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# NORMAL AUTOMORPHISMS OF FREE METABELIAN LEIBNIZ ALGEBRAS

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Abstract. Let $\mathfrak{M}$ be a free metabelian Leibniz algebra with generating set $X = \{x_1, ..., x_n\}$ over the field $\mathfrak{K}$ of characteristic 0. An automorphism $\phi$ of $\mathfrak{M}$ is said to be normal automorphism if each ideal of $\mathfrak{M}$ is invariant under $\phi$. In this work, it is proven that every normal automorphism of $\mathfrak{M}$ is an IA-automorphism and the group of normal automorphisms coincides with the group of inner automorphisms.

## 1. Introduction

Leibniz algebras were discovered in 1965 by A. Bloh [2] and forgotten for nearly thirty years. In the early 1990s Leibniz algebras were rediscovered by Loday as a generalization of Lie algebras [8]. In 1993, Loday and Pirashvili studied these algebras and they described the free Leibniz algebras [9]. In 2001, Mikhalev and Umirbaev obtained some important results on subalgebras of free Leibniz algebras [11]. Then automorphisms of free Leibniz algebras of rank two were described by Abdykhalykov at.al. [1]. In [13], the author studied on automorphic orbits of free Leibniz algebras of rank two. In [16], Hall bases of free Leibniz algebras were defined by Shahryari. In 2002, it was given a description of free metabelian Leibniz algebras by Drensky and Cattaneo [3]. Let $\mathfrak{M}$ be a free metabelian Leibniz algebra of rank $n$. Denote by $\mathfrak{M}'$, the commutator ideal of $\mathfrak{M}$. We write $\mathrm{Aut}(\mathfrak{M})$ for the automorphism group of $\mathfrak{M}$. Let

$$\pi : \mathrm{Aut}(\mathfrak{M}) \to \mathrm{Aut}(\mathfrak{M}/\mathfrak{M}')$$

be the canonical homomorphism with kernel consisting of automorphisms that induce the identity mapping on $\mathfrak{M}/\mathfrak{M}'$. The kernel of $\pi$ is called the IA-automorphism group and denoted by $\mathrm{IAut}(\mathfrak{M})$. In [17,18], the author and Taş Adıyaman described a generating set for $\mathrm{IAut}(\mathfrak{M})$ of rank three and $n$, respectively. Recently, symmetric

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polynomials of $\mathfrak{M}$ were considered in [7]. An automorphism $\theta$ of $\mathfrak{M}$ is said to be a normal automorphism if $\theta(I) = I$ for each ideal $I$ of $\mathfrak{M}$. Normal automorphism group $\mathrm{Aut}(\mathfrak{N})$ is a normal subgroup of $\mathrm{Aut}(\mathfrak{M})$. For an element $u$ of $\mathfrak{M}'$ the adjoint operator

$$\mathrm{ad}u : \mathfrak{M} \longrightarrow \mathfrak{M}$$

defined by $\mathrm{ad}u(v) = [v, u]$, for every $v \in \mathfrak{M}$ is nilpotent since $\mathrm{ad}^2 u = 0$. Hence $\exp(\mathrm{ad}u) = 1 + \mathrm{ad}u$ is an automorphism of $\mathfrak{M}$ called an inner automorphism. Denote by $\mathrm{Inn}(\mathfrak{M})$, the inner automorphism group of $\mathfrak{M}$. It is known that $\mathrm{Aut}(\mathfrak{N})$ contains $\mathrm{Inn}(\mathfrak{M})$. There exist many groups whose normal automorphisms are inner. See the papers [5, 10, 14, 15, 19]. In [4], Endimioni studied normal automorphisms of a free metabelian nilpotent group. Normal automorphisms are important for algebras. In [6], normal automorphisms of free metabelian nilpotent Lie algebras were considered. In [12], Öğüşlü proved that each normal automorphism of the metabelian product of abelian Lie algebras is an IA-automorphism and acts identically on the commutator algebra. It is natural to generalize results of Lie algebras to Leibniz algebras.

In this work, an analogue of the result in [12] is established for Leibniz algebras over a field of characteristic 0 and it is proven that each normal automorphism of $\mathfrak{M}$ is an IA-automorphism. Then it is proven that $\mathrm{Aut}(\mathfrak{N}) = \mathrm{Inn}(\mathfrak{M})$.

## 2. Preliminaries

Let $\mathfrak{K}$ be a field of characteristic 0. The vector space $\mathfrak{L}$ over $\mathfrak{K}$ equipped with a bilinear map $[,] : \mathfrak{L} \times \mathfrak{L} \longrightarrow \mathfrak{L}$ is called a Leibniz algebra if it satisfies the Leibniz identity

$$[x, [y, z]] = [[x, y], z] - [[x, z], y]$$

for all $x, y, z \in \mathfrak{L}$. In the general case a Leibniz algebra $\mathfrak{L}$ is a non-associtive and non-commutative algebra. If the condition $[x, x] = 0$ for all $x \in \mathfrak{L}$ is satisfied, then $\mathfrak{L}$ is a Lie algebra. Every commutator is reduced to a linear combination of left normed commutators by the Leibniz identity. Denote by $\mathrm{Ann}(\mathfrak{L})$, the ideal of $\mathfrak{L}$ generated by elements $\{[a, a] : a \in \mathfrak{L}\}$. It is known (see [9]) that $r_z = 0 \Leftrightarrow z \in \mathrm{Ann}(\mathfrak{L})$, where $r_z = \mathrm{ad}z$.

Let $\mathfrak{F}$ be the free Leibniz algebra with a generating set $\{x_1, \ldots, x_n\}$ over the field $\mathfrak{K}$ of characteristic 0 (see [9]) and let $\mathfrak{F}'$ and $\mathfrak{F}''$ be the commutator subalgebras of $\mathfrak{F}$ and $\mathfrak{F}'$, respectively. Then $\mathfrak{F}/\mathfrak{F}'$ and $\mathfrak{F}'/\mathfrak{F}''$ are abelian Leibniz algebras over $\mathfrak{K}$. We fix the notation $\mathfrak{M} = \mathfrak{F}/\mathfrak{F}''$ for the free metabelian Leibniz algebra over the field $\mathfrak{K}$. Then $\mathfrak{M}' = \mathfrak{F}'/\mathfrak{F}''$. Denote by $\langle \mathfrak{S} \rangle$, the ideal of $\mathfrak{M}$ generated by a set $\mathfrak{S}$.

The generators of $\mathrm{Aut}(\mathfrak{M})$ are given in the following theorem from [18].

**Theorem 1.** *Let $\mathfrak{M}$ be the free metabelian Leibniz algebra with a generating set $\{x_1, \ldots, x_n\}$. Then $\mathrm{Aut}(\mathfrak{M})$ is generated by the general linear group together with the inner automorphisms and the following IA-automorphisms*

$$\phi \quad : \quad x_1 \to x_1 + [z, x_1]$$

$$x_j \to x_j - [x_j, z]$$

*where* $z \in \mathfrak{M}'$ *and* $z \in \langle x_2 \rangle \oplus \ldots \oplus \langle x_n \rangle$,

$$\sigma \quad : \quad x_j \to x_j + [z, x_j]$$

*where* $z$ *is generated by the elements of the form* $[x, y] - [y, x]$ *where* $x, y \in \{x_1, \ldots, x_n\}$,

$$\tau \quad : \quad x_1 \to x_1 + u$$
$$x_i \to x_i$$

*where* $i \neq 1$, $u \in \mathrm{Ann}(\mathfrak{M})$ *depends on* $x_t$'s, $t \in \{2, \ldots, n\}$,

$$\psi \quad : \quad x_1 \to x_1 + v$$
$$x_i \to x_i$$

*where* $v \in \langle [x_j, x_k] \rangle$, $j \neq k \neq 1$,$i \neq 1$.

## 3. Normal Automorphisms

**Theorem 2.** *Let* $\theta \in \mathrm{Aut}(\mathfrak{N})$. *Then* $\theta \in \mathrm{IAut}(\mathfrak{M})$.

*Proof.* Let $\mathfrak{M}$ be a free metabelian Leibniz algebra with the generating set $\{x_1, \ldots, x_n\}$. Every automorphism $\theta$ of $\mathfrak{M}$ is defined by

$$\theta \quad : \quad x_i \to k_{i1}x_1 + k_{i2}x_2 + ... + k_{in}x_n + u_i,$$

where the linear part is invertible, $u_i \in \mathfrak{M}', i = 1, \ldots, n$, $k_{ij} \in \mathfrak{K}$ [18]. Let $\theta \in \mathrm{Aut}(\mathfrak{N})$. Consider the ideal $\langle x_i \rangle$ of $\mathfrak{M}$. We have $\theta(x_i) \in \langle x_i \rangle$. Then $k_{i1}x_1 + \ldots + k_{ii}x_i + \ldots + k_{in}x_n + u_i \in \langle x_i \rangle$. By grading $k_{i1}x_1 + \ldots + k_{ii}x_i + \ldots + k_{in}x_n \in \langle x_i \rangle$ and $u_i \in \langle x_i \rangle$ are obtained. Since $x_1, x_2, \ldots, x_n$ are free generators, we obtain $k_{ij} = 0$ for $i \neq j$. Hence we have

$$\theta \quad : \quad x_i \to k_{ii}x_i + u_i,$$

where $k_{ii} \in \mathfrak{K}$. Consider the ideal $\langle \sum_{i=1}^{n} x_i \rangle$ of $\mathfrak{M}$. We obtain $\theta(\sum_{i=1}^{n} x_i) \in \langle \sum_{i=1}^{n} x_i \rangle$. Clearly

$$\theta(x_1 + x_2 + ... + x_n) = k_{11}x_1 + k_{22}x_2 + \ldots + k_{nn}x_n + u_1 + u_2 + \ldots + u_n$$

and

$$k_{11}x_1 + k_{22}x_2 + \ldots + k_{nn}x_n + u_1 + u_2 + \ldots + u_n \in \langle x_1 + x_2 + \ldots + x_n \rangle.$$

By grading we have

$$k_{11}x_1 + k_{22}x_2 + \ldots + k_{nn}x_n = k(x_1 + x_2 + \ldots + x_n)$$

for a coefficient $k \in \mathfrak{K}$. It implies

$$(k_{11} - k)x_1 + (k_{22} - k)x_2 + \ldots + (k_{nn} - k)x_n = 0,$$

by the linearly independence $k_{ii} - k = 0$, and $k_{ii} = k$ for $i = 1, 2, \ldots, n$. Therefore,

$$\theta \quad : \quad x_i \to kx_i + u_i.$$

Consider the ideal $\langle x_i + [x_i, x_i] \rangle$ of $\mathfrak{M}$,

$$\theta(x_i + [x_i, x_i]) = kx_i + k^2[x_i, x_i] + u_i + k[u_i, x_i] + k[x_i, u_i] \in \langle x_i + [x_i, x_i] \rangle.$$

By Theorem 1, $u_i \neq [x_i, x_i]$. Clearly it yields

$$kx_i + k^2[x_i, x_i] + u_i + k[u_i, x_i + k[x_i, u_i] = c(x_i + [x_i, x_i]) + z$$

where $c \in \mathfrak{K}, z \in \langle x_i + [x_i, x_i] \rangle$. By this equality, we obtain $k = c$, $k^2 = c$. Then we see that $k = k^2$ and $0 = k - k^2 = k(1 - k)$. Hence $k = 1$. □

**Theorem 3.** $\mathrm{Aut}(\mathfrak{N}) = \mathrm{Inn}(\mathfrak{M})$.

*Proof.* Let $\theta \in \mathrm{Aut}(\mathfrak{N})$. Then $\theta$ is an IA-automorphism by Theorem 2. Hence, it can be defined by

$$\theta \quad : \quad x_i \to x_i + u_i$$

where $u_i \in \mathfrak{M}'$. Using the generating set of IA-automorphisms by Theorem 1, we can write the elements $u_i, i = 1, 2, \ldots, n$ as in the following forms;
**Case 1.** $u_i = [x_i, w]$ for $i = 1, 2, \ldots, n$ and $w \in \mathfrak{M}'$. In this form, $\theta$ is an inner automorphism.
**Case 2.** $u_1 = [w, x_1]$, $u_j = -[x_j, w]$, for $j = 2, \ldots, n$, where $w \in \mathfrak{M}'$ and $w \in \langle x_2 \rangle \oplus \ldots \oplus \langle x_i \rangle \oplus \ldots \oplus \langle x_n \rangle, i \neq 1$. Now take $[x_1, x_2] \in \mathfrak{M}'$. Consider the ideal $\langle [x_1, x_2] \rangle$ of $\mathfrak{M}$. Then

$$\theta([x_1, x_2]) = [x_1, x_2] + [u_1, x_2] + [x_1, u_2] = [x_1, x_2] + [[w, x_1], x_2] - [x_1, [x_2, w]].$$

Since $[[w, x_1], x_2] - [x_1, [x_2, w]] \notin \langle [x_1, x_2] \rangle$, then $\theta([x_1, x_2]) \notin \langle [x_1, x_2] \rangle$. This is a contradiction.
**Case 3.** $u_i = [w, x_i]$, for $i = 1, 2, \ldots, n$, where $w$ is generated by the elements of the form $[x, y] - [y, x]$, for $x, y \in \{x_1, ..., x_n\}$. Consider the ideal $\langle [x_1, x_2] \rangle$ of $\mathfrak{M}$. Then

$$\theta([x_1, x_2]) = [x_1, x_2] + [u_1, x_2] + [x_1, u_2] = [x_1, x_2] + [[w, x_1], x_2] + [x_1, [w, x_2]].$$

Since $[[w, x_1], x_2] + [x_1, [w, x_2]] \notin \langle [x_1, x_2] \rangle$, then $\theta([x_1, x_2]) \notin \langle [x_1, x_2] \rangle$. This is a contradiction.
**Case 4.** $u_1 \in \mathrm{Ann}(\mathfrak{M})$ depends on $x_t$'s, $t \in \{2, \ldots, n\}$, and $u_j = 0$ for $j = 2, \ldots, n$. We have

$$\theta([x_1, x_2]) = [x_1, x_2] + [u_1, x_2].$$

Since $[u_1, x_2] \notin \langle [x_1, x_2] \rangle$, this automorphism is not a normal automorphism.
**Case 5.** $u_1 = \langle [x_j, x_k] \rangle$, $j \neq k \neq 1$, and $u_j = 0$, for $j = 2, \ldots, n$. We obtain

$$\theta([x_1, x_2]) = [x_1, x_2] + [u_1, x_2].$$

Since the element $[u_1, x_2] \notin \langle [x_1, x_2] \rangle$, then $\theta([x_1, x_2]) \notin \langle [x_1, x_2] \rangle$. This is a contradiction.

Therefore, the elements $u_i$ are only as in Case 1. Hence $\theta$ is an inner automorphism. □

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