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The One Dimensional Keller-Segel Model

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Abstract

In this paper, the Keller-Segel model is analysed. The work presented will focus on the mass criticality results for the Chemotaxis model. Subsequently the relative stability of stationary states are analysed using the Keller-Segel system for the Chemotaxis with linear diffusion. In this analysis, the techniques of 'separation of variables' and 'standard linearization' were used. Also, the graphics illustrate stability or instability in all the cases analysed.

Keywords: Chemotaxis, Keller-Segel Model

Introduction

The randomly-determined motion of an entity will be discussed here. For example, cells, bacteria, chemicals, and animals generally move around randomly. Microscopic movement analysis shows that many individual particles move irregularly. We know that diffusion is one of several transport phenomena that occur in nature. Reaction-diffusion systems influence local chemical reactions in which objects are transformed into each other. This system also affects diffusion, whereby the objects spread out over a surface (MURRAY, 2002).

We denote $a(x,t)$ as the gradient in attractant, which prompts a movement. The resulting flux of cells will rise with the number of cells, $\rho(x,t)$. It can be written,

$$J = \rho x(a)\nabla a \tag{1}$$

where J is chemotactic flux and $x(a)$ is a function of the concentration of the attractant HILLEN and PAINTER (2009).

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The equation can be written generally for $\rho(x,t)$:

$$\frac{\partial \rho}{\partial t} = div\, J = f(n)$$

where $f(n)$ introduces the growth term for the cells, the total flux,

$$J_{tot} = J_{diff} + J_{chem}$$

where $J_{diff} = -D\nabla\rho$. Hence,

$$\rho_t = D\rho_{xx} - (\rho\chi(a)a_x)_x + f(n) \tag{2}$$

where D is diffusion coefficient of the cells. The (2) equation is called the *reaction-diffusion-chemotaxis* equation. It is known that $a(x,t)$ is a chemical term, and in general we may write $a(x,t)$:

$$a_t = D_a a_{xx} + g(a,\rho) \tag{3}$$

where $g(a,\rho)$ is the kinetics term and D_a is a diffusion coefficient of a. This term depend upon ρ and a.

$$\begin{cases} \rho_t = D\rho_{xx} - \chi(\rho a_x)_x \\ a_t = D_a a_{xx} + g(a,\rho) \end{cases} \tag{4}$$

According to KELLER and SEGEL (1970), the kinetics terms would be $g(a,n) = h\rho - ka$ where h,k are positive constant. While $(h\rho)$ is rational to the number of amoebae n, $(-ka)$ introduces decay of attractant activity. One simple model is $f(n) = 0$, which means that we ignored the amoebae production rate. The chemotactic term $\chi(a)$ can be taken as a constant χ. Then the nonlinear system is written with the linear form to $g(a,n)$.

$$\begin{cases} \dfrac{\partial \rho}{\partial t} = D\Delta\rho - \chi div(\rho\nabla \mathrm{a}) \\ \dfrac{\partial a}{\partial t} = D_a\Delta a + h\rho - ka \end{cases} \tag{5}$$

where a is the food which it consumed and ρ refers to a bacterial population.

This suggests that diffusion is commonly stabilizing while chemotaxis is commonly destabilizing because a and ρ have a Laplacian contribution but with different sign.

The first equation of (5) introduces the cell dynamics. This equation describes a diffusive flux model of the random motion of cells, with flux modelling directed cell movement and velocity proportional to the concentration gradient of the chemical. The second equation of (5) is a reaction-diffusion equation that represents the chemical kinetics, with linear production and degeneration at constant rates $h, k > 0$ [2]. The system (5) is called the 'minimal chemotaxis model'. This model contains strong dynamics such as the universal existence of solution and spatial pattern formation. HORSTMAN (2003) and PERTHAME (2007) have previously studied this subject.

Definition and Problem

In this section one-dimensional chemotaxis model will be analysed. Let us consider the system (5), namely

$$\begin{cases} \rho_t = D\rho_{xx} - \chi(\rho a_x)_x \\ a_t = D_a a_{xx} + h\rho - ka, \end{cases} \tag{6}$$

The parameters D, D_a, χ are constants. D and D_a are the diffusion coefficient of the cells and a, respectively h and k are positive constants. The first term in the first equation in (6) involves a Laplacian, representing the random spatial motion of the cells. The second term models the chemotactic motion of the cells. In the second equation in (6) the first term represent diffusion of the chemoattractant similar to that of the equation (5). The second term models the production of the chemoattractant by the cells, and the third term represents linear decay. (KELLER and SEGEL, 1971)

Initial and Boundary Conditions

The initial conditions for the system (6) are

$$\begin{cases} \rho(x,0) = \rho_0(x) \\ a(x,0) = a_0(x), \end{cases} \tag{7}$$

The boundary conditions with no flux for ρ are

$$\begin{cases} \rho_x(0,L) = 0 \\ a_x(0,L) = 0, \end{cases} \tag{8}$$

This system can be analysed for the linear stability of the constant steady states. Let us first consider whether the system has any steady states, and in particular whether there are any spatially homogenous steady states. Let (ρ^∞, a^∞) be a constant steady state. The equation (6) equation yields,

$$h\rho^\infty - ka^\infty = 0$$

and the steady state

$$(\rho^\infty, a^\infty) = \left(\rho^\infty, \frac{h\rho^\infty}{k}\right)$$

From the conservation of total mass,

$$\int \rho(x,t)dx = \int \rho_0(x)dx$$

Then the steady state ρ^∞ will be determined by

$$\int \rho^\infty dx = M$$

This gives us,

$$(\rho^\infty, a^\infty) = \left(\frac{M}{L}, \frac{hM}{kL}\right).$$

Linear Analysis

We now consider a perturbation of the linear system for $\rho(x,t)$ and $a(x,t)$

$$\rho(x,t) = \rho^\infty + u(x,t)$$
$$a(x,t) = a^\infty + v(x,t)$$

when the system (6) is arranged for u and v,

$$\begin{cases} \dfrac{\partial u}{\partial t} = D\dfrac{\partial^2 u}{\partial x^2} - \chi\rho^\infty \dfrac{\partial^2 v}{\partial x^2} \\ \dfrac{\partial v}{\partial t} = D_a \dfrac{\partial^2 v}{\partial x^2} + \mathrm{hu} - kv, \end{cases} \tag{9}$$

These coupled PDEs are linear in u and v and so should be easier to deal with. The solution may be found by the technique of the separation of variables but we need to "normal modes" the solution,

$$u(x,t) = u_n(x,t) = \alpha_n(t) f_m(x)$$
$$v(x,t) = v_n(x,t) = \beta_n(t) g_m(x)$$

where $f_m(x) = g_m(x) = \cos(\mu_m x)$ and $\mu_m = \frac{n\pi}{L}$. The solution will be a linear homogeneous differential equation in x and t. Thus we need to substitute this into the differential equation and then

$$\dot{\alpha_n}(t) f_m(x) = D\alpha_n(t)\ddot{f_m}(x) + \chi\rho^\infty \beta_n(t)\ddot{g}_m(x)$$
$$\dot{\beta_n}(t) g_m(x) = D_a\beta_n(t)\ddot{g}_m(x) + h\alpha_n(t) f_m(t) - k\beta_n(t) g_m(x)$$

Then the system will be such that

$$\alpha_n(t) = -D\mu_n^2\alpha_n(t) + \chi\rho^\infty\mu_n^2\beta_n(t)$$
$$\beta_n(t) = -D_a\mu_n^2\beta_n(t) + h\alpha_n(t) - k\beta_n(t)$$

which can be re-written in matrix form as

$$\partial_t U_n(t) = A_n U_n(t)$$
$$A_n := \begin{pmatrix} -D\mu_n^2 & \chi\rho^{\infty}\mu_n^2 \\ h & -D_a\mu_n^2 - k \end{pmatrix} \quad (10)$$

where

$$U_n(t) = \begin{pmatrix} \alpha_n(t) \\ \beta_n(t) \end{pmatrix}$$

We will seek solution to (10) of the form:

$$\det(\sigma I - A_n) = 0$$

This gives us,

$$\begin{pmatrix} \alpha + D\mu_n^2 & -\chi\rho^{\infty}\mu_n^2 \\ -h & \sigma + D_a\mu_n^2 - k \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \beta_0 \end{pmatrix} e^{\sigma t} = 0 \quad (11)$$

where $\alpha_n(t) = \alpha_0 e^{\sigma t}$ and $\beta_n(t) = \beta_0 e^{\sigma t}$. Solutions of the linearised system exist if the determinant of this matrix is zero, i.e.

$$(\sigma + D\mu_n^2)(\sigma + D_a\mu_n^2 + k) - h\chi\rho^{\infty}\mu_n^2 = 0 \quad (12)$$

where $\mu_n = \mu$. Since our solutions are of the form given by (11), in order to detect stable solutions corresponding to values of σ such that $Re(\sigma) < 0$, since these solutions do not decay over time and may therefore result in a high density of cells somewhere in our domain. Accordingly, let us re-write as a quadratic in σ:

$$\sigma^2 - (trA_n)\sigma + \det(A_n) = 0 \quad (13)$$

where

$$\begin{cases} Tr(A_n) = -D\mu_n^2 - D_a\mu_n^2 - k \\ \det(A_n) = \mu_n^2(DD_a\mu_n^2 - Dk - \chi\rho^{\infty}\mathrm{h}) \end{cases} \quad (14)$$

Thus there are two possibilities:

a) σ_1 and σ_2 are negative,
b) σ_1 is negative and σ_2 is positive.

Therefore, it is clearly seen that the conditions σ_1 and σ_2 are negative if and only if $\det(A_n)$ is positive. Thus, where $\rho^{\infty} = \frac{M}{L}$. Then we get,

$$M < \frac{DLk}{\chi h} \quad (15)$$

On the other hand, if

$$M > \frac{DLk}{\chi h} \quad (16)$$

there exists an interval $\mu \in [0, \hat{\mu}]$ on which A_n has one positive eigen value, which implies linear instability. We have therefore obtained a threshold condition for stability, which involves ρ^{∞}, D, χ and k. One way to see such a condition is that if the ratio D/χ is sufficiently large, then diffusion dominates and the system is stable, whereas if D/χ is sufficiently small then chemotaxis dominates and the system is unstable.

Results and Conclusion

It can be clearly seen that, the eigenvalues of (A_n) are both strictly negative for all μ. Therefore the system is stable for both eigenvalues in Figure 1 and 2.

When two graphs are compared, we have two critical results. Firstly, the blue line always remains while the ratio of D/χ changes. On the other hand, the red line slightly reduces while the ratio of D/χ increases.

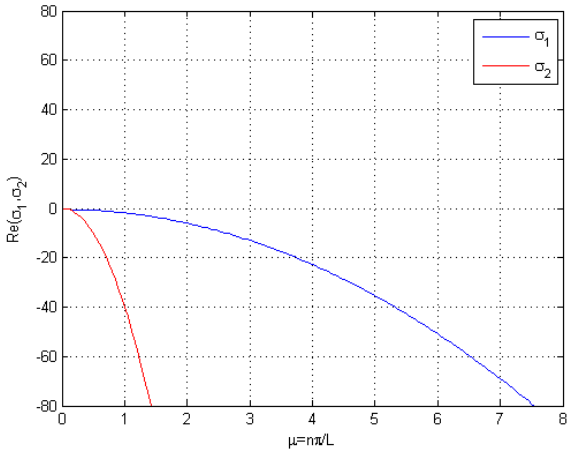

Figure1. $D = 3.8, h = 0.4, \chi = 0.6, \frac{M}{L} = 4, D_a = 1.4$

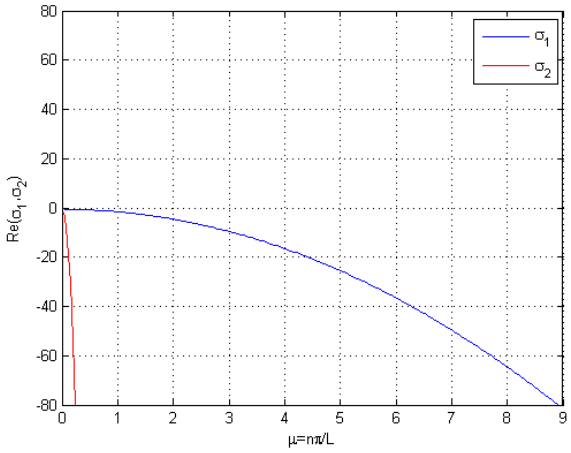

Figure2. $D = 9.8, h = 0.1, \chi = 0.6, \frac{M}{L} = 4, D_a = 1.4$

It can be seen that both of the systems in the graphs are stable. The ratio of D/χ in the system are 6.33 and 16.33, respectively. Even though they are stable, the eigenvalue of the second system shows a faster reduction than in the first graph.
On the other hand, if the ratio of D/χ is sufficiently small then we have an unstable situation as in the graph below.

In Figure 3, even though the red line is still stable, the blue line is unstable for small variables of μ. After a certain point, it starts to become stable again.

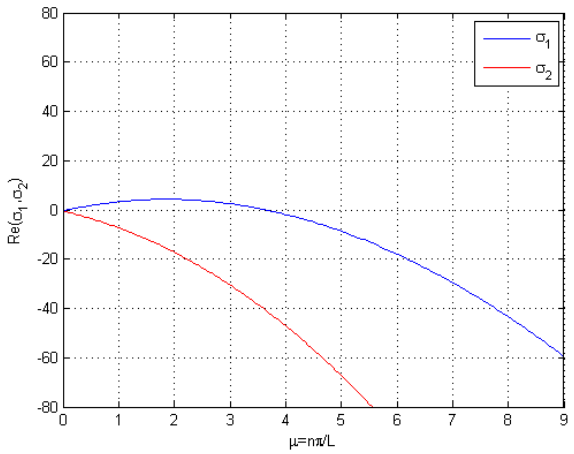

Figure3. $D = 0.8, h = 1.9, \chi = 3.3, \frac{M}{L} = 4, D_a = 2$

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