre-service teachers, mathematical concepts in modeling activities are carefully embedded in a series of related tasks so that students can develop mathematical understanding and meaningful learning. Ideas are explored in depth necessary for students to understand them through these mathematical tasks. Treating an idea superficially leads to cursory understanding and does not support the connection of ideas. Therefore, students must make connections between mathematical ideas, knowledge, and procedures to gain a deep mathematical understanding. Pre-service teachers also added that it is important to make connections between previous knowledge and newly learned knowledge to establish the relationship between mathematical concepts and facts. In other words, to make sense of a new concept, it is essential to integrate newly learned knowledge with existing mathematical understanding in various ways. In this context, mathematical modeling activities are an effective means of linking previously learned mathematical concepts to newly learned concepts. For example, the modeling problem titled "*Hazelnut Field Inheritance Sharing*" posed by participant P21 is as follows:

Beyza and Meryem are two sisters. They want to fairly divide the hazelnut field they inherited from their father. After they divide it among themselves, the boundaries are determined by the land registry and the cadastre. The field slopes along the border. The *shape of the field is trapezoidal (as shown in Figure 2), and the angle between the borders AC and CD is 120 degrees.*

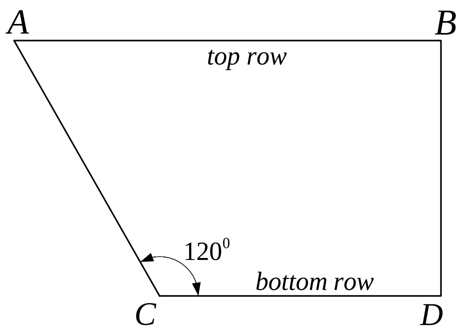

Figure 2: *The Shape of the Hazelnut Field Drawn by Participant 21*

There are a total of 550 hazelnut trees in the field. There are 25 rows from bottom to top. 15 hazelnut trees are in the lowest row, and 27 hazelnut trees are in the top row. From the 12th row, the slope of the field increases. In the area to the right of the field, the hazelnut trees are not yet mature. Since the right side of the field is exposed to more sunlight, some of the hazelnut flowers dry up, which reduces the yield of the trees.

The sisters do not know how to divide the field based on the information given, so they need help. How do you think two sisters should divide the hazelnut field so that the division is fair? Find a way to help the sisters. Prepare a detailed report so the sisters can understand how your proposed method works. Also, make sure your explanations are clear so the sisters can make a clear decision. Justify whether the model you are proposing to the sisters is effective and can be used in similar situations. (P21)

Participant P21 made the following comments about the mathematical connection involved in the modeling activity she wrote:

This modeling problem allows students to explore the area of the trapezoid using the areas of different regions. In other words, it is possible to find the area formula of a trapezoid using the areas of the triangle and rectangle they have previously learned. So, this task, in which the area of the trapezoid was to be related to the area of a triangle and the area of a rectangle, attempted to develop the ability to make connections between different mathematical concepts. (P21)

3.3. Connections Between Different Modes of Representation

A large number of the pre-service teachers (*n*=12) reported that the ability to use multiple representations is the ability to represent a given situation in a different form to clarify and better understand a complex situation, and it plays an important role in transforming real-world problems given in modeling activities into mathematical statements. They noted that mathematical modeling often involves different representations of mathematical situations. In other words, mathematical modeling requires developing the ability to think and experience different representations of the situation in order to find out the necessary information about the given problem. This means that different representations are used to understand a situation, mathematize a problem, or offer a solution. According to them, the more multiple representations and the relationships between them are revealed, the better the modeling problem can be expressed. Representations used may include diagrams, formulas, equations, charts, tables, pictures, etc. When solving modeling problems, it is important to demonstrate skills such as finding the type of representation that best fits the situation, modifying an existing representation as needed, and knowing when and where it is appropriate to use different representations. Therefore, modeling activities in mathematics classrooms enhance the ability to explore and use different representations of mathematical facts and facilitate a better understanding of those facts. For instance, the modeling problem titled "*Buying the Best Diapers for Two Babies*" written by participant P24 is as follows:

> *Sibel and Tugay have been married for three years. Their baby named Tunay was born in the first year of their marriage. Then, 21 months later, they had another baby named Asya. Tunay has not yet gotten used to the toilet and continues to use diapers. It is supposed that he will also use diapers in the next six months. There are many diaper brands on the market, and their prices vary according to quality features such as soft and breathable, thin and lightweight, comfortable and highly flexible, wetness indicator, and high absorbency. The family wants to buy the best diapers for both babies that will last for the next six months. You are expected to help this family with their preferences. Which solution offers the best choice of diapers? Formulate your solution clearly and understandably. Verify the accuracy of the solution you received at each stage and present it in a report. Also, justify why your solution is the most effective and efficient. Can the solution you developed be generalized to children of different ages who use diapers? Please explain this in detail. (P24)*

Participant P24 commented as follows on the mathematical connection in her task:

> *...choosing the best diapers for the babies in the family is an extremely confusing task. Apart from quality, the size and weight of the baby also play an important role in choosing the best-fitting diaper when buying diapers. Therefore, to help the family choose the perfect diaper for babies with different weights, different diaper sizes should be tabulated considering their quality features and price to solve the problem. Moreover, the information in the table should be expressed in mathematical equations and given in the form of useful notations for the solution. In this way, the problem is attempted to be made more understandable through various forms of representation. (P24)*

3.4. Connections to Other Disciplines

Some of the pre-service teachers (*n*=10) explained that the importance of mathematical modeling has increased not only because of the use of mathematics as a versatile tool in science, engineering, technology, medicine, business, and many other fields but also because of the increasing need for mathematics in these fields. They underlined that mathematical modeling is an activity that lends itself to the use of knowledge and skills from various disciplines other than mathematics because the real-world situation has a complex structure and often involves different branches of science simultaneously. They expressed that engaging students in mathematical modeling activities can provide them with an interdisciplinary perspective by showing the relationship between mathematics and other disciplines. For this reason, mathematical modeling has been referred to as an approach in which different disciplines are treated together. According to pre-service teachers, in this approach, problems related to real-life situations are solved with the help of models created by using different disciplines together with mathematics. In this way, a holistic educational opportunity is provided by introducing concepts or achievements from different disciplines through modeling activities. For example, the modeling problem titled "*What to Do to Reduce Global Warming?*" written by participant P6 is as follows:

> *In 8th-grade science class, Adem realized that pollution caused by human activities is causing climate change as a result of global warming. From this point of view, Adem has learned through in-depth research that global warming is the process of artificially increasing the temperature of the near-earth layers of the atmosphere due to the sharp increase of various gases in the atmosphere, called greenhouse gases, produced by various human activities. In addition, Adem has obtained data showing that the average temperature of the lowest layer of our atmosphere has increased by 0.12 °C to 0.135 °C in each decade since 1979, according to satellite temperature measurements. He understood that the main gases causing the greenhouse effect and global warming are 30-70 % water vapor (H2O), 4-9 % methane (CH4), 9-26 % carbon dioxide (CO2), and 3-7 % ozone (O3). Adem wonders which of the man-made factors, such as fossil fuel use, transportation, population growth, urbanization and industrialization, deforestation and logging, industrial agriculture and livestock, and consumerism, are causing more global warming, and needs help in this regard. Find a way to help him make this decision. Also, make sure your explanations are clear so that Adem can decide whether your method is useful and effective. To do this, prepare a detailed report on how the method was developed and justify whether the model you propose can be used in similar situations. (P6)*

Participant P6 made the following comments about the mathematical connection that exists in the modeling problem she wrote:

> *In this activity, it is not enough for students to just use their mathematical knowledge to solve the problem. In order to develop models that are appropriate for real life, connections must be made to these disciplines by having students use their knowledge and skills from different disciplines. Therefore, in addition to mathematics, other subjects such as chemistry, biology, meteorology, and ecology are used in solving this model-eliciting activity. (P6)*

4. DISCUSSION and CONCLUSION

The evidence from this study suggests that prospective mathematics teachers have four dominant, but also intertwined and mutually supportive, identifications of mathematical connections in model-eliciting activities. Firstly, most of the pre-service teachers in this study referred to one of the dimensions of mathematical connections in modeling problems as the relation of mathematics to the real world. They supported the view that connecting what is taught in school to what is happening in the real or imagined world can provide exciting opportunities for students to understand why what they are learning in mathematics is meaningful and useful beyond school. For this reason, it is very important to select, plan, and implement tasks or activities that relate mathematics to real life in order to increase the quality of mathematics education (Korkmaz & Kertil, 2023). Pre-service teachers also reported that mathematical modeling can be defined either as the mathematical expression of preconceived models in real-world situations or as the creation of a new model by linking real-world situations to mathematical expressions and representations. Therefore, the way in which the relationship between mathematics and the real world is established in mathematics education can be considered an important element in determining different approaches to mathematical modeling. In this sense, there is a trend from mathematics to real-life situations in approaches that emphasize real-life situations related to mathematical concepts and models provided directly to students. On the other hand, there is another trend from real life to mathematics in other approaches that emphasize the creation of mathematical concepts by students using real-life situations (Niss et al., 2007). The Model and Modeling Perspective (MMP) in mathematics education (Lesh & Doerr, 2003) and Realistic Mathematics Education (emergent modeling) (Gravemeijer, 2002) are examples of approaches with this perspective. In this study, it can be assumed that the goal is to teach or develop certain mathematical concepts by using real-life situations in modeling activities from a model and modeling perspective. Pre-service teachers in this study also agreed that making connections to real life when modeling problems increases engagement in the learning process as individuals gain valuable insight into real-life applications of the mathematics being taught. Baki (2014) notes that relating mathematics to real-life situations helps students develop positive attitudes toward mathematics. Similarly, Ji (2012) argues that connections to real life are seen as beneficial to the application of mathematical concepts in daily life and that the contexts associated with students' real or imagined experiences outside the context of formal classroom activities can make learning more meaningful and increase their interest, motivation, and participation. In this context, it is also important to recognize that real situations are not only those that students experience themselves in daily life but also those that they can imagine (Van den Heuvel-Panhuizen & Wijers, 2005).

Many pre-service teachers considered making connections between mathematical ideas as another type of mathematical connection seen in modeling activities. They expressed that making connections between mathematical ideas in modeling problems is important for establishing relationships between mathematical entities or processes to strengthen understanding of mathematical knowledge and principles. They put forward that since the acquisition of new knowledge and skills is linked to existing knowledge and skills, a key element in developing integrated knowledge is to create a classroom environment where learning means building knowledge and skills based on prior knowledge. Thus, pre-service teachers emphasized the need to construct and present new information in model-eliciting activities in a way that connects it to learners' previous mathematical structures. Otherwise, as Fennema and Romberg (1999) mentioned, students with low mathematical coherence are more likely to overlook new knowledge and fail to transfer it to new situations. According to Klosterman (2018), when students explain their thinking, they often do not indicate how their arguments relate to prior understanding. This prevents essential mathematical connections between newly acquired and existing knowledge and increases the likelihood that new mathematical ideas will be perceived by the class as isolated chunks of knowledge. However, the realization of meaningful learning in the mathematics classroom depends on the amount and strength of the relationships the individual establishes between new and old knowledge. The more the individual can link the newly discovered knowledge to the previously learned knowledge on a solid basis, the stronger his or her learning will be (Yavuz-Mumcu, 2023). Considering that learning increases by making connections between mathematical concepts (Osmanoğlu, 2023), it is important for prospective teachers to develop modeling activities in the classroom where students can reflect on the connections between mathematical concepts and share their views. In other words, the student learns to the extent that he or she can make connections. Therefore, pre-service teachers' ability to integrate mathematical connections between mathematical ideas into model-eliciting activities can prepare their students for the increasingly complex and formal mathematical thinking they will encounter in the future.

Some of the prospective teachers also pointed out the importance of using different representations when connecting mathematical concepts to the real world through model-eliciting activities. They underlined the critical role of representations in formulating a real-life problem into a well-expressed mathematical form that can be appropriately interpreted using various mathematical techniques. They stated that a single representation is not sufficient to reveal the structures of mathematical modeling in the mind. They emphasized the necessity of using different representations, performing mathematical operations, and establishing relationships between them to best reflect the characteristics of real-life situations. In other words, the mathematical models created in the modeling process cannot contain all the features of the real situation (Lehrer & Schauble, 2003). Pre-service teachers noted that different mathematical representations should be treated together to express and deal with real-world problems mathematically because representations also have a kind of modeling function (Kaput, 1998). As Sokolowski (2018) indicates, real-life situations can be presented as an excellent basis for developing students' ability to transform between different representations. While abstract concepts are embodied in the real world with representations, a relationship between objects and symbols is established to facilitate understanding of mathematical situations. A similar argument can be illustrated with the Realistic Mathematics Education approach (Freudenthal, 1973). In this approach, which is based on real-life situations, the process of structuring mathematical knowledge is called mathematization. Whereas horizontal mathematization focuses on organizing, translating, and transforming a contextual problem into a mathematical form that represents the underlying mathematical ideas, vertical mathematization focuses on developing mathematical structures by navigating the world of mathematical diagrams, schemes, models, and symbols (Treffers, 1987). Vertical mathematization refers to a process in which students construct new abstract knowledge by transforming previous mathematical structures within mathematics and through mathematical means (Dreyfus et al., 2015). The transitions between different representations during the mathematization process prevent mathematics from being viewed as a collection of separate entities (Deniz & Uygur-Kabael, 2017). When knowledge is transformed into different representations, its cognitive value is not diminished, but merely transformed into different and more accessible forms that can be retrieved and reused (Sokolowski, 2018). This process allows students to easily recognize meaningful information and relate it to their past experiences (Cook, 2006). In this regard, model-eliciting activities in which students engage with real-world situations by formulating graphical or symbolic representations enable them to construct and shape the mathematical knowledge they seek as mathematical objects in their minds. Conceptualizing the real world in this way helps learners focus on connecting previously formed mathematical knowledge to a new mathematical structure.

Moreover, making connections to other disciplines is another type of mathematical connection that is distinguished by some pre-service teachers in model-eliciting activities. They explained that since daily life may involve problems from different disciplines, it is inevitable to use knowledge and skills from other disciplines in solving real-life problems. In other words, since the complexity of the real-life situation often requires that different scientific fields be involved simultaneously when addressing a modeling problem in a real-life context, it is naturally necessary to relate it to other disciplines. Moreover, problems in real life contain many different correct answers depending on assumptions, showing us that learning in many disciplines is interrelated and connected (Hıdıroğlu, 2023). Hence, Fung (2017) emphasizes the importance of students having opportunities to connect their subjects to other disciplines. Prospective teachers also reported that making connections to other disciplines can be helpful not only in formulating different representations but also in enriching learning or deepening mathematical understanding. Indeed, the effectiveness of mathematics is demonstrated by its ability to reveal seemingly different things as one and the same, which explains why mathematics can easily solve problems that arise in other disciplines (Dreher, 2015). Thus, using multiple representations of mathematical ideas to connect concepts within subject areas (algebra, calculus, geometry, trigonometry, etc.) with each other and other disciplines makes it easier for students to perceive mathematics as a network of interconnected concepts and procedures, which allows them to construct knowledge in meaningful ways (Flores et al., 2018). Accordingly, interdisciplinary studies should be encouraged in accordance with the demands of daily life rather than memorizing subjects and acquiring relatively mechanical skills in mathematics learning environments (Yavuz-Mumcu, 2023). In this way, students can confidently use mathematics to explain complex phenomena in various fields and the outside world (NCTM, 2014). Therefore, it is important for prospective teachers to engage in model-eliciting activities and practices at all grade levels that demonstrate both the hierarchical relationships of mathematics within itself and its relationships to other domains and real life as a branch of science.

Overall, the mathematical connections identified by pre-service mathematics teachers in model-eliciting activities do not seem to be mutually exclusive, but rather mutually supportive, compatible, and sometimes overlapping. Their identification of the various mathematical connections in modeling problems reflects both their awareness of mathematical connections and their perception that the skills needed for the mathematical modeling process are directly related to the skills of mathematical connections. Recognition of mathematical connections by pre-service teachers in model-eliciting activities can also be addressed in two major categories. The first is connecting mathematics to contexts other than mathematics, while the second is connecting mathematics to itself. Therefore, the mathematical connections identified by prospective teachers in model-eliciting activities tend to fall into one of two main mathematical connections: intra-mathematical connections that focus on relationships among mathematical ideas, notions, definitions, paradigms, principles, and representations, and extra-mathematical connections that focus on relationships to other disciplines or real-life situations (García-García & Dolores-Flores, 2018). Although intra-mathematical connections include nine dimensions, such as instruction-oriented connections, different representations, procedural, implication, part-whole, meaning, feature, reversibility, and metaphorical (Rodríguez-Nieto et al., 2021), the pre-service teachers in this study identified only two dimensions of intra-mathematical connections including different representations and instruction-oriented connections, which are manifested in linking new knowledge to previous knowledge. Therefore, if various dimensions of mathematical connections are to be effectively integrated into model-eliciting activities along with relevant mathematical, pedagogical, and curricular knowledge, it is suggested that pre-service mathematics teachers should have a more comprehensive understanding of mathematical connections and be convinced that this is an essential element of the modeling process. Thus, a worthwhile quest for further research is to investigate pre-service mathematics teachers' views about the barriers to integrating various aspects of mathematical connections in model-eliciting activities in order to develop appropriate strategies and take action. In this context, classroom observations and follow-up interviews can shed light on the reasons for possible obstacles to making mathematical connections in model-eliciting activities. In addition, practice-based studies examining how pre-service teachers' model-eliciting activities affect students' mathematical connection skills in real classroom settings can make an important contribution to research on this topic. Moreover, long-term follow-up studies can be conducted that allow us to more thoroughly assess prospective teachers' mental processes and thinking strategies to understand how their mathematical connection skills develop over time through model-eliciting activities. Such studies can also be used to evaluate the effectiveness of mathematical modeling courses in teacher education. Furthermore, studies that examine how pre-service teachers' mathematical connection skills are affected during the mathematical modeling process, particularly in a technology-enhanced environment, can also be focused.

Disclosure Statement

No potential conflict of interest was reported by the author.

APPENDIX

Appendix 1: *List of Mathematical Modeling Problems Designed by Pre-service Mathematics Teachers*

Pre-service Teacher	List of Mathematical Modeling Problems
P1	MP election
P2	Unity neighborhood event
P3	Surprise heritage
P4	The housing problem
P5	A takeover of a restaurant
P6	What to do to reduce global warming?
P7	Preference for the city in graduate education
P8	Win or lose
P9	Material for hospital indoor wall cladding
P10	What to do with these tomatoes?
P11	Door design
P12	The cheapest but funniest birthday party
P13	Fishing net order
P14	Tutoring preferences
P15	Production of free-range geese
P16	Job application
P17	Home selection problem
P18	Livestock breeding and farming
P19	Bodybuilding nutrition program
P20	KPSS exam preparation course preference
P21	The problem of inheritance
P22	Dangerous roads for water supply
P23	Bus ticket at the best price
P24	Buying the best diapers for two babies
P25	The best smart TV to buy
P26	The amount of oxygen in the classroom
P27	Deciding where to work
P28	Buying a summer house
P29	Online sales in a boutique
P30	Productivity of workers
P31	Buying the most suitable car
P32	Which cell phone plan?
P33	Buying hay bales and feed
P34	Natural dairy products
P35	Animal shelters
P36	Banks and interest system
P37	Movie theaters

P38	Summer vacation work
P39	Production of cardboard boxes
P40	Let us dig a well
P41	Iftar menu
P42	Making salad at the best price

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