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Research Article

On Reciprocity Formula for Certain Character Hardy–Berndt Sums

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Abstract

Applying character analogue of the Euler–MacLaurin summation formula to certain functions, we define sums which generalize character Hardy–Berndt sums $s_{3,p}(b,c:\chi)$ and $s_{4,p}(b,c:\chi)$, and prove the corresponding reciprocity formulas. We also obtain integral representations for these sums.

Keywords: Dedekind sum, Hardy-Berndt sum, Bernoulli polynomial, Euler-MacLaurin summation formula.

Bazı Karakter Hardy–Berndt Toplamlarının Resiprosite Formülü Üzerine

Öz

Euler–MacLaurin toplama formülünün karakter benzerini bazı fonksiyonlara uygulayarak, $s_{3,p}(b,c:\chi)$ and $s_{4,p}(b,c:\chi)$ karakter Hardy–Berndt toplamlarının genelleştirmeleri olan toplamlar tanımladık ve karşılık gelen resiprosite formüllerini ispatladık. Ayrıca, bu toplamlar için integral gösterimleri elde ettik.

Anahtar kelimeler: Dedekind toplamı, Hardy-Berndt toplamı, Bernoulli polinomu, Euler-MacLaurin toplama formülü.

Introduction

For the logarithms of the classical theta functions, Berndt [2] and Goldberg [7] derived transformation formulas. In these formulas, six different arithmetic sums arise that are known as Berndt's arithmetic sums or Hardy-Berndt sums. Two of these sums are defined for $c > 0$ by

$$s_3(d,c) = \sum_{n=1}^{c-1} (-1)^n \overline{B}_1\left(\frac{dn}{c}\right),$$

$$s_4(d,c) = \sum_{n=1}^{c-1} (-1)^{[dn/c]} = -4\sum_{n=1}^{c-1} \overline{B}_1\left(\frac{dn}{2c}\right),$$

where $\overline{B}_n(x)$ are the Bernoulli functions given by

$$\overline{B}_n(x) = B_n(\mathrm{x}-[\mathrm{x}]), \quad n>1,$$
$$\overline{B}_1(x) = \begin{cases} 0, & x \text{ integer}, \\ B_1(\mathrm{x}-[\mathrm{x}]), & \text{otherwise}. \end{cases}$$

Here, [x] denotes the largest integer $\leq x$ and $B_n(x)$ are the Bernoulli polynomials

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defined by means of the generating function [1]

$$\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}\quad(|t|<2\pi)$$

and $B_n(0)=B_n$ are the Bernoulli numbers with $B_0=1$, $B_1=-1/2$ and $B_{2n-1}\left(\frac{1}{2}\right)=B_{2n+1}=0$ for $n\geq 1$.

These sums also obey reciprocity formulas. For instance, for coprime positive integers d and c we have ([2, 7])

$$2s_3(d,c)-s_4(c,d)=1-\frac{d}{c},\ \text{if } c \text{ is odd.}$$

The generalizations of these sums in the sense of Apostol have been given in [4] by

$$s_{3,p}(b,c)=\sum_{m=1}^{c-1}(-1)^m\overline{B}_p\left(\frac{bm}{c}\right),$$

$$s_{4,p}(b,c)=-4\sum_{m=1}^{c-1}\overline{B}_p\left(\frac{bm}{2c}\right),$$

which satisfy certain reciprocity formulas (see [4]).

The character generalization of these sums has been given in [5] by

$$s_{3,p}(b,c:\chi)=\sum_{n=1}^{ck}(-1)^n\chi(n)\overline{B}_{p,\chi}\left(\frac{bn}{c}\right),\quad(1.1)$$

$$s_{4,p}(b,c:\chi)=\sum_{n=1}^{ck}\chi(n)\overline{B}_{p,\chi}\left(\frac{bn}{2c}\right),\quad(1.2)$$

where $\overline{B}_{p,\chi}(x)$ are the generalized Bernoulli functions defined by Berndt [3] as

$$\overline{B}_{p,\chi}(x)=k^{p-1}\sum_{j=0}^{k-1}\overline{\chi}(j)\overline{B}_p\left(\frac{j+x}{k}\right)$$

for $p\geq 1$. Also, the corresponding reciprocity formula is established in [5].

The reciprocity formulas are proved by employing various techniques and theories such as transformation formulas, residue theory, Franel integral and arithmetic methods.

Recently, the authors in [6] systematically generalize the Dedekind sum $s_p(b,c:\chi)$ (analogues of Hardy-Berndt sums) to sums involving two primitive characters and prove the corresponding reciprocity formulas applying the character analogue of the Euler–MacLaurin summation formula, which is presented here in the following form.

Theorem 1 *([3, Theorem 4.1]) Let $f\in C^{(l+1)}[\alpha,\beta]$, $-\infty<\alpha<\beta<\infty$. Then,*

$$\sum_{\alpha\leq n\leq\beta}{}'\chi(n)f(n)=\chi(-1)\sum_{j=0}^{l}\frac{(-1)^{j+1}}{(j+1)!}\times\left(\overline{B}_{j+1,\overline{\chi}}(\beta)f^{(j)}(\beta)-\overline{B}_{j+1,\overline{\chi}}(\alpha)f^{(j)}(\alpha)\right)+\chi(-1)\frac{(-1)^l}{(l+1)!}\int_{\alpha}^{\beta}\overline{B}_{l+1,\overline{\chi}}(u)f^{(l+1)}(u)du,$$

where the dash indicates that if $n=\alpha$ or $n=\beta$, then only $\frac{1}{2}\chi(\alpha)f(\alpha)$ or $\frac{1}{2}\chi(\beta)f(\beta)$ is counted, respectively.

In the present paper, by the same motivation of [6], the character Hardy–Berndt sums involving primitive characters χ_1 and χ_2 of modulus k are defined by

$$s_{3,p}(b,c:\chi_1,\chi_2)=\sum_{n=1}^{ck}(-1)^n\chi_1(n)\overline{B}_{p,\chi_2}\left(\frac{bn}{c}\right),$$

$$s_{4,p}(b,c:\chi_1,\chi_2)=\sum_{n=1}^{ck}\chi_1(n)\overline{B}_{p,\chi_2}\left(\frac{bn}{2c}\right),$$

which are the natural generalizations of the sum $s_{3,p}(b,c:\chi)$ and $s_{4,p}(b,c:\chi)$ given by (1.1) and (1.2), i.e., $s_{3,p}(b,c:\chi,\chi)=s_{3,p}(b,c:\chi)$ and $s_{4,p}(b,c:\chi,\chi)=$

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$s_{4,p}(b,c:\chi)$.

Utilizing the power of the character analogue of the Euler–MacLaurin summation formula, the following reciprocity formula is derived.

Theorem 2 *Let b and c be coprime positive integers with odd c. For odd $p \geq 1$, we have*

$$(p+1)\{\chi_2(-1)c(2b)^p s_{4,p}(c,b:\overline{\chi}_2,\overline{\chi}_1) + \overline{\chi}_1(-2)bc^p s_{3,p}(b,c:\chi_1,\chi_2)\}$$
$$= 2\sum_{j=1}^{p}\binom{p+1}{j}(-1)^j b^j c^{p+1-j} \times \{\overline{\chi}_2(2)-2^j\}B_{j,\overline{\chi}_1}B_{p+1-j,\chi_2}$$
$$-(p+1)\overline{\chi}_1(-2)bck^{p-1} \times \sum_{h=1}^{k-1}\sum_{j=1}^{k-1}\chi_1(h)\overline{\chi}_2(j)\overline{B}_p\left(\frac{cj}{k}+\frac{bh}{k}\right).$$

Secondly, we define sums $\hat{S}_{3,p}(b,c:\chi_1,\chi_2)$ and $\hat{S}_{4,p}(b,c:\chi_1,\chi_2)$ for primitive characters χ_1 and χ_2 having modulus k_1 and k_2, which need not to be same, by

$$\hat{S}_{3,p}(b,c:\chi_1,\chi_2) = \sum_{n=1}^{ck_1k_2}(-1)^n\chi_1(n)\overline{B}_{p,\chi_2}\left(\frac{bn}{c}\right),$$
$$\hat{S}_{4,p}(b,c:\chi_1,\chi_2) = \sum_{n=1}^{ck_1k_2}\chi_1(n)\overline{B}_{p,\chi_2}\left(\frac{bn}{2c}\right),$$

which reduce to $s_{3,p}(b,c:\chi_1,\chi_2)$ and $s_{4,p}(b,c:\chi_1,\chi_2)$ for $k_1=k_2$.

Finally, for primitive characters χ_1 and χ_2 having modulus k_1 and k_2, we consider following sums

$$\tilde{S}_{3,p}(b,c:\chi_1,\chi_2) = \sum_{n=1}^{ck_1}(-1)^n\chi_1(n)\overline{B}_{p,\chi_2}\left(\frac{bnk_2}{ck_1}\right),$$
$$\tilde{S}_{4,p}(b,c:\chi_1,\chi_2) = \sum_{n=1}^{ck_1}\chi_1(n)\overline{B}_{p,\chi_2}\left(\frac{bnk_2}{2ck_1}\right),$$

which generalize the previous sums. More clearly, $\tilde{S}_{3,p}(b,c:\chi_1,\chi_2) = s_{3,p}(b,c:\chi_1,\chi_2)$, $\tilde{S}_{4,p}(b,c:\chi_1,\chi_2) = s_{4,p}(b,c:\chi_1,\chi_2)$ for $k_1=k_2$, and $\tilde{S}_{3,p}(bk_1,ck_2:\chi_1,\chi_2) = \hat{S}_{3p}(b,c:\chi_1,\chi_2)$ and $\tilde{S}_{4,p}(bk_1,ck_2:\chi_1,\chi_2) = \hat{S}_{4,p}(b,c:\chi_1,\chi_2)$. We obtain the following reciprocity formula for these sums.

Theorem 3 *Let b, c be positive integers with $gcd(b,c)=1$ and $p>1$. Let χ_1 and χ_2 be non-principal primitive characters of modulus k_1 and k_2, respectively. For $(-1)^{p+1}\chi_1(-1)\chi_2(-1)=1$ the following reciprocity formula holds*

$$\chi_2(-1)ck_1(2bk_2)^p\ \tilde{S}_{4,p}(c,b:\overline{\chi}_2,\overline{\chi}_1) + \overline{\chi}_1(-2)bk_2(ck_1)^p\ \tilde{S}_{3,p}(b,c:\chi_1,\chi_2)$$
$$= \frac{2}{p+1}\sum_{j=1}^{p}\binom{p+1}{j}(-bk_2)^j(ck_1)^{p+1-j} \times \{\overline{\chi}_2(2)-2^j\}B_{j,\overline{\chi}_1}B_{p+1-j,\chi_2}$$
$$-\overline{\chi}_1(-2)bck_1k_2^p \times \sum_{h=1}^{k_1}\sum_{j=1}^{k_2}\chi_1(h)\overline{\chi}_2(j)\overline{B}_p\left(\frac{cj}{k_2}+\frac{bh}{k_1}\right).$$

The reciprocity formula for $\hat{S}_{3,p}(b,c:\chi_1,\chi_2)$ and $\hat{S}_{4,p}(b,c:\chi_1,\chi_2)$ follows as a result of Theorem 3 as

$$\chi_2(-1)ck_1(2bk_2)^p\ \hat{S}_{4,p}(c,b:\overline{\chi}_2,\overline{\chi}_1) + \overline{\chi}_1(-2)bk_2(ck_1)^p\ \hat{S}_{3,p}(b,c:\chi_1,\chi_2)$$
$$= \frac{2}{p+1}\sum_{j=1}^{p}\binom{p+1}{j}(-bk_2)^j(ck_1)^{p+1-j}$$

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$$\times \{\overline{\chi}_2(2) - 2^j\} B_{j,\overline{\chi}_1} B_{p+1-j,\chi_2}.$$

Before establishing the proofs, we recall some properties that we need in the sequel.

$$\frac{d}{dx} B_m(x) = mB_{m-1}(x), m \geq 1$$
$$\frac{d}{dx} B_{m,\chi}(x) = mB_{m-1,\chi}(x), m \geq 1$$
$$\frac{d}{dx} \overline{B}_{m,\chi}(x) = m\overline{B}_{m-1,\chi}(x), m \geq 2$$

and for $m \geq 0$

$$\overline{B}_{m,\chi}(k) = \overline{B}_{m,\chi}(0) = B_{m,\chi},$$
$$B_{m,\chi}(-x) = (-1)^m \chi(-1) B_{m,\chi}(x). \quad (1.3)$$

The Gauss sum $G(z,\chi)$ is defined by

$$G(z,\chi) = \sum_{m=0}^{k-1} \chi(m) e^{2\pi imz/k}.$$

If n is an integer, then [1, p. 168]

$$G(n,\chi) = \overline{\chi}(n) G(\chi),$$

where $G(\chi) = G(1,\chi)$.

Proofs of Theorems 2 and 3

We apply character analogue of the Euler–MacLaurin summation formula to generalized Bernoulli function in order to obtain identities involving integrals for $s_{3,p}(b,c:\chi_1,\chi_2)$ and $s_{4,p}(b,c:\chi_1,\chi_2)$. These identities lead the reciprocity formula given by Theorem 2.

In view of (1.3), it is easy to see that

$$s_{3,p}(b,c:\chi_1,\chi_2) = (-1)^{p+1}\chi_1(-1)\chi_2(-1)s_{3,p}(b,c:\chi_1,\chi_2)$$

and

$$s_{4,p}(b,c:\chi_1,\chi_2) = (-1)^{p+1}\chi_1(-1)\chi_2(-1)s_{4,p}(b,c:\chi_1,\chi_2),$$

which entails $s_{r,p}(b,c:\chi_1,\chi_2) = 0$ $(r = 3,4)$ for $(-1)^{p+1}\chi_1(-1)\chi_2(-1) = -1$.

Let $f(x) = \overline{B}_{p,\chi}(xy)$, $y \in \mathbb{R}$. The property ([3, Corollary 3.3])

$$\frac{d}{dx} \overline{B}_{m,\chi}(x) = m\overline{B}_{m-1,\chi}(x), m \geq 2$$

entails that

$$\frac{d^j}{dx^j} f(x) = \frac{d^j}{dx^j} \overline{B}_{p,\chi}(xy) = y^j \frac{p!}{(p-j)!} \overline{B}_{p-j,\chi}(xy)$$

for $0 \leq j \leq p-1$ and $f \in C^{(p-1)}[\alpha,\beta]$.

Proof of Theorem 2

We consider the following three cases;

I) $y = b/c$, $\alpha = 0$ and $\beta = ck$,

II) $y = b/(2c)$, $\alpha = 0$ and $\beta = ck$,

III) $y = 2b/c$, $\alpha = 0$ and $\beta = ck/2$,

where $c > 0$, separately.

For $\alpha = 0$, $\beta = ck$ and $1 \leq l+1 \leq p-1$, Theorem 1 can be written as

$$\sum_{n=1}^{ck} \chi_1(n)\overline{B}_{p,\chi_2}(ny) = \frac{\chi_1(-1)}{p+1} \sum_{j=0}^{l} \binom{p+1}{j+1} (-1)^{j+1} y^j \times \{\overline{B}_{p-j,\chi_2}(cky) - B_{p-j,\chi_2}\} B_{j+1,\overline{\chi}_1} - \chi_1(-1)(-y)^{l+1} \binom{p}{l+1} \times \int_0^{ck} \overline{B}_{l+1,\overline{\chi}_1}(u)\overline{B}_{p-l-1,\chi_2}(yu)du. \quad (2.1)$$

I) Let $y = b/c$, $\alpha = 0$ and $\beta = ck$. Then, (2.1) becomes

$$\sum_{n=1}^{ck} \chi_1(n)\overline{B}_{p,\chi_2}\left(\frac{b}{c}n\right) = -\chi_1(-1)\binom{p}{l+1}\left(-\frac{b}{c}\right)^{l+1} \times \int_0^{ck} \overline{B}_{l+1,\overline{\chi}_1}(u)\overline{B}_{p-l-1,\chi_2}\left(\frac{b}{c}u\right)du. \quad (2.2)$$

- If $b = c$, using the fact that [6,

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Remark 7]:

$$\sum_{n=1}^{ck-1} \chi_1(n)\overline{B}_{p,\chi_2}\left(\frac{bn}{c}\right) = -\varepsilon\frac{p!}{(2\pi i)^p}\left(\frac{k}{c}\right)^{p-1} \times G(b,\chi_1)G(c,\overline{\chi_2})L(p,\overline{\chi_1}\chi_2)$$

for $\gcd(b,c)=1$ with $c>0$, (2.2) becomes with $b=c=1$ as

$$\int_0^k \overline{B}_{r,\overline{\chi}_1}(u)\overline{B}_{m,\chi_2}(u)du = \varepsilon\frac{r!\,m!\,(-1)^r\overline{\chi}_1(-1)}{(2\pi i)^p}k^{p-1} \times G(\chi_1)G(\overline{\chi_2})L(r,\overline{\chi_1}\chi_2),$$

by setting $l+1=r$, $p-r=m$. Here $\varepsilon = 1+(-1)^p\chi_1(-1)\chi_2(-1)$ and $\overline{\chi_1}\chi_2$ is a Dirichlet character modulo k, and $L(p,\overline{\chi_1}\chi_2)$ stands for the Dirichlet $L-$function.

- If $b=ck$, it follows from the fact $\sum_{n=0}^{ck}\chi(n)=0$ and (2.2) that

$$\int_0^k \overline{B}_{l+1,\overline{\chi}_1}(u)\overline{B}_{p-l-1,\chi_2}(ku)du = 0.$$

- Now assume that $(b,c)=1$. Then, it follows from [6, Remark 7] and (2.2) that

$$\int_0^k \overline{B}_{l+1,\overline{\chi}_1}(cu)\overline{B}_{p-l-1,\chi_2}(bu)du = \varepsilon\frac{\overline{\chi}_1(-1)(-1)^{l+1}}{\binom{p}{l+1}}\frac{c^{l+1-p}}{b^{l+1}}\frac{p!}{(2\pi i)^p}k^{p-1} \times G(b,\chi_1)G(c,\overline{\chi_2})L(p,\overline{\chi_1}\chi_2).$$

II) Let k and b be odd and consider $y=b/2c$ with $(b,c)=1$. From the fact that [5, Eq. (3.13)]

$$\chi(r)r^{1-m}\overline{B}_{m,\chi}(rx) = \sum_{j=0}^{r-1}\overline{B}_{m,\chi}\left(x+\frac{jk}{r}\right),$$

we have, for odd b,

$$\overline{B}_{p-j,\chi}\left(\frac{bk}{2}\right) = \overline{B}_{p-j,\chi}\left(\frac{k}{2}\right) = \left\{2^{j+1-p}\chi(2)-1\right\}B_{p-j,\chi}. \quad (2.3)$$

Therefore, (2.1) becomes

$$\begin{aligned} &s_{4,p}(b,c:\chi_1,\chi_2) \\ &= \sum_{n=1}^{ck}\chi_1(n)\overline{B}_{p,\chi_2}\left(\frac{b}{2c}n\right) \\ &= -2^{1-p}\frac{\chi_1(-1)}{p+1}\sum_{m=p-l}^{p}\binom{p+1}{m}\left(-\frac{b}{c}\right)^{p-m} \times\{\chi_2(2)-2^m\}B_{m,\chi_2}B_{p+1-m,\overline{\chi}_1} \\ &\quad -\chi_1(-1)c\binom{p}{l+1}\left(-\frac{b}{2c}\right)^{l+1} \times\int_0^k \overline{B}_{l+1,\overline{\chi}_1}(cu)\overline{B}_{p-l-1,\chi_2}\left(\frac{b}{2}u\right)du \end{aligned} \quad (2.4)$$

by setting $j=p-m$.

For $l=0$ we have the following integral representation

$$\begin{aligned} &\overline{\chi}_1(-1)s_{4,p}(b,c:\chi_1,\chi_2) \\ &= \frac{pb}{2}\int_0^k \overline{B}_{1,\overline{\chi}_1}(cu)\overline{B}_{p-1,\chi_2}\left(\frac{b}{2}u\right)du \\ &\quad -\{2^{1-p}\chi_2(2)-2\}B_{p,\chi_2}B_{1,\overline{\chi}_1}. \end{aligned} \quad (2.5)$$

- Let p be odd and put $l+1=p-1$ in (2.4). Then,

$$\begin{aligned} &\overline{\chi}_1(-1)(p+1)b(2c)^p s_{4,p}(b,c:\chi_1,\chi_2) \\ &= 2\sum_{m=2}^{p}\binom{p+1}{m}(-1)^m b^{p+1-m}c^m \times\{\chi_2(2)-2^m\}B_{m,\chi_2}B_{p+1-m,\overline{\chi}_1} \\ &\quad -2cb^p p(p+1)c \times\int_0^k \overline{B}_{p-1,\overline{\chi}_1}(cu)\overline{B}_{1,\chi_2}\left(\frac{b}{2}u\right)du. \end{aligned} \quad (2.6)$$

III) Let k and c be odd and consider $y=2b/c$, $\alpha=0$, $\beta=ck/2$ with $(b,c)=1$. Then, from Theorem 1 and

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Eq. (2.3) we have

$$\begin{aligned}&\sum_{0\le n\le ck/2}\chi_1(n)\overline{B}_{p,\chi_2}\left(\frac{2b}{c}n\right)\\&=\sum_{n=1}^{(ck-1)/2}\chi_1(n)\overline{B}_{p,\chi_2}\left(\frac{2b}{c}n\right)\\&=-\frac{\chi_1(-1)}{p+1}\sum_{j=1}^{l+1}\binom{p+1}{j}\left(-\frac{b}{c}\right)^{j-1}\\&\quad\times\{\overline{\chi}_2(2)-2^j\}B_{j,\overline{\chi}_1}B_{p+1-j,\chi_2}\\&\quad-\chi_1(-1)\left(-\frac{2b}{c}\right)^{l+1}\binom{p}{l+1}\frac{c}{2}\\&\quad\times\int_0^k\overline{B}_{l+1,\overline{\chi}_1}\left(\frac{c}{2}u\right)\overline{B}_{p-l-1,\chi_2}(bu)du.\end{aligned}\tag{2.7}$$

Now consider the sum $s_{3,p}(b,c:\chi_1,\chi_2)$.

$$\begin{aligned}&s_{3,p}(b,c:\chi_1,\chi_2)\\&=\sum_{n=1}^{ck}(-1)^n\chi_1(n)\overline{B}_{p,\chi_2}\left(\frac{bn}{c}\right)\\&=2\chi_1(2)\sum_{n=1}^{\frac{ck-1}{2}}\chi_1(n)\overline{B}_{p,\chi_2}\left(\frac{2bn}{c}\right)\\&\quad-\sum_{n=1}^{ck}\chi_1(n)\overline{B}_{p,\chi_2}\left(\frac{bn}{c}\right).\end{aligned}\tag{2.8}$$

• Let p be odd. Put $l+1=p-1$ in (2.7). Then, (2.7), (2.8) and [6, Eq. (21)] yield

$$\begin{aligned}&\overline{\chi}_1(-2)(p+1)bc^ps_{3,p}(b,c:\chi_1,\chi_2)\\&=2\sum_{j=1}^{p-1}\binom{p+1}{j}(-1)^jb^jc^{p+1-j}\\&\quad\times\{\overline{\chi}_2(2)-2^j\}B_{j,\overline{\chi}_1}B_{p+1-j,\chi_2}\\&\quad-2^{p-1}b^pc^2p(p+1)\\&\quad\times\int_0^k\overline{B}_{p-1,\overline{\chi}_1}\left(\frac{c}{2}u\right)\overline{B}_{1,\chi_2}(bu)du\\&\quad-(p+1)\overline{\chi}_1(-2)bck^{p-1}\\&\quad\times\sum_{h=1}^{k-1}\sum_{j=1}^{k-1}\chi_1(h)\overline{\chi}_2(j)\overline{B}_p\left(\frac{cj}{k}+\frac{bh}{k}\right).\end{aligned}\tag{2.9}$$

Putting $l=0$ in (2.7) gives an integral representation for $s_{3,p}(b,c:\chi_1,\chi_2)$:

$$\begin{aligned}&\frac{1}{2}\overline{\chi}_1(-2)s_{3,p}(b,c:\chi_1,\chi_2)\\&\quad+\frac{\overline{\chi}_1(-2)}{2}\left(\frac{k}{c}\right)^{p-1}\\&\quad\times\sum_{h=1}^{k-1}\sum_{j=1}^{k-1}\chi_1(h)\overline{\chi}_2(j)\overline{B}_p\left(\frac{cj}{k}+\frac{bh}{k}\right)\\&=bp\int_0^k\overline{B}_{1,\overline{\chi}_1}\left(\frac{c}{2}u\right)\overline{B}_{p-1,\chi_2}(bu)du\\&\quad-\{\overline{\chi}_2(2)-2\}B_{1,\overline{\chi}_1}B_{p,\chi_2}.\end{aligned}\tag{2.10}$$

Combining (2.5) and (2.9), we arrive at the reciprocity formula

$$\begin{aligned}&\chi_2(-1)c(2b)^ps_{4,p}(c,b:\overline{\chi}_2,\overline{\chi}_1)\\&\quad+\overline{\chi}_1(-2)bc^ps_{3,p}(b,c:\chi_1,\chi_2)\\&=\frac{2}{p+1}\sum_{j=1}^{p}\binom{p+1}{j}(-1)^jb^jc^{p+1-j}\\&\quad\times\{\overline{\chi}_2(2)-2^j\}B_{j,\overline{\chi}_1}B_{p+1-j,\chi_2}\\&\quad-\overline{\chi}_1(-2)bck^{p-1}\\&\quad\times\sum_{h=1}^{k-1}\sum_{j=1}^{k-1}\chi_1(h)\overline{\chi}_2(j)\overline{B}_p\left(\frac{cj}{k}+\frac{bh}{k}\right).\end{aligned}$$

Proof of Theorem 3

We first note that similar to $s_{3,p}(b,c:\chi_1,\chi_2)$ and $s_{4,p}(b,c:\chi_1,\chi_2)$ the sums $\tilde{S}_{3,p}(b,c:\chi_1,\chi_2)$ and $\tilde{S}_{4,p}(b,c:\chi_1,\chi_2)$ vanishes when $(-1)^{p+1}\chi_1(-1)\chi_2(-1)=-1$. Let $\gcd(b,c)=1$ with $c>0$. Let $y=bk_2/ck_1$, $y=bk_2/2ck_1$ and $y=2bk_2/ck_1$ instead of $y=b/c$, $y=b/2c$ and $y=2b/c$ in **I)**, **II)** and **III)** respectively. Then adopting the arguments in the proof of Theorem 2, and using [6, Eq. (22)] instead of [6, Eq. (21)], the desired result follows.

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