Elementary Education Online, 2017; 16(2): 566-583,
İlköğretim Online, 16(2), 566-583, 2017. [Online]:http://ilkogretim-online.org.tr
doi: 10.17051/ilkonline.2017.304717

Integration of Algebraic Habits of Mind into the Classroom Practice

Zihnin Cebirsel Alışkanlıklarının Sınıf Ortamına Entegrasyonu

Deniz Eroğlu, Arş. Gör. Dr., Mehmet Akif Ersoy Üniversitesi, Eğitim Fakültesi, denizc.eroglu@gmail.com
Dilek Tanışlı, Doç. Dr., Anadolu Üniversitesi, Eğitim Fakültesi, dtanisli@anadolu.edu.tr

ABSTRACT. Getting students adopt the habits of mind specific to many disciplines can be seen as an anchor that will support them to solve the problems in their daily lives. Starting from this premise, a new subject, featuring the habits of mind and how to use and improve them in educational environments seems to worth thinking on. This article describes an activity designed to integrate the components of algebraic habits of mind into the classroom practice. Data were collected in three weeks and analyzed by descriptive method within the analytical framework for algebraic habits of mind. As the results of the classrroom practice, students showed various ways of algebraic thinking, used multiple representations, generalized their solutions and made abstractions. They were also able to transform their relational and generalized thoughts, which have been verbally expressed at the beginning, to symbolic arguments and mathematical statements that allow development of formal algebra.
Keywords: Algebraic Habits of Mind, Classroom Practice, Middle School Students

ÖZ. Öğrencilere bir çok disipline özgü zihinsel alışkanlıkları kazandırmanın gerçek yaşamlarında problem çözmelerini destekleyecek bir dayanak noktası olabileceği ifade edilmektedir. Bu öneriden hareketle zihinsel alışkanlıkların ne olduğu ve öğretim ortamlarında nasıl kullanılıp geliştirileceği düşünmeye değer bir konu olarak görülmektedir. Bu çalışmanın amacı zihnin cebirsel alışkanlıklarının bileşenlerinin bir sınıf içi öğretim uygulamasına entegre edildiği bir etkinliğin tanıtımını yapmaktır. Veriler 3 haftalık bir süreçte toplanmış ve zihnin cebirsel alışkanlıkları çerçevesi bağlamında betimsel yöntemle analiz edilmiştir. Öğretim uygulaması sonucunda, öğrencilerde farklı cebirsel düşünme yolları ortaya çıktığı, öğrencilerin çoklu temsil kullandıkları, çözümlerini genelleyebildikleri, başlangıçta sözel olarak ifade ettikleri ilişkisel ve genellenmiş düşüncelerini formel cebirin gelişimini sağlayacak sembolik argümanlara ve matematiksel açıklamalara dönüştürdükleri ve işlemleri soyutlayabildikleri görülmüştür.
Anahtar Sözcükler: Zihnin Cebirsel Alışkanlıkları, Öğretim Uygulaması, Ortaokul Öğrencileri

ÖZET

Amaç ve Önem: Bu çalışmanın amacı zihnin cebirsel alışkanlıklarının bileşenlerinin bir öğretim uygulamasına entegre edildiği bir etkinliği uygulamak ve değerlendirmektir. Cebirsel düşünmenin geliştirilebilmesinde zengin ve ilişkisel bağlamlar oluşturmak, öğrencilerin istenilen düzeyde ve derinlikte cebirsel düşünmelerinin geliştirebilmesi için önemli görülmektedir (Driscoll, 1999; Kaput, 1999). Bu yüzden bu çalışmanın zihnin cebirsel alışkanlıklarının bir öğretim uygulamasındaki bir etkinliğe nasıl entegre edilebileceğine ilişkin hem öğretmenlere hem de matematik eğitimcilerine örnek teşkil edebileceği düşünülmektedir. Ayrıca zihnin cebirsel alışkanlığı üzerine özellikle Türkiye'de çalışmalara rastlanılmaması cebirsel düşünme bağlamında zihinsel alışkanlıkların göstergelerinin tanıtılması bakımından da çalışmanın alana katkı sağlayacağı söylenebilir.

Yöntem: Bu araştırmada yorumlayıcı yaklaşıma dayalı bir model olan (Wood, Cobb, Yackel, 1990), öğretim deneyi araştırma deseni olarak kullanılmıştır. Araştırma, ortaokul yedinci sınıf öğrenci grubu ve cebirsel düşünmenin gelişimine yönelik sınıf ortamı oluşturmada altı yıl deneyime sahip bir öğretmen ile gerçekleştirilmiştir. Çalışmada kullanılan öğretim etkinliği 2014-2015 eğitim-öğretim yılı güz dönemi başında üç haftalık bir süreçte toplam 31 öğrenci ile uygulanmıştır. Uygulama haftada bir saatlik periyodlarda devam etmiştir. Öğretim uygulamasında 'ardışık sayıların toplamı' başlıklı etkinlik kullanılmıştır. Tüm öğretim uygulamaları video kamera ile kayıt altına alınmış, veriler zihnin cebirsel alışkanlıkları çerçevesi bağlamında üç periyotta analiz edilmiştir.

Geliş tarihi: 12/04/2016, Kabul tarihi: 25/11/2016, Yayımlanma tarihi: 01/04/2017

Bulgular: Birinci öğretim uygulamasının sonucunda öğrencilerin özellikle örüntü arama becerisi konusunda yetersiz kaldıkları belirlenmiştir. Bunun nedeni olarak tablo, liste gibi görsel temsil kullanımına yönelik öğretmen yönlendirmesinin işlemediği söylenebilir. Bununla birlikte öğrencilerin açıklamalarında daha çok sözel ve sayısal temsilleri tercih ettikleri de görülmüştür. Bu ilk uygulama ve analizi sonucunda ikinci öğretim uygulaması sürecinde öğrencilerin tablo kullanmaları ve sözel ifadeden cebirsel ifadeye geçiş yapmaları sağlanarak çoklu temsil kullanımı da desteklenmiştir. İkinci öğretim uygulamasında öğretmenin gerçekleştirdiği tüm işlemler zihnin cebirsel alışkanlığının, fonksiyonel kural oluşturma bileşenine karşılık gelmektedir. İkinci dersin uygulamasından sonra yapılan makro-analizler sonucunda üçüncü dersin planlaması yapılırken öğrencilerin iki veya üç ardışık sayının toplamına ilişkin ortaya koydukları ilişkileri diğer toplamlara da uyarlamaları, aynı zamanda hangi sayıların ardışık sayıların toplamı şeklinde yazılamadığını belirlemeleri planlanmıştır. İlk iki ders öğrencilerin iki veya üç ardışık sayının toplamına ilişkin ifade ettikleri genellemelerini ve bu bağlamda kazandıkları farklı düşünme yollarını diğer ardışık sayıların toplamlarına ilişkin genellemelerine kolayca entegre ettikleri fark edilmiştir. Üç periyotta gerçekleştirilen çalışma sonucunda öğrencilerin başlangıçta daha kısır bir düşünceye sahipken süreçte farklı düşünme yollarının farkına vardıkları ve elde ettikleri sonuçları farklı bir duruma transfer edebildikleri söylenebilir.

Sonuç, Tartışma, ve Öneriler: Bu çalışmada üç periyodla sınırlanan bir süreçte sınıf sorgulaması ve çeşitli yönlendirme sorularıyla, öğrencilerde farklı cebirsel düşünme yolları ortaya çıktığı ve uzun bir süreçte ise bu düşünme yollarının alışkanlık haline dönüştürülebileceği söylenebilir. Sorgulama süreci öğrencilerin problemle ilgili çeşitli düşünme yolları geliştirmelerini, çoklu temsil kullanmalarını ve çözümlerini genellemelerini sağlamıştır. Zihnin cebirsel alışkanlıkları bağlamında ise öğrencilerin kendi genellemelerini, sözel ifadelerini, ya da kullandıkları diğer türdeki temsilleri cebirsel olarak sembolleştirebilmelerinin, onların uzun vadede yapılacak diğer etkinliklerle ortaya çıkan düşünme yollarını alışkanlık haline dönüştürebilmelerine katkısının olacağı söylenebilir. Sonuç olarak zihnin cebirsel alışkanlıkları öğrencilere kısa zamanda kazandırılabilecek türden düşünme yolları olmadığından öğretmenler süreç içerisinde gerçekleştireceği tartışmalar ve yönlendirmelerle öğrencilerin gelişimlerine katkı sağlayabilirler. Üç periyottan oluşan uygulamalı bu çalışmada öğrencilerin cebirsel alışkanlıklar kazandıkları iddia edilmemektedir. Ancak bu çalışmanın zengin öğretim-öğrenme ortamları oluşturulduğunda öğrencilerde cebirsel alışkanlıkların geliştirilebileceği yönünde farkındalık sağladığı ve örnek bir öğretim uygulaması olduğu söylenebilir. Dolayısıyla öğrencilerin zihnin cebirsel alışkanlıklarının gelişimine yönelik daha uzun süreli çalışmaların yapılmasına gereksinim vardır.

INTRODUCTION

Many researchers have addressed algebraic habits of mind (AHM) in different ways (Bass, 2008; Cuoco, Goldenberg and Mark, 1997; Matsuura, Sword, Piecham, Stevens, & Cuoco, 2013; Lim and Selden, 2009). However, the first systematic and in-depth representation was performed by Cuoco, Goldenberg and Mark (1996), where they discussed about the status of habits of mind in school mathematics and suggested to point out AHM as an organizing principle of the curriculum. In another study, Bass (2008) considered AHM as the practices performed by the mathematicians and listed them as posing questions, looking for a pattern, consulting to experts, making connections, refinement of mathematic language, examining evidences, making generalizations. Similarly, AHM is also defined as approaching mathematics problems in a customized ways and thinking mathematical concepts as done by the mathematicians (Cuoco, Goldenberg, & Mark, 1997, 2010; Goldenberg, Mark, & Cuoco, 2010; Mark, Cuoco, Goldenberg, & Sword, 2010). It has been suggested that AHM consists of the actions performed by the mathematicians in their works, such as continuous reasoning, studying extremes, performing thought experiments and using abstraction (Cuoco, Goldenberg and Mark, 1996). From this perspective, AHM can be defined as the skills where mathematical knowledge is used by different ways of thinking, which become habits.

Several studies have suggested that AHM should be a component of mathematics teaching knowledge (Matsuura, Sword, Piecham, Stevens, & Cuoco, 2013), a principle for mathematic curriculum and school culture, and a framework to be used in the development of students' reasoning (Cuoco, Goldenberg, & Mark, 1996) and an item that should affect the field courses of teacher education programs (Seaman & Szydlik, 2007). In addition, Common Core State Standards for Mathematics (CCSS-M, 2012) have stated that habits of mind consist of the skills that would lead to success in both academic and daily life. Similar to habits of mind, mathematical practices include the characteristics that enable people to perform mathematics in their routine work (Lim & Selden, 2009). Besides CCSS-M, many researchers have associated mathematical practices with habits of mind and they have suggested that these two concepts are equivalent and replaceable (Selden and Selden, 2005; Bass, 2008). Furthermore, Driscoll (1999) has developed a conceptual framework for AHM and made suggestions that habits of mind can help students in the development of algebraic thinking.

Review of national and international research literature has revealed that students have troubles and misconceptions concerning concepts such as variable, algebraic expressions, equation, (Dede, 2004; Küchemann, 1978; Soylu, 2008). Additionally, it was stated in the results of the PISA reports that Turkish students have difficulties in reflecting on their real-life problems and in expressing their thoughts through using these concepts (MEB, 2012). However, apart from algebraic knowledge, getting students to gain algebraic habits of mind can increase their success in exams like PISA and TIMMS. Although there is a strong emphasis that AHM can contribute to students' success in mathematics and in their daily lives, there is no clear consensus on how students can acquire algebraic habits of mind in the classroom contexts. This study aims to implement and evaluate an activity whose components were designed to be adapted to the classroom environment in order to set an example of its integration into the classroom context. To explain the lesson periods, teacher's actions to develop students' AHM and students' reactions and reflections to teachers' actions were determined within the classroom context. In addition, the deficiencies of teachers and student teachers in questioning skills (Tanisli, 2013) can be regarded as an obstacle to students' acquisition of algebraic habits of mind. Because pedagogical approaches that support the habits of mind are viewed as essential to develop flexible problem solving (CCSS-M, 2012; NCTM, 2000), this study makes a contribution to both teachers and mathematic educators about how to integrate the analytical framework for the AHM into their practice. Moreover, in-class implementations and the methods applied can pay contributions to mathematics educators with respect to the activities designed for improving mathematics teachers' and student teachers' professional development.

In the following section, the conceptual framework of AHM adapted using the components of Driscoll's (1999) framework and supported by many algebraic thinking research (CCSS-M, 2012; Herbert & Brown, 1997; Schoenfeld, 2014, 2015a, 2015b) will be discussed in detail.

Algebraic Habits of Mind

AHM brings many algebraic thinking components together and it allows working on the activities and discussions used on the algebraic thinking of the students in the classroom in a wider context. As can be seen from Table 1, it consists of three main components, namely "doing-undoing, building rules to represent functions and abstraction from computation".

Doing-undoing algebraic habit is considered as a roof component for the other two algebraic habits. This habit is always present in a problem solving case of the students, changes occur only in the indicators. *Doing* habit is considered with understanding the problem and it has some sub-components. Similarly, undoing habit has an inclusive nature for the other habits and it can occur in all problem-solving cases embedded in the other habits. The second habit, *building rules to represent functions*, consists of pattern seeking, pattern recognition and generalization components, which were used by Herbert and Brown (1997) in the analysis of problem solving process. Finally, the indicators of the third habit, *abstracting from computation*, are the using structures (presented as mathematical practices by CCSS-M, 2012) and formulation of generalization about computation (Driscoll, 1999).

Table 1. *Analytical Framework for AHM*

	DOING	
	Understanding the Problem - Reading, interpreting and understanding the problem within its context - Defining the quantities and the relationships among them - Developing the representation	
	BUILDING RULES TO REPRESENT FUNCTIONS	ABSTRACTION FROM COMPUTATION
D O I N G & U N D O I N G	Pattern Seeking - Revealing the pattern Pattern Recognition - Search repeating information outlining how the pattern works - Use of multiple representation - Predicting the pattern - Analyzing the change Generalization - Defining the rule - Validating the rule	Using structures - Finding computational short cuts - Writing an equivalent statement depending on need Formulation of generalization about computation - Testing whether the solution works on different conditions - Using computational short cuts to understand how number systems work - Thinking about computations freed from the particular numbers - Generalizing beyond examples - Expressing the generalizations about operations symbolically - Justifying computational short cuts
	UNDOING	
	Undoing - Finding input from output - Working backward	

The first algebraic habit: Doing – undoing

Effective algebraic thinking includes doing and undoing mathematical processes. In other words, taking a process from the starting point and forward it or to be able to think backward and reverse it. Driscoll (1999) described doing-undoing as the capacity not only to use a process to get to a goal, but also to understand the process well enough to work backward from the answer to the starting point. In other words, doing-undoing habit consider forward and backward analysis of a mathematical task as a natural process and it forms the basis for various mathematical practices, such as, finding solution of the equation, inverse function, derivatives, anti-derivatives and factorization (Moyer, Huinker, & Cai, 2004). So, for example, algebraic thinkers don't limit themselves with solving an equation such as $9x^2 - 16 = 0$, they can also answer the question, "What is an equation with solutions 4/3 and -4/3?"

Within the conceptual framework of this study, doing – undoing algebraic habit is embedded in the other two algebraic habits, and it starts with the actions of the students when they encounter a problem. These actions start with understanding the problem (Schoenfeld, 2014; 2015a, 2015b), which can be explained as describing the meaning of the problem. This ability is considered together with the following components; reading, interpreting and understanding the meaning of the problem within its context, defining the quantities and the relationships among them and developing the representation. Let's see understanding the problem and the components of this skill in detail using the problem below (Driscoll, 1999):

> *Locker Problem:* Imagine you are at a school that still has student lockers. There are 1000 lockers, all shut and unlocked, and 1000 students. Suppose the first student goes along the row and opens every locker. The second student then goes along and shuts every other locker beginning with number two. The third student changes the state of every third locker beginning with number three. (If the locker is open the student shuts it, and if the locker is closed the student opens it.) The fourth student changes the state of every fourth locker beginning with number four. Imagine that this continues until the 20 students have followed the pattern with the 20 lockers. At the end, which lockers will be open and which will be closed? Why? Which lockers were touched the most? After the 200th student opens/closes lockers, which lockers are open? Which are closed? Which lockers would be touched the most?

Understanding the problem is considered as the first step of the doing-undoing skill. This skill can be defined as the student explaining what the problem means to him. Here, the student should *read, interpret and understand the context of the problem*. In the locker problem, understanding can be expressed as defining one or more terms given in the problem, such as explaining the change of the state for the door. Another component of understanding the problem is *defining the quantities and the relationships among them*. Regarding the locker problem, this can be said as the first student will open all doors, the second student will close the doors that are the multiples of two and the third student will change the state of the doors that are the multiples of three. And finally, *developing the representation* is the representation of the identified quantities or terms and the relationships among them with symbol, picture, word, table and algebra. For instance, the data in the locker problem can be organized by using a representation as seen in Table 2.

Table 2. *The table that can be used in the solution of locker problem*

Student/Door	1	2	3	4	5	6	7	8	9	10	...
1	O	O	O	O	O	O	O	O	O	O	
2		C		C		C		C		C	
3			C			O			C		
...											

(O: Open, C: Close)

Undoing should be considered with *finding input from output* and *working backward* components. For example, determining the state of the door when 20th student follows the pattern or identifying numbers with three factors and six factors can be considered as undoing in locker problem. Doing – undoing is a habit continuing during the problem solving process, which is embedded in the other algebraic habits as well. Doing habit, which starts with understanding the problem, is completed by building rules to represent functions or abstraction from computation. Undoing can also appear during solution process.

The second algebraic habit: Building rules to represent functions

Another ability that is important for algebraic thinking is defining the pattern and organizing the data. Organizing the data means that showing the cases containing input numbers related to the outputs and using well-defined functions. For example "a number multiplied by 3 minus 4" can be written as $f(x) = 3x - 4$. This habit of mind is a natural complementary of doing-undoing habit. Algebraic thinking is expressed as the use of mathematical symbols and tools for revealing the terms and quantities of the problems; interpreting, applying and analyzing mathematical findings; and showing this information mathematically via representations (Herbert & Brown, 1997). Based on this statement, pattern seeking, pattern recognition and generalization are considered as the indicators of building rules to represent functions. These components are explained in details, using the example below (Driscoll, 1999):

> *Crossing the river:* There are eight adults and two children in a group. This group needs to cross a river, and they have one small boat available to help them. The boat can hold either one adult, or one or two children. Everyone in the group is able to row the boat. How many trips does it take for the whole group to cross the river? How many trips does it take for two children and any number of adults to cross the river?

While solving this problem, in order to *seek a pattern,* students may model the situation given in the problem via various representations, such as concrete materials, pictures, diagrams and tables and they can develop various methods to record the data that they obtained. With the accumulation of data, students can recognize the pattern and predict the rule, *search the repeating information outlining how the pattern works*, show various pieces of information by multiple representations and test different representations. For example, in a case students have first modeled the problem using concrete material, revealed the pattern presented in the problem and predicted a rule. Afterwards they explained the pattern of the problem as, "First two children go by boat and one of them comes back. Than one adult passes the river and the child brings the boat back. Even though this process lasts for the whole day, the number of adults is irrelevant, but there must be two children" (Herbert & Brown, 1997, p.125). After the occurrence of such a situation, students may seek the pattern (two children cross, one rows back, one adult crosses, child rows back…) and they can reach the *generalization* that four trips are needed to cross an adult to the other side. Students who reached such a generalization may define the rule (verbal, algebraic, etc.) and look for the validation of the rule.

The third algebraic habit: Abstraction from computation

Abstraction from computation habit can be explained as the ability of thinking about the operations freed from the numbers. For example, while teaching factorization of second degree functions, algebra tiles can be used to ensure that students understand how to model multiplication using area model. Afterwards, students should abstract this understanding from the arithmetic operation (Foster, 2007). In this abstraction, students should understand how the area model was used to explain the relationship between multiplier and multiplication and make a generalization what if variables were used instead of numbers. Since abstraction is one of the most important and significant characteristics of algebra, it should be noted that algebraic thinking includes the ability of thinking the operations freed from the numbers.

Abstraction from computation habit has two main components, namely using structures and formulation of generalization about computation. In CCSS-M (2012) it has been stated that mathematically capable students can recognize a structure, they can express that they see complex issues such as algebraic statements, as an object or a combination of many objects. For example, some students who are new to algebra, may attempt to solve 3(x+5)-2=20 equation by performing some operations step by step. Although this is sufficient to enable them to find the first step of the solution, students should be able to see the whole structure. It has been suggested that working on the structure will help students to see the logic of algebra (CCSS-M, 2012). For example, students who are working on the structure can easily read the equation noted as $11-\frac{50}{3x-2}$ (if we obtain five by dividing 50 to a number (3x-2), then 3x-2=10). Students who reached such a conclusion seek for computational short cuts (based on their understanding about how the operation works); they can find the solution in a fast and accurate way by using relational thinking skills.

Generalization about computation is the ability of thinking the operations freed from the numbers. A student having this skill; tests if the solution works on different conditions, uses computational short cuts to understand how number systems work, thinks about computations freed from the numbers, goes beyond the examples and develops generalizations, defines sets of numbers, expresses the cases where particular mathematical statements are valid or makes assumptions for these cases, expresses generalizations about operations using mathematical language (symbolically) and can justify computational short cuts using generalizations about the operations. For example, a student who calculate the sum of 1+2+3+...+100 can find the solution easily by grouping the numbers that summing up to 101 together, in other word by using a computational shortcut (Driscoll, 1999).

METHODS

In this research, the teacher experiment research design, a model grounded in the interpretive approach, (Wood, Cobb, & Yackel, 1990) was used. Teaching experiment can be described as a conceptual instrument that researcher use while designing their own activities. In teaching experiments, the focus can be not only on the success of students, either individually or as a group, in rich learning environments, or on the development of their thoughts, but also on the development of educational activities. Considering that this study aims to describe three lesson periods that were designed to integrate the components of AHM into classroom practice, the teaching experiment was chosen as the research design of the study. By this purpose, researchers completed the consecutive number task with seventh grade students and analyzed the lesson periods according to the AHM indicators introduced above. In this section, setting and task, and data sources and analysis are described in detail.

Setting and Task

In the typical curriculum, students learn variable and algebraic expression concepts in the sixth grade; and one-unknown equations and linear equations in the seventh grade. Thus this grade level was selected because by the end of the seventh grade, students were ready to produce different ways of algebraic thinking and acquire the algebraic habits in solving various problems. The task used in the study was completed at the end of 2014-2015 academic year through three weeks in a period of one hour per week. The task (adapted from Driscoll, 1999) requires students to explore the numbers that can or cannot be written as the sum of consecutive numbers. The following sums of consecutive numbers task (as shown in Table 3) were presented to a group of 31 seventh-grade students.

Table 3. *The sum of consecutive numbers task*

- 7=3+4
- 9=2+3+4
- 22=4+5+6+7

Above, there are some numbers written as the sum of consecutive numbers. Seven is written as the sum of two consecutive numbers; nine as the sum of three consecutive numbers; and 22 as the sum of four consecutive numbers. Accordingly:

a) Find all the ways that can be used to write each number between1 and 35, as the sum of two or more consecutive numbers.
b) Find a rule that can be used to write the numbers as the sum of two or more consecutive numbers.

The participants were invited to find all ways that can be used to write each number between 1 and 35 and a rule that can be used to write the numbers as the sum of two or more consecutive numbers. This task is selected because it paved the way to support all indicators of AHM. The teacher who has six year experience with the seventh graders implemented the lesson plan. She is currently pursuing a PhD at mathematics education program and she works with elementary students and is interested in developing classroom environments for fostering algebraic thinking. During the problem solving process, the teacher encouraged and supported students to use a variety of strategies, to communicate and explain their ideas, and to reflect upon their solutions.

Data Sources and Analysis

Since the purpose of the study is the description of components of AHM occurred into the classroom practice, videos of the courses was set as the main data source. Each of the lesson periods was videotaped. In addition, during the implementation another mathematics teacher observed the lesson and took field notes. These notes were analyzed after each period of lessons. Within each lesson, components of AHM were identified and plans of other lessons were prepared according to this macro analysis. After all video-records were transcribed, descriptive analysis was performed considering all three periods of lessons.

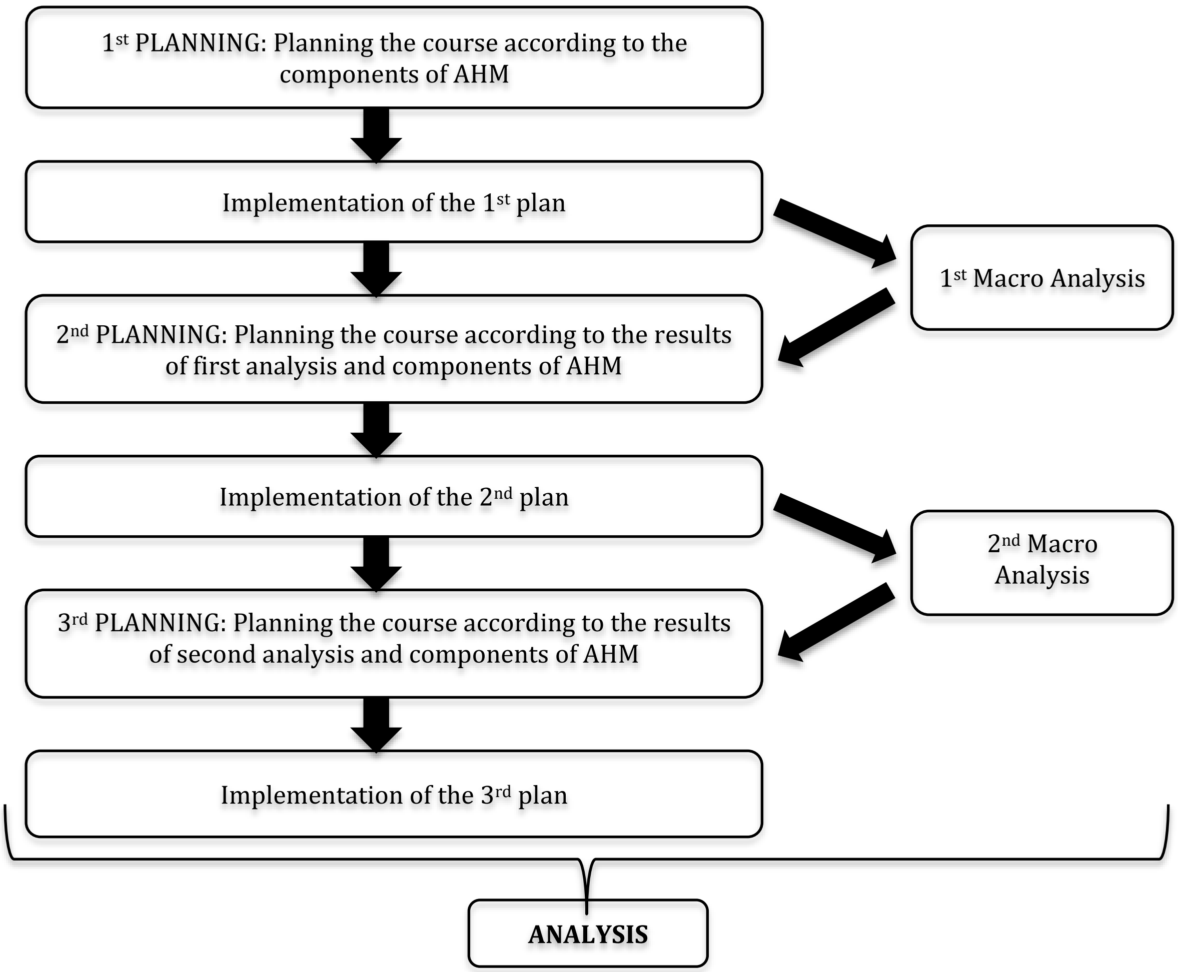

Figure 1. *The Implementation Process in The Context of AHM*

The data were analyzed in three periods as illustrated in Figure 1. Both teacher's and students' actions were analyzed according to the indicators AHM which is in the analytical framework. The first phase of coding involved identifying the indicators of each AHM which is adapted as an analytical framework. Whole dialogues were identified as an indicator of AHM such as read and understanding problem, using of representation and finding input from output. In the first phase of coding, all teachers' observable activities and actions were coded as such through a process code (Saldana, 2009). After the first cycle coding, the categories from the analytical framework were determined. The codes and categories that arised from data were shown in Table 4.

Table 4. *Categories and codes for each type of AHM*

Categories	Codes
Ability to understand the problem	- Reading and understanding the problem - Developing the representation
Pattern Seeking	- Revealing the pattern
Pattern Recognition	- Using multiple representations - Searching repeating information outlining how the pattern works - Analyzing the change
Generalization	- Defining the rule - Validating the rule
Formulation of Generalization about Computation	- Testing whether the solution works on different conditions - Generalizing beyond examples - Using computational short cuts to understand how number systems work - Thinking about computations freed from the particular numbers - Expressing the generalizations about operations symbolically
Undoing	- Finding input from output - Working backward

To obtain inter-rater agreement, each of the researchers independently coded the whole transcribed text according to the analytical framework. After the coding process was completed, each researcher compared all codes and they reached acceptable agreement on the codes and categories.

RESULTS

The findings of the study are presented for all three periods, in terms of the components and sub-components of AHM.

The First Period

Asking questions to students and letting them to ask questions during the problem-solving process are quite important for the development of AHM, because they are creating clues to the solution. Since understanding the problem and planning various ways to solve it are the first two steps of AHM development, it has been decided to start the periods with doing-undoing habit. Thought-provoking and guiding questions towards the development of AHM were conceived in detail and several questions were added to the course programs to support inquiry.

During the implementation phase of the plan, first the teacher introduced the sum of consecutive number problem to the students; then he asked them to form groups of two and write the numbers between 1 and 35 as the sum of two, three, four, five and six consecutive numbers. Before letting students to work in groups, teachers asked students to *read and understand the problem*, and to give the example for the sum of consecutive numbers. Then, students worked in groups of two for about fifteen minutes and they made efforts to understand the problem. Starting from this moment, *understanding the problem* skill of the students, which is one of the skills of *doing* habit, has been emerged. In order to understand the problem, students picked up some numbers randomly and attempted to write them as the sum of consecutive numbers. The teachers walked across the classroom, checked the performance of the students, and cleared the misunderstandings and unclear points. Students encountered some difficulties in the process of understanding the problem. For example one of the students has taken four and five as consecutive numbers and used seven to sum up them to 16. However, with the objection of other students in the classroom, it was agreed that this sum could not be accepted. Another challenge encountered in the process of problem understanding is obtaining the desired number as the sum of two consecutive odd numbers. For example,

> Emre: Sir, for example sixteen … we add one of the numbers before it … nine plus this number, for example we add one to six, get seven. Nine plus seven is equal to sixteen.
> Teacher: Nine and seven, are they consecutive numbers?
> Emre: Nine plus seven makes sixteen.
> Teacher: Let me ask again. Seven and nine, are they consecutive numbers?
> Emre: They are consecutive odd numbers
> Teacher: Ok, the problem asks you to write consecutive numbers.

After discussing false examples, the teacher got sure that the students understood how to handle consecutive numbers within the problem. In this process, some students have worked on the sum of two or three consecutive numbers by picking up random numbers. In this case, the teacher directed students to *use representations*, such as forming a table, listing (as shown below) in order to prevent random selection of numbers. The purpose of this guidance was ensuring students *to reveal the existing patterns and to use the representations*. These instructions are for the development of *pattern seeking and pattern recognizing*, which are the indicators *building rules to represent functions*.

1	0+1
2	
3	1+2
...	
35	

Figure 2. *The multiple representation for listing number*

After finishing the processes of understanding the problem and group discussions, some discussions featuring the ways of writing the numbers as the sum of consecutive numbers and some generalizations took place. Teacher saw that some students encountered difficulties in undoing and generalizing, thus he stopped walking around and started a class discussion. During this discussion, teacher gave students a voice to share their different ways of thinking that was the indicators of the algebraic habits in order to allow other students to be aware of these thinking ways. This decision focused on the development of *undoing and formulation of generalization about computation.* These two skills are the indicators of *Undoing* and *Abstraction from Computation* habit. The student stated her way that is subtracting one from the number and divide two to find one of the consecutive numbers. After she expressed her way, the teacher asked her to show the way with an example. Below, there is a student dialogue showing an indicator of AHM.

> Melisa: I found a way to do it. We subtract one from the number; than we divide the remaining number by two, it gives one of the consecutive numbers.
> Teacher: How? Give me an example.
> Melisa: Let's take thirteen for example, thirteen minus one is twelve. We get six by dividing it by two. Six plus seven makes thirteen.
> Teacher: So you write thirteen as the sum of six and seven
> Melisa: I can do the same operation for all odd numbers.

The student has also showed her works on board while explaining her way of solving. The student *used a computational shortcut* to *find the input from the output* and she expressed the valid generalization for all odd numbers using this shortcut. After a while, the teacher recognized that one of the students was investigating whether the shortcut was also valid for even numbers or not. Then he asked the following question in order to allow the whole class to think this question. Students gave an even number examples and show the rule did not work with them.

Emre: Is the way we found valid for even numbers?
Teacher: (Turn to the class) Do you think that the way that we have found is valid for even numbers as well?
Students: No.
Teacher: Why?
Alp: If we subtract one from sixteen, we get fifteen. Two does not divide fifteen.
Emre: It doesn't work sir, for example for thirty-four.

By this discussion, teacher aimed to enable all students to *test whether the result works in different situations or not* and *to generalize beyond examples*. In addition, as shown in the dialogue, teacher's encouragement led students to show that the rule is not valid for all cases by giving counter examples. After this discussion, teacher opened the floor for discussion whether they could find a shortcut for the sum of three consecutive numbers. However, students could not express the shortcut for three consecutive numbers as in two consecutive numbers. On the contrary, students tended to obtain the sum by using randomly picked up three consecutive numbers. That is to say, they used random inputs in order to get the result of adding numbers. This fact prevented students to reveal the existing pattern and consequently to find the shortcut. At this moment, the teacher stated that the numbers written as the sum of three consecutive numbers should have something in common in order to lead students to find a pattern. Students recognized that the numbers increased by six by looking through the random numbers written on the board. However, this pattern could not lead them to find the shortcut writing a number as the sum of three consecutive numbers, it just gave a clue. Even though this class discussion mainly aimed to foster students' *formulations of generalization about computation*, it can be said that it has contributed to the improvement of *pattern seeking, pattern recognition and generalization* as well, which are the indicators of building rules to represent functions. The dialogue that exemplifies this line of inquiry is as follows:

Onur: Sir, five plus six plus seven is eighteen.
Teacher: Yes, it sums up to eighteen. Think about how I can continue to write the others?
Hakan: Sir, we can subtract two from all numbers (showing 5+6+7 at the board), two from this, two from this, two from this. So we obtain three plus four plus five.
Students: It makes twelve.
Teacher: What about other numbers?
Hakan: Seven plus eight plus nine.
Teacher: Which one?
Ecem: Twenty-four
Teacher: Let's look at these numbers. The numbers that can be written as the sum of three consecutive numbers should have something in common.
Mine: They increase by six.
Hakan: Sir, I found thirty-six: eleven plus twelve plus thirteen.
Teacher: You didn't say a rule for the sum of three consecutive numbers.
Hakan: Sir I found it.
Teacher: What is it?
Hakan: I found for six, twelve, eighteen, twenty-four, thirty, thirty-six.
Teacher: What is the numbers resulting with thirty?
Hakan: I already told them.
Teacher: You told them for 36.
Hakan: Sorry. Nine plus ten plus eleven.

Teacher: OK, all of these increase by six. You caught a clue. Think more about it.

After this discussion, the first instruction finished. The macro analysis conducted at the end of the application revealed the working and non-working parts of the course within the context of AHM. Accordingly, pattern seeking was not the preference of the students in order to solve the problem. Moreover, the students mostly preferred verbal and numerical representations in their explanation.

The Second Period

As a result of the macro-analysis conducted after the first instruction, it has been decided to distribute tables to the students, because most of the students have encountered difficulties in finding the pattern composed by the numbers that was written as the sum of consecutive numbers. The table, shown as Table 5, has allowed students to reveal the *pattern*. At the same time, using the table could enable them to acquire *use of representation* habit. The teacher started instruction by asking students to write 15 as a sum of two consecutive numbers so as to recall them shortcut found in the instruction before. During this inquiry, students remembered the rules and generalizations that they have previously built and explained them verbally. Then, students filled the table with the numbers written as the sum of two consecutive numbers.

Table 5. *A table that can be used to record the ways of writing a number as the sum of several consecutive numbers*

Number/Consecutive number	2	3	4	5	6	Cannot be written
1						
2						
3	1+2					
...						

Afterwards, the teacher questioned the numbers that was written as the sum of three consecutive numbers and asked students to record them at the appropriate places in the table. After he ensured that all students found all consecutive number sums, he asked students about a variety of ways to write the number as the sum of three consecutive numbers. As a result of the discussions students stated that 6, 9, 12, 18, 24, 27, 30 and 36 was written as the sum of three consecutive numbers. Students easily *revealed the pattern* when they used the table; they *discovered the information about how the pattern works*; and they *defined the rule verbally*. As can be seen from the below discussion, after recognizing the pattern, students tend to *validate the rule of* the pattern that they have expressed verbally with the guidance of the teacher.

Teacher: So, what are the numbers that I can write as the sum of three consecutive numbers?
Emre: They increase by three, multiples of three.
Teacher: How do you show that all the multiples of three are written as the sum of three consecutive numbers?
Melisa: Shall we subtract three first, and then divide by three?
Teacher: Why?
Melisa: Because … for example three consecutive numbers goes like one, two, three. The difference of the second number and the first is one and the difference of the third and first is two. So, to write it as the sum of three consecutive numbers, first we subtract three and then divide by three.

One of the students expressed the way to find one of the numbers for the sum of three consecutive numbers. He stated that multiples of three could be written as the sum of three consecutive numbers. After teachers asked the reason for this fact, another student give an example and expressed the reason that subtracting the increase between numbers and dividing it by three gave one of the numbers. Until this point in the second instruction, all these practices were for the development of *building rules to represent functions*. In the continuation of this discussion, the teacher asked students to find consecutive numbers of nine. By doing this, the teacher wanted students to *work backward* and attempted to develop the habit of *undoing*.

> Teacher: So, what did you find when you subtract three and divide by three? As an example apply your rule to nine.
> Melisa: I subtracted three, get six; I divided six by three and get two.
> Teacher: OK you found two, which number is two?
> …
> Onur: We write the number that is consecutive to two.
> Teacher: Yes, than which number is two?
> Melisa: The first one.

Here, students attempted *to find the input from the output* and they expressed that they have found the first consecutive number by applying the operation. In order to take this discussion a little further and to allow the students to describe the rule algebraically, the teacher asked how students express this rule algebraically.

> Teacher: How can we express the sum of three consecutive numbers?
> Melisa: x minus three divided by three, as an equation?
> Teacher: What does x mean when you write x minus three divided by three?
> Melisa: The number that is the multiples of three.
> ...
> Teacher: If I call the sum as x, what do I find by x minus three divided by three?
> Melisa: ıı ... the first consecutive number
> Teacher: So, if I call the first consecutive number "n", how can I write the sum?
> Onur: n plus one and n plus two.
> ...
> Teacher: Than, what is the sum?
> Emre: Three n plus three (3n+3).

A student expressed the rule algebraically as $\frac{(x-3)}{3}$, after stating it verbally at the beginning, which he has reached using the ability of finding input from output. Then, when the teacher assigned a variable to the first number and asked the rule again, the student expressed the rule as 3n+3.

During the second instruction, suggesting students a table and shifting from verbal to the algebraic statement reinforced students' usage of multiple representations. All the operations of the second period correspond to b*uilding rules to represent functions* component of AHM. In addition, in order to foster their *undoing* habit, they were asked to illustrate how to write 24 as the sum of three consecutive numbers, they solved 3n+3=24, thus *doing-undoing* component was also taken into account in the process.

The Third Period

As a result of the macro-analysis conducted after the second period, it has been decided that in the third period students should start to get adopt the relationships that they have revealed for the sum of two and three numbers to the remaining sums and discover the numbers that cannot be written as the sum of consecutive numbers. In this context, the teacher started the course by asking the numbers that cannot be written as the sum of consecutive numbers. The purpose of this question was to foster *pattern recognition and generalization* skills of the students for the numbers that are powers of two.

> Teacher: Now, what are the numbers that cannot be written as the sum of consecutive numbers?
> Students: Two, four, eight, sixteen, thirty-two, sixty-four… (all students in the class)
> Özlem: Sir, I couldn't find one … we start with one. For example one times two is two, missing; tow times two is four, missing; four times two is eight, missing. It goes like that.
> Students: Multiples of two are missing
> Teacher: Can you tell me the numbers that are multiples of two?
> Students: Two, four, six, eight, ten.
> Teacher: But, six and ten are not in the numbers that you have listed before?
> Onur: These are exponents of two.
> Teacher: Exponents of two yes. How do we call exponents of two?
> Melisa: Powers of two.
> Teacher: Than the powers of two cannot be written as the sum of consecutive numbers.

In the discussion above, it has been observed that the student reached the generalization using the abilities of *pattern seeking* and *pattern recognition*. This student recognized *the repeating information*, he stated that the numbers increases through multiplying by two; meanwhile he *analyzed the change* occurred between the terms of the pattern. Thus, all the students in the class were attempted to gain ways of thinking within the scope of *pattern recognition and generalization* ability. After this discussion, the other decision considered in the planning was realized.

After the first two instruction that students generalized about the sum of two or three consecutive numbers and found different ways to find the consecutive numbers, they could easily adapted the generalizations for the sums of other consecutive numbers. Accordingly, students expressed the rules such as 2n+1, 3n+3, 4n+6, 5n+10 and 6n+15 in order to use all numbers between 1 and 35. For example, one of the students said that if he write n for one of the consecutive number and n+1 for the other one, he could find the rule for two consecutive numbers. Some of the students have showed different ways of thinking while expressing the rules. For example one of the student was more flexible in telling the numbers represented by the symbols and she played with the symbols. She stated that the rule for two consecutive numbers could depend on place of the variables that she gave on the numbers.

> Selin: The difference between two consecutive numbers is always one. For example if we call the first one as n, the second became (n+1). On the other hand if we have called the second one as n, the first on would be (n-1).
> Teacher: So, what would be the rule then?
> Selin: 2n-1
> Teacher: Which number is n for (2n-1) and for (2n+1)?
> Selin: There (pointing 2n-1) n is the second number, there (pointing 2n+1) n is the first number.

As can be seen above, student started the relationship between two consecutive numbers and she discovered that the only way to state the rule is not 2n+1, it could be expressed as 2n-1 as well. At this point it can be said that the student reached a generalization by *thinking about computations freed from the particular numbers* and *expressed it symbolically*, and consequently *abstracted computation*. On the other hand, another student *revealed the pattern* formed by the numbers 1,3,6,10, by considering the constants used in the rules 2n+**1,** 3n+**3,** 4n+**6,** 5n+**10** and he continued the pattern as 6n+**15** by recognizing the *repeating information in the pattern.*

> Tayfun: Sir may I say something? I found a simple method.
> Teacher: Yes.
> Tayfun: Sir, for two numbers we said plus one, for three we said plus three, for four we said six. If you look at here (pointing the numbers in the rules) it increases by two, here it increases by three. After this it should increase by four. Every time it increases plus one than the previous increase.
> Teacher: So, what is the rule to write the sum of five consecutive numbers?
> Tayfun: According to my rule, it is 5n+10.
> Teacher: What is the rule for the sum of six consecutive numbers?
> Onur: 6n+15

After writing down all the rules, teacher asked students to write bigger numbers, such as 45, 57, 62, 75, 80 and 123, as the sum of consecutive numbers, in order to foster *working backward*. For example:

> Teacher: How can you write hundred and twenty-three as the sum of consecutive numbers? What are these numbers?
> Onur: I found as sixty-one plus sixty-two
> Teacher: How do you find it?
> Onur: I subtract one and divide by two.
> Tayfun: Two n plus one is equal to hundred and twenty-three (2n+1 =123)
> Onur: We take plus one to the other side, it becomes two n is equal to hundred and twenty-two (2n=122). I divide both sides by two and n is sixty-one.
> Teacher: OK, what are the other consecutive numbers?
> A student wrote 3n+3=123 to the board and found n as 40.
> Teacher: Which number is n?
> Selin: The first one. We can write it as forty plus forty-one plus forty-two.

As can be seen from this class discussion, while writing 123 as the sum of two and three consecutive numbers students applied the established rules to the given numbers and obtained the consecutive numbers by *working backward*, which is an indicator of *undoing.*

As a result of the study implemented in three periods, it can be said that even though students had initially more unproductive thoughts, they realized different ways of thinking in the process and they could be able to transfer the results that they have found to a different situation.

DISCUSSION, CONCLUSION and RECOMMENDATION

Mathematics educators reached a consensus about teaching AHM instead of teaching algebra as a subject area (Driscoll, 1999; Kaput, 1999; Kieran, 2004; Molina, Castro, & Ambrose, 2005). Regarding present study, it was revealed that different algebraic ways of thinking have been emerged among students as a result of questionning and various guiding questions and these ways of thinking might become habits in a longer process. Indeed, Driscoll (1999) argued that students may gain algebraic thinking habits but teachers play a crucial role in fostering these habits. In this study, the questioning approach of the teacher affected algebraic reasoning of the students (Windsor and Norton, 2011). The questionning process allowed students to develop various ways of thinking about the problem, use of multiple representations and generalize their solutions. In addition, it has been observed that students transformed their relational and generalized thoughts, which they have expressed verbally at the beginning, to symbolic arguments and mathematical statements that allow the development of formal algebra. Conducted studies revealed that students have deficiencies on the algebraic statement of the cases that they have stated verbally (Schonfeld, 1988; Zazkis & Liljedahl, 2002). It has been suggested that students who have difficulties about algebraic generalizations cannot express their solution completely (Zazkis and Liljedahl, 2002). Unlike the existing studies in the literature, the instruction and guiding questions of the teacher revealed students' ability of making generalization and stating them algebraically, which is the base of students' algebraic thinking (Charbonneau, 1996). Indeed, it was also stated that the theoretical (algebraic) generalization requires a precise symbolic definition (Dörfler, 1991). In terms of AHM, it can be said that students' algebraic symbolization of their own generalization, verbal statements or other types of representations that they use, will make a contribution to the transformation of their ways of thinking, which will emerge via the activities to be performed in the long term, into habits.

Regarding the ability of abstraction from computation, the ability of formulation of generalization about computation, in other words finding computational short cuts, the indicators emerged during the class discussions were thinking about computations freed from the numbers and abstraction from computation. In a similar study (Blanton and Kaput, 2011), examining students' transition to the use of symbols via an instruction starting from their thoughts and going towards the introduction of symbolic presentation, it has been argued that meaningful symbolic reasoning will better prepare students to the abstractions that will be performed in advanced mathematic courses. It has been suggested that students who can think algebraically should be able to use unknown quantities as known quantities (Swafford and Langrall, 2000), not only focus on the operation, but to be able to focus on the inverse operation as well (doing-undoing) and consider both numbers and letters (Kieran, 2004). From this perspective, the mentioned ways of thinking built on students' own algebraic understanding are considered to be important for providing an entry point to more complex algebra (Blanton & Kaput, 2011; Brenner et al., 1997; Kysh, 1991; Windsor & Norton, 2011).

As a result, since the AHM are not the ways of thinking that students can gain in a short time, long term discussions and guidance of the teachers may contribute to the development of the students. This study was shaped according to the classroom discussions and the students' answers. The researchers do not claim that students gained algebraic habits during this study which consists of only three periods. However this study creates awareness that providing rich classroom environments (that includes teacher-student and student-student interaction, using multiple representations, questioning, functional thinking, thinking independently) could develop students' algebraic habits. Hence, long-term studies, featuring the development of students' AHM, are needed. Meanwhile, as stated by Cuoco, Goldenberg and Mark (1996), habits of mind are recommended to be an organizing principle of the curriculum in the process of the development of content areas.

REFERENCES

Bass, H. (2008, January). *Helping Students Develop Mathematical Habits of Mind.* Paper presented at a Project NExt Session on Joint Mathematics Meetings, San Diego, CA.

Blanton, M., & Kaput, J. (2011). Building mathematical generality into curriculum and instruction. In J. Cai and E. Knuth (Eds*.), Early algebraization: A global dialogue from multiple perspectives. Advances in Mathematics Education Monograph Series* (pp.5-23). New York: Springer.

Brenner, M. E., Mayer, R. E., Moseley, B., Brar, T., Duran, R., Reed, B. S.,... & Webb, D. (1997). Learning by understanding: The role of multiple representations in learning algebra. *American Educational Research Journal*, *34*(4), 663-689.

Charbonneau, L. (1996). From Euclid to Descartes: Algebra and its relation to geometry. In N. Bednarz, C. Kieran & L. Lee (Eds.), *Approaches to Algebra: Perspectives for Research and Teaching* (pp.15-37). Dordrecht: Kluwer Academic Publishers.

Common Core State Standards Initiative [CCSM] (2012). *Common Core State Standards for Mathematics.* Retrieved at June 25, 2014 from http://www.corestandards.org/Math.

Cuoco, A., Goldenberg, E. P., & Mark, J. (1997). Habits of mind: An organizing principle for mathematics curriculum. *Journal of Mathematical Behavior, 15*(4), 375–402.

Cuoco, A., Goldenberg, E. P., & Mark, J. (2010). Contemporary curriculum issues: Organizing a curriculum around mathematical habits of mind. *Mathematics Teacher, 103*(9), 682-688.

Dede, Y. (2004). Değişken kavramı ve öğrenimindeki zorlukların belirlenmesi [The concept of variable and identification its learning difficulties]. *Kuram ve Uygulamada Eğitim Bilimleri Dergisi, 4* (1), 24-56.

Dörfler, W. (1991). Forms and means of generalization in mathematics. In A. J. Bishop (Ed.), *Mathematical Knowledge: Its Growth through Teaching* (pp.63-85). Dordrecht: Kluwer Academic Publishers.

Driscoll, M. (1999). *Fostering algebraic thinking: A guide for teachers, Grades 6-10.* Portsmouth, NH: Heinemann.

Foster, D. (2007). Making meaning in algebra examining students' understanding and misconceptions. In A. H. Schoenfeld (Ed.), *Assessing Mathematical Proficiency* (pp.163-176). Berkeley: Cambridge University Press.

Goldenberg, E. P. (1997). "Habits of mind" as an organizer for the curriculum. *Learning and Teaching Mathematics, 178*(1), 13-34.

Herbert, K., & Brown, R. H. (1997). Patterns as tools for algebraic reasoning. *Teaching Children Mathematics, 3*(6), 340-344.

Kaput, J. J. (1999). Teaching and learning a new algebra. In E. Fennema & T. A. Romberg (Eds.), Mathematics classrooms that promote understanding (pp. 133-155). Mahwah, NJ: Lawrence Erlbaum Associates.

Kieran, C. (2004). Algebraic thinking in the early grades: What is it?. *The Mathematics Educators, 8*(1), 139-151.

Küchemann, D. (1978). Children's understanding of numerical variables. Mathematics in School, 7(4), 23-26.

Kysh, J. (1991). Implementing the curriculum and evaluation standards: First year algebra. *Mathematics Teacher, 84*, 715-722.

Lim, K. H., & Selden, A. (2009). Mathematical habits of mind. In S. L. Swars, Stinson, D. W., & Lemons-Smith, S. (Eds.), *Proceedings of the Thirty-first Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1576-1583). Atlanta: Georgia State University.

Mark, J., Cuoco, A., Goldenberg, E. P., & Sword, S. (2010). Contemporary curriculum issues: Developing mathematical habits of mind. *Mathematics Teaching in the Middle School, 15*(9), 505-509.

Matsuura, R., Sword, S., Piecham, M., B., Stevens, G., & Cuoco, A. (2013). Mathematical habits of mind for teaching: Using language in algebra classrooms. *The Mathematics Enthusiast, 10*(3), 735-776.

MEB, (2012). PISA 2012 Araştırması Ulusal Nihai Rapor. http://pisa.meb.gov.tr/?page_id=22 internet sitesinden 15.05.2015 tarihinde alınmıştır.

Molina, M., Castro, E., & Ambrose R. (2005). Enriching arithmetic learning by promoting relational thinking. *The International Journal of Learning, 12*(5), 265 – 270.

Moyer, J., Huinker, D., & Cai, J. (2004). Developing algebraic thinking in the earlier grades: A case study of the U.S. Investigations Curriculum. *The Mathematics Educator, 8*(1), 6-38.

Saldana, J. (2009). *The coding manual for qualitative researchers.* Los Angeles, CA: SAGE.

Schoenfeld, A. H. (1988). 'When good teaching leads to bad results: The disaster of well taught mathematics courses'. *Educational Psychologist, 23*(2), 145-166.

Schoenfeld, A. H. (2014). What makes for powerful classrooms, and how can we support teachers in creating them? *Educational Researcher, 43*(8), 404-412. DOI: 10.3102/0013189X1455

Schoenfeld, A.H. (2015a). Thoughts on scale. *ZDM, The International Journal of Mathematics Education, 47*(1), 161-169. DOI 10.1007/s11858-014-0662-3

Schoenfeld, A.H. (2015b). *The teaching for robust understanding framework.* Retrieved April 20, 2015, from http://ats.berkeley.edu/tools/The%20Teaching%20for%20Robust%20Understanding%20(TRU)%20Framework.pdf.

Seaman, C. E., & Szydlik, J. E. (2007). Mathematical sophistication among preservice elementary teachers. *Journal of Mathematics Teacher Education, 10*, 167-182.

Selden, A. & Selden, J. (2005). Perspectives on advanced mathematical thinking. *Mathematical Thinking and Learning, 7*, 1-13.

Soylu, Y. (2008). 7. sınıf öğrencilerinin cebirsel ifadeleri ve harf sembollerini (değişkenleri) yorumlamaları ve bu yorumlamada yapılan hatalar [7th grade students' interpretation of algebraic expression and symbol of letters while doing these interpretations]. *Selçuk Üniversitesi Ahmet Keleşoğlu Eğitim Fakültesi Dergisi, 23*, 237-248.

Swafford, J. W. & Langrall, C. W. (2000). Grade 6 students' preinstructional use of equations to describe and represent problem situations. *Research in Mathematics Education, 131*(1), 89-112.

Tanışlı, D. (2013). İlköğretim Matematik Öğretmeni Adaylarının Pedagojik Alan Bilgisi Bağlamında Sorgulama Becerileri ve Öğrenci Bilgileri, *Eğitim ve Bilim, 38* (169), 80-95.

Windsor, W., & Norton, S. (2011). Developing algebraic thinking: Using a problem solving approach in a primary school context. In J. Clark, B. Kissane, J. Mousley, T. Spencer & S. Thornton (Eds.), *Proceedings of the 34th annual conference of the Mathematics Education Research Group of Australia* (pp.813-820). Adelaide: MERGA.

Wood, T., Cobb, P. & Yackel, E. (1990). The contextual nature of teaching: Mathematics and reading instruction in one second-grade classroom. JSTOR: The Elementary School Journal, 90(5), 497-513.

Zazkis, R., & Liljedahl, P. (2002). Generalization of patterns: The tension between algebraic thinking and algebraic notation. *Educational Studies in Mathematics, 49,* 379-402.