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Some Aspects on a Special Type of (α, β)-metric

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(Dedicated to the memory of Prof. Dr. Krishan Lal DUGGAL (1929 - 2022))

ABSTRACT

The aim of this paper is twofold. Firstly, we will investigate the link between the condition for the functions $\phi(s)$ from (α, β)-metrics of Douglas type to be self-concordant and k-self concordant, and the other objective of the paper will be to continue to investigate the recently new introduced (α, β)-metric ([17]):

$$F(\alpha, \beta) = \frac{\beta^2}{\alpha} + \beta + a\alpha$$

where $\alpha = \sqrt{a_{ij}y^iy^j}$ is a Riemannian metric; $\beta = b_iy^i$ is a 1-form, and $a \in \left(\frac{1}{4}, +\infty\right)$ is a real positive scalar. This kind of metric can be expressed as follows: $F(\alpha, \beta) = \alpha \cdot \phi(s)$, where $\phi(s) = s^2 + s + a$. In this paper we will study some important results with respect to the above mentioned (α, β)-metric such as: the Kropina change for this metric, the Main Scalar for this metric and also we will analyze how the condition to be self-concordant and k-self-concordant for the function $\phi(s)$, can be linked with the condition for the metric F to be of Douglas type.

Keywords: *Finsler (α, β)-metric, self-concordant functions, Kropina change, main scalar.*
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1. Introduction

The main purpose of this paper is to obtain some new results for the (α, β)-metric:

$$F(\alpha, \beta) = \frac{\beta^2}{\alpha} + \beta + a\alpha \tag{1.1}$$

where $\alpha = \sqrt{a_{ij}y^iy^j}$ is a Riemannian metric; $\beta = b_iy^i$ is a 1-form, and $a \in \left(\frac{1}{4}, +\infty\right)$ is a real positive scalar. First, let's recall some important results regarding the (α, β)-metrics:

As we know, (see [4]), the (α, β)-metric is defined in the following form: $F = \alpha\phi(s)$, where $s = \frac{\beta}{\alpha}$.
The function $\phi = \phi(s)$ is a C^∞ positive function on an open interval $(-b_0, b_0)$ and it satisfies the following conditions (see [4])

$$\phi(s) > 0$$
$$\phi(s) - s\phi'(s) > 0$$
$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \quad |s| \leq b < b_0$$

Also its a well known fact that F is a Finsler metric if and only if $||\beta_x||_\alpha < b_0$ for any $x \in M$.
The relationship between the geodesic coefficients of an (α, β)-metric F and α, namely G^i and G^i_α is presented in [11] in the following form:

$$G^i = G^i_\alpha + \frac{F_{|k}y^k}{2F}y^i + \frac{F}{2}g^{ij}\left(\frac{\partial F_{|k}}{\partial y^j}y^k - F_{|j}\right) \tag{1.2}$$

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Definition 1.1. A Finsler space $F^n = (M, F(x,y))$ is endowed with an (α, β) metric, if there exist a 2-homogeneous function L of two variables such that the Finsler metric $F : TM \to \mathbb{R}$ is given by:

$$F^2(x,y) = L(\alpha(x,y), \beta(x,y)), \tag{1.3}$$

where $\alpha^2(x,y) = a_{ij}y^iy^j$, α is a Riemannian metric and $\beta(x,y) = b_i(x)y^i$ is a 1-form on M.

More generally, a Finsler metric on a manifold M, is a function $F : TM \to [0, \infty)$, satisfying the following properties:

1. F is smooth on TM_0.
2. F is positevely 1-homogenous on the fibres of tangent bundle TM.
3. the Hessian of F^2 with elements $g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ is positevely defined on TM.

A smooth manifold M, equipped with the Finsler metric F is called Finsler manifold and the corresponding space denoted by $F^n = (M, F)$ is called Finsler space.
The supporting element l_i, angular metric tensor h_{ij}, and the metric tensor g_{ij} of F^n, are defined by:

$$l_i = \frac{\partial F}{\partial y^i}; h_{ij} = \frac{1}{2}\frac{\partial^2 F}{\partial y^i \partial y^j}; g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}.$$

Some interesting and important results regarding Finsler (α, β)-metrics are presented in the following papers: [3], [12], [9], [10], [18], [20].
In paper [19] is presented a special β-change for the (α, β)-metrics, called Kropina change, defined by:

$$\bar{F} = \frac{F^2}{\beta} = f(F, \beta) \tag{1.4}$$

and for this kind of change, in the same paper are presented the following results: differention of the above relation, gives:

$$f_1 = \frac{\partial \bar{F}}{\partial F} = \frac{2F}{\beta}; f_2 = \frac{\partial \bar{F}}{\partial \beta} = \frac{-F^2}{\beta^2}$$

$$f_{11} = \frac{\partial^2 \bar{F}}{\partial F^2} = \frac{2}{\beta}; f_{22} = \frac{\partial^2 \bar{F}}{\partial \beta^2} = \frac{2F^2}{\beta^3}.$$

$$f_{12} = \frac{\partial^2 \bar{F}}{\partial \beta \partial F} = -\frac{2F}{\beta^2}$$

Then, $\bar{l_i} = \overline{F_{y^i}}$, gives:

$$\bar{l_i} = f_1 l_i + f_2 b_i = -\frac{F^2}{\beta^2}\left(b_i - \frac{2\beta}{F^2}y_i\right)$$

and because $\overline{g_{ij}} = \frac{1}{2}\left(\bar{F}^2\right)_{y^iy^j}$, one obtains:

$$\overline{g_{ij}} = pg_{ij} + p_0 b_i b_j + p_{-1}\left(b_i y_j + b_j y_i\right) + p_{-2} y_i y_j$$

$$= \frac{2F^2}{\beta^2}g_{ij} + \frac{3F^4}{\beta^4}b_i b_j - \frac{4F^2}{\beta^3}\left(b_i y_j + b_j y_i\right) + \frac{4}{\beta^2}y_i y_j.$$

We introduced a very special type of (α, β)-metrics in some previous papers [14], [15], [16], [17] and we analyzed some results regarding the properties for this metric. This special (α, β)-metric is given in the following form:

$$F = \beta + \frac{a\alpha^2 + \beta^2}{\alpha} = \alpha \cdot \phi(s) \tag{1.5}$$

with $\phi(s) = s^2 + s + a; s = \frac{\beta}{\alpha}$, where $\alpha = \sqrt{a_{ij}y^iy^j}$ is a Riemannian metric; $\beta = b_i y^i$ is a 1-form, and $a \in \left(\frac{1}{4}, +\infty\right)$ is a real positive scalar.

Remark 1.1. A very special type of (α, β)-metrics in Finsler geometry are the Douglas type metrics.

In the paper [8], the following important theorem is presented:

Theorem 1.1. *([8]) Let $F = \alpha\phi(s)$, with $s = \frac{\beta}{\alpha}$, be a regular (α, β)-metric on an open subset $U \subset \mathbb{R}^2$, where $\phi(0) = 1$. Suppose that β is not parallel with respect to α and F is not of Randers type. Let F be a Douglas metric. Then: $\phi(s)$ satisfies:*

$$\left\{1 + (k_1 + k_3)s^2 + k_2 s^4\right\} \phi''(s) = (k_1 + k_2 s^2)(\phi(s) - s\phi'(s)) \tag{1.6}$$

where k_1, k_2, k_3 are constants, satisfying: $k_2 \neq k_1 k_3$ and β must be closed.

Next, we will recall some classical results regarding the **self-concordance** and **k-self concordance** for convex functions. According to [13], we know the following definition:

Definition 1.2. ([13]) A self-concordant function is a function $f : \mathbb{R} \to \mathbb{R}$ for which

$$|f'''(x)| \leq 2\left(f''(x)\right)^{\frac{3}{2}} \tag{1.7}$$

or, equivalently, a function $f : \mathbb{R} \to \mathbb{R}$ that, wherever $f''(x) > 0$, satisfies $\left|\frac{d}{dx}\frac{1}{\sqrt{f''(x)}}\right| \leq 1$ and which satisfies $f'''(x) = 0$ elsewhere. More generally, a multivariate function $f(x) : \mathbb{R}^n \to \mathbb{R}$ is self-concordant if

$$\left.\frac{d}{d\alpha}\nabla^2 f(x + \alpha y)\right|_{\alpha=0} \preceq 2\sqrt{y^T \nabla^2 f(x)\, y}\, \nabla^2 f(x)$$

Definition 1.3. ([13]) A k-self-concordant function is a function $f : \mathbb{R} \to \mathbb{R}$ for which

$$|f'''(x)| \leq 2k\left(f''(x)\right)^{\frac{3}{2}}$$

where $k \geq 0$.

We recall the following proposition from [1]:

Proposition 1.1. *([1]) The Gaussian curvature of the Finslerian wavefront I_x, is:*

$$K_x = g\left[\sum_{i=1}^{n}\left(g_{ij}y^j\right)^2\right]^{-\frac{n+1}{2}}$$

From this proposition, in the same paper [1], for the (α, β)-metrics is given the following form for Gaussian curvature:

$$K_x = \frac{\phi^{n+1}\left(\phi - s\phi'\right)^{n-2}\left[\phi - s\phi' + (\|\beta\|_x^2 - s^2)\phi''\right] det\ a}{\left[\sum_{i=1}^{n}\left(\frac{\tilde{y_i}}{\alpha}(\phi - s\phi') + b_i\phi'\right)^2\right]^{\frac{n+1}{2}}}$$

2. Main Results

First, we want to investigate the link between the Douglas metrics from Finsler geometry with the theory of self-concordant functions and also k-self concordant functions. In this respect, we will investigate this link using the metric (1.1) because this metric could be written as $F = \alpha \cdot \phi(s)$, where $\phi(s) = s^2 + s + a$ is a convex function and $a \in \left(\frac{1}{4}, +\infty\right)$ is a real positive scalar, $s = \frac{\beta}{\alpha}$. So, we have here a convex function and could do further investigations in this respect. But, first of all we may formulate the following general result:

Theorem 2.1. *Let $F = \alpha \cdot \phi(s)$ be a regular (α, β)-metric on an open subset $U \subset \mathbb{R}^2$, with $s = \frac{\beta}{\alpha}$ and where $\phi(s)$ is a convex function with $\phi(0) = 1$. Suppose that β is not parallel with respect to α and F is not of Randers type. If F is of Douglas type and $\phi(s)$ is a self-concordant function then, the following inequality take place:*

$$\sqrt[3]{4}\left(k_1 + k_2 s^2\right)\left(\phi(s) - s\phi'(s)\right) \geq \left(\phi'''(s)\right)^{\frac{2}{3}}\left(1 + (k_1 + k_2)s^2 + k_2 s^4\right) \tag{2.1}$$

Proof. Starting with the condition for the metric $F = \alpha \cdot \phi(s)$ to be of Douglas type (see (1.6)), we can impose the condition for the convex function $\phi(s)$ to be self-concordant, namely $|\phi'''(s)| \leq 2\left(\phi''(s)\right)^{\frac{3}{2}}$. This is equivalent with $\phi''(s) \geq \frac{(\phi'''(s))^{\frac{2}{3}}}{\sqrt[3]{4}}$. Using this inequality and replacing above in (1.6), one obtains:

$$\sqrt[3]{4}\left(k_1 + k_2 s^2\right)\left(\phi(s) - s\phi'(s)\right) \geq \left(\phi'''(s)\right)^{\frac{2}{3}}\left(1 + (k_1 + k_2)s^2 + k_2 s^4\right)$$

So, the proof is done. □

As we say before the above Theorem, let us now consider the following example:

Example 2.1. Taking the metric (1.1), with $\phi(s) = s^2 + s + a$, where $a \in \left(\frac{1}{4}, +\infty\right)$ is a real positive scalar, and $s = \frac{\beta}{\alpha}$, we observe that is of Douglas type if

$$3k_2 s^4 + (3k_1 + 2k_3 - ak_2)s^2 + 2 - ak_1 = 0.$$

If it is of Douglas type, imposing the conditions of the above Theorem 2.1, we get:

$$\begin{cases} \phi(s) = s^2 + s + a \\ \phi'(s) = 2s + 1 \\ \phi''(s) = 2 \\ \phi'''(s) = 0 \end{cases} \quad \Rightarrow \sqrt[3]{4}(k_1 + k_2 s^2)(a - s^2) \geq 0$$

where $k_1, k_2 \in \mathbb{R}$ and $a \in \left(\frac{1}{4}, +\infty\right)$. Imposing the condition from the Douglas type of metrics, namely $\phi(0) = 1$, we get in this case the value of the scalar $a = 1$. Then, we can discuss the above inequality

$$\sqrt[3]{4}(k_1 + k_2 s^2)(1 - s^2) \geq 0$$

and we observe that we can have some different cases depending how we choose the values of k_1 and k_2. Let us remark, just two of this cases:

Case I Suppose $k_1 < 0$ and $k_2 > 0$ and $1 > -\frac{k_1}{k_2}$, we get from the above inequality: $s \in \left(-1, -\sqrt{-\frac{k_1}{k_2}}\right) \cup \left(\sqrt{-\frac{k_1}{k_2}}, 1\right)$.

Case II Suppose $k_1 < 0$ and $k_2 > 0$ and $1 < -\frac{k_1}{k_2}$, we get from the above inequality: $s \in \left(-\sqrt{-\frac{k_1}{k_2}}, -1\right) \cup \left(1, \sqrt{-\frac{k_1}{k_2}}\right)$.

Starting with a Douglas type of metric, from [21], we know also the following results

$$\phi''(s) = \frac{k_1 + k_2 s^2}{1 + (k_1 + k_2 s^2)s^2 + k_3 s^2}\left[\phi(s) - s\phi'(s)\right] \tag{2.2}$$

$$\phi'''(s) = \frac{2(k_2 - k_1 k_3) - 3(k_1 + k_2 s^2)^2}{\left[1 + (k_1 + k_2 s^2)s^2 + k_3 s^2\right]^2}\left[\phi(s) - s\phi'(s)\right] \tag{2.3}$$

These results, together with the condition for the function $\phi(s)$ to be convex and self-concordant or k-self concordant, can lead us to some inequalities regarding this kind of functions.
More generally, if we consider the more general class of (α, β)-metrics, as in paper [2], satisfying:

$$\phi(s) - s\phi'(s) = (p + rs^2)\phi''(s) \tag{2.4}$$

after derivation, one obtains:

$$-s\phi''(s) = 2rs\phi''(s) + (p + rs^2)\phi'''(s)$$

and from this, we get:

$$-s(1 + 2r)\phi''(s) = (p + rs^2)\phi'''(s).$$

Then, imposing the condition for the function $\phi(s)$ to be convex and self-concordant, i.e. $|\phi'''(s)| \leq 2\left(\phi''(s)\right)^{\frac{3}{2}}$, one obtains:

$$-s(1 + 2r)\phi''(s) = (p + rs^2)\phi'''(s) \leq 2(p + rs^2)\left(\phi''(s)\right)^{\frac{3}{2}}$$

$$\Rightarrow \phi''(s)\left(2\phi''(s)^{\frac{1}{2}}(p+rs^2)+s(1+2r)\right)\geq 0$$

This means that $\phi''(s)\geq 0$ and

$$\left(2\phi''(s)^{\frac{1}{2}}(p+rs^2)+s(1+2r)\right)\geq 0$$

The case when the both quantities are negative is out of discussion , because of the positivity of $\phi''(s)$.
So, we can now formulate the following lemma:

Lemma 2.1. *For the general (α,β)-metrics, which satisfies the condition $\phi(s)-s\phi'(s)=(p+rs^2)\phi''(s)$, the condition for the function $\phi(s)$ to be self-concordant can be reduced to the following properties:*

- $\phi''(s)\geq 0$
- $\left(2\phi''(s)^{\frac{1}{2}}(p+rs^2)+s(1+2r)\right)\geq 0$

Even more, for the general (α,β)-metrics $F=\alpha\phi(s)$, where $\phi=\phi(s)$ satisfies

$$\phi(s)-s\phi'(s)=(p+rs^2)\phi''(s).$$

According to [2], we know that β satisfies

$$b_{i|j}=2\tau\left\{(p+b^2)a_{ij}+(r-1)b_ib_j\right\},$$

where $\tau=\tau(x)$ is a scalar function, then F is a Douglas metric.

Remark 2.1. For our metric (1.1), it is easy to verify that β satisfies:

$$b_{i|j}=2\tau\left\{(\pm a+2b^2)a_{ij}-3b_ib_j\right\},$$

where $\tau=\tau(x)$ is a scalar function, and $a\in\left(\frac{1}{4},+\infty\right)$ is a real positive scalar, then we can remark that F is a Douglas metric, when we choose $a=1$.

Next, we will analyze the Kropina change for the metric (1.1) and we will analyze if the function $\phi_1(s)$ obtained from the original $\phi(s)$ function will remains self-concordant, starting with the initial assumption that $\phi(s)$ is self concordant. We know that for the metric (1.1), that $F=\alpha\phi(s)=\alpha(s^2+s+a)$, where $a\in\left(\frac{1}{4},+\infty\right)$ is a real positive scalar. Then, using the Kropina change $\bar{F}=\frac{F^2}{\beta}$, for this metric, we get the following quantities:

$$f_1=\frac{2F}{\beta}=2\frac{\beta}{\alpha}+2+2a\frac{\alpha}{\beta}=2s+2+\frac{2a}{s}$$

$$f_2=-\frac{F^2}{\beta^2}=-\left(\frac{\beta}{\alpha}+1+\frac{a\alpha}{\beta}\right)^2=-\left(s+1+\frac{a}{s}\right)^2$$

$$f_{11}=\frac{2}{\beta};\ f_{22}=\frac{2F^2}{\beta^3}=\frac{2}{\beta}\left(s+1+\frac{a}{s}\right)^2$$

$$f_{12}=-\frac{2F}{\beta^2}=-\frac{1}{\beta}\left(2s+1+\frac{a}{s}\right)$$

And using this relations, we get for the metric (1.1), the following link between $\overline{g_{ij}}$ and g_{ij}:

$$\overline{g_{ij}}=2\left(s+1+\frac{a}{s}\right)^2g_{ij}+3\left(s+1+\frac{a}{s}\right)^4b_ib_j-\frac{4}{\beta}\left(s+1+\frac{a}{s}\right)^2(b_iy_j+b_jy_i)+\frac{4}{\beta^2}y_iy_j.$$

Now we can observe that the function $\phi(s)$ for our metric (1.1), was of the form $\phi(s)=s^2+s+a$ and after Kropina change this function became $\phi_1(s)=\frac{(s^2+s+a)^2}{s}$. Initially, the function $\phi(s)$, fulfilled the condition to be self-concordant and then, after Kropina change, the new function obtained after, namely $\phi_1(s)$, fullfil the condition to be self-concordant, but the condition to remain a Douglas metric for $\overline{F}$ endowed with the function $\phi_1(s)$is not fulfilled. Even more, even the function $\phi_1(s)$ is k-self-concordant, the condition to remain a Douglas metric for $\overline{F}$ endowed with the function $\phi_1(s)$is not fulfilled. Next, we will try to see how the self-concordance condition and k-self concordance condition on the function $\phi(s)$ from an (α,β)-metric, could change some conditions for the bounded Gaussian curvature of such Finsler metrics. Generally, as we see before in this paper in Proposition 1.1, we know that for the Gaussian curvature we have the following expression for the (α, β)-metrics:

$$K_x = \frac{\phi(s)^{n+1}\left(\phi(s) - s\phi'(s)\right)^{n-2}\left[\phi(s) - s\phi'(s) + (\|\beta\|_x^2 - s^2)\phi''(s)\right] det\ a}{\left[\sum_{i=1}^n \left(\frac{\tilde{y}_i}{\alpha}(\phi(s) - s\phi'(s)) + b_i\phi'(s)\right)^2\right]^{\frac{n+1}{2}}}$$

In this expression of the Gaussian curvature, we noticed the following term which will be denoted by $A(s)$:

$$A(s) = \phi(s) - s\phi'(s) + (\|\beta\|_x^2 - s^2)\phi''(s).$$

Remark 2.2. We can observe that for our metric (1.1), $\phi(s) = s^2 + s + a$, we get after computations

$$A(s) = a + 2\|\beta\|_x^2 - 3s^2.$$

Starting with

$$A(s) = \phi(s) - s\phi'(s) + (\|\beta\|_x^2 - s^2)\phi''(s),$$

imposing the condition of self-concordant functions, one obtains:

$$A(s) \geq \phi(s) - s\phi'(s) + (\|\beta\|_x^2 - s^2)\left(\frac{1}{2}\phi'''(s)\right)^{\frac{2}{3}}.$$

Even more, for the k-self-concordant functions, we get

$$A(s) \geq \phi(s) - s\phi'(s) + (\|\beta\|_x^2 - s^2)\left(\frac{1}{2}k\phi'''(s)\right)^{\frac{2}{3}}.$$

So, we can observe that using the conditions for the function $\phi(s)$ to be self-concordant and k-self-concordant, this conditions can help us to obtain a boundary for the Gaussian curvature of (α, β)-metrics which contains convex functions $\phi(s)$.

Next, we will continue our investigation on the metric (1.1) and we will compute the Main Scalar for this kind of metric in the case of two dimensional Finsler spaces.
For a two dimensional Finsler space, let $\Gamma = \left\{\gamma_{jk}^i(x^1, x^2)\right\}$ be the Levi-Civita connection for the associated Riemannian space R^2 and the Finsler connection $\Gamma^S = \left(\gamma_{jk}^i, \gamma_{0j}^i, 0\right)$. Now, let h and v-covariant differentiation in Γ^* be denoted by $(; i, (i))$, respectively, where the (0) index denote the contraction with y^i. Then $y^i_{;j} = 0$, $\alpha_{;i} = 0$ and $a_{(i);j} = 0$. From [7], we know that the Main Scalar I of a two dimensional Finsler space $F^2 = (M^2, L(\alpha, \beta))$ endowed with an (α, β)-metric could be expressed as follows:

$$\epsilon I^2 = \left(\frac{L}{\alpha}\right)^4\left[\frac{\gamma^2(T_2)^2}{4T^3}\right]$$

where ϵ is the signature of the space, $\gamma^2 = b^2\alpha^2 - \beta^2$;

$$T = P(P + P_0 b^2 + P_{-1}\beta) + \left(P_0 P_{-2} - (P_{-1})^2\right)\gamma^2$$

and $T_2 = \frac{\partial T}{\partial \beta}$; $P = LL_\alpha \alpha^{-1}$; $P_0 = LL_{\beta\beta} + (L_\beta)^2$;

$$P_{-1} = (LL_{\alpha\beta} + L_\alpha L_\beta)\,\alpha^{-1},$$

$$P_{-2} = L\alpha^{-2}\left(L_{\alpha\alpha} - L_\alpha\alpha^{-1}\right) + L_\alpha^2\alpha^{-2}$$

For the metric (1.1), we can now compute the Main Scalar. In our case, $L = \alpha(s^2 + s + a)$, where $a \in \left(\frac{1}{4}, +\infty\right)$ is a real positive scalar. So, in our case,

$$L = \alpha\left(\frac{\beta^2}{\alpha^2} + \frac{\beta}{\alpha} + a\right) = \frac{\beta^2}{\alpha} + \beta + a\alpha.$$

Next, we get:

$$L_\alpha = -\frac{\beta^2}{\alpha^2} + a; L_\beta = \frac{2\beta}{\alpha} + 1; L_{\alpha\alpha} = \frac{2\beta^2}{\alpha^3}$$
$$L_{\alpha\beta} = -\frac{2\beta}{\alpha^2}; L_{\beta\beta} = \frac{2}{\alpha}.$$

Then, after computations, we get:

$$P = LL_\alpha\alpha^{-1} = \frac{(a\alpha^2 + \beta\alpha + \beta^2)(a\alpha^2 - \beta^2)}{\alpha^4}$$
$$P_0 = \frac{(2\,a+1)\,\alpha^2 + 6\,\beta\,\alpha + 6\,\beta^2}{\alpha^2}; P_{-1} = \frac{a\alpha^3 - 3\,\alpha\,\beta^2 - 4\,\beta^3}{\alpha^4}$$
$$P_{-2} = -\frac{\beta\,\left(a\alpha^3 - 3\,\alpha\,\beta^2 - 4\,\beta^3\right)}{\alpha^6}$$
$$T = \frac{\left(\left(2\,b^2 + a\right)\alpha^2 - 3\,\beta^2\right)\left(\alpha^2 a + \beta\,\alpha + \beta^2\right)^3}{\alpha^8}$$
$$\frac{\partial T}{\partial \beta} = 3\,\frac{\left(\left(2\,b^2 + a\right)\alpha^3 + 4\,\alpha^2 b^2 \beta - 5\,\alpha\,\beta^2 - 8\,\beta^3\right)\left(\alpha^2 a + \beta\,\alpha + \beta^2\right)^2}{\alpha^8}$$

and also, from

$$\epsilon I^2 = \left(\frac{L}{\alpha}\right)^4 \left[\frac{\gamma^2 (T_2)^2}{4T^3}\right].$$

We are now ready to give the following theorem:

Theorem 2.2. *The Main Scalar of a two dimensional Finsler space endowed with the (α, β)-metric (1.1), is given by:*

$$\epsilon I^2 = \frac{9\,\left(2\,\alpha^3 b^2 + 4\,\alpha^2 b^2 \beta + a\alpha^3 - 5\,\alpha\,\beta^2 - 8\,\beta^3\right)^2 \left(\alpha^2 b^2 - \beta^2\right)}{\left(4\,\alpha^2 a + 4\,\beta\,\alpha + 4\,\beta^2\right)\left(\left(2\,b^2 + a\right)\alpha^2 - 3\,\beta^2\right)^3}$$

In paper [5], the authors have noticed that the metric $F = \alpha\phi(s)$, where $\phi(s) = k_1 s + k_2\sqrt{k_3 s^2 + 1}$, $k_1 > 0$ is a Finsler metric of Randers type. Also in paper [5], the authors have remarked that for a certain choose of the constant k_3, the metric tensor is singular $(det(g_{ij}) = 0)$.
Generally, for (α, β)-metrics, we know that the metric tensor could be written as follows:

$$det(g_{ij}) = \phi^{n+1}(s)\,(\phi(s) - s\phi'(s))^{n-2}\left(\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s)\right) det(a_{ij}).$$

Next we will give the following Lemma:

Lemma 2.2. *For the (α, β)-metric $F = \alpha\phi(s)$, of the form $\phi(s) = k_1 s + k_2\sqrt{k_3 s^2 + 1}$, $k_1 > 0$, the function $\phi(s)$ is self-concordant if the following inequality holds:*

$$\frac{3\,|k_3|^2\,|k_2 s|}{|k_2 s^2 + 1|^{\frac{5}{2}}} \le 2\left(\frac{k_2 k_3}{(k_3 s^2 + 1)^{\frac{3}{2}}}\right)^{\frac{3}{2}}$$

where k_1, k_2, k_3 are constants with $k_1 > 0$.

Proof. It is easy to see that:

$$\phi'(s) = k_1 + \frac{k_2 k_3 s}{\sqrt{k_3 s^2 + 1}}$$
$$\phi''(s) = \frac{k_2 k_3}{\sqrt{(k_3 s^2 + 1)^3}}$$
$$\phi'''(s) = \frac{-3 k_2 k_3^2 s}{\sqrt{(k_3 s^2 + 1)^5}}$$

Next, imposing the condition for the function $\phi(s)$ to be self-concordant, we get the desired inequality. Even more, from the condition $\phi(0) = 1$, we get $k_2 = 1$. □

Finally, let us give the following example:

Example 2.2. If we take in the above Lemma, $k_3 = 1$, $k_2 = 1$ we get the metric tensor null for this (α, β)-metric, so this metric is singular and $\phi(s)$ is self-concordant. In this case we get the inequality:

$$2\left(\frac{1}{(1+s^2)^{\frac{3}{2}}}\right)^{\frac{3}{2}} \geq \left|\frac{3s}{(1+s^2)^{\frac{5}{2}}}\right|$$

Solving this inequality we get $s \in \left[-\frac{2\sqrt{2+\sqrt{85}}}{9}, \frac{2\sqrt{2+\sqrt{85}}}{9}\right]$. In this case the metric is singular and the function $\phi(s)$ is self-concordant.

3. Conclusion

In this paper we have continued our investigation on the (α, β)-metric (1.1) and we succeed to obtain new results for this kind of metric. Also, we have obtained some important results concerning some inequalities using the self-concordant and k-self concordant conditions for some special (α, β)-metrics. As we have seen this could lead us to establish some results regarding the bounded Gaussian curvature of such Finsler metrics.

Another important result established in this paper, was to investigate the link between Douglas type of metrics under Kropina change and the self-concordant functions. In this respect, we observe that for the function $\phi(s) = s^2 + s + a$ for our metric (1.1), after Kropina change this function became $\phi_1(s) = \frac{(s^2+s+a)^2}{s}$. Even the functions $\phi(s)$ and respectively $\phi_1(s)$ fulfilled the conditions to be self-concordant and respectively k-self-concordant, but the condition to remain a Douglas metric remains just in the case of the metric F endowed with the function $\phi(s)$. For $\overline{F}$ endowed with the function $\phi_1(s)$, this condition is not fulfilled. In our future papers we will try to investigate other types of Finsler (α, β)-metrics from the above points of view and to find some new families of (α, β)-metrics which satisfies this conditions.

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Author's contributions

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