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Research Article

Local distance antimagic cromatic number of join product of graphs with cycles or paths

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Abstract

Let G be a graph of order p without isolated vertices. A bijection $f : V \to \{1, 2, 3, \ldots, p\}$ is called a local distance antimagic labeling, if $w_f(u) \neq w_f(v)$ for every edge uv of G, where $w_f(u) = \sum_{x \epsilon N(u)} f(x)$. The local distance antimagic chromatic number $\chi_{lda}(G)$ is defined to be the minimum number of colors taken over all colorings of G induced by local distance antimagic labelings of G. In this paper, we determined the local distance antimagic chromatic number of some cycles, paths, disjoint union of 3-paths. We also determined the local distance antimagic chromatic number of join products of some graphs with cycles or paths.

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1. Introduction

Let $G = (V, E)$ be a simple graph. For graph-theoretic terms, we refer to Bondy and Murty [4]. For integers $a < b$, let $[a, b]$ be the set of integers from a to b. For a vertex v, let $N(v)$ be the set of all the neighbors of v and $\deg(v) = |N(v)|$.

Hartsfield and Ringel [8] introduced antimagic labeling, which is defined as a bijection $f : E \to \{1, 2, \ldots, |E|\}$, for each vertex $u \in V(G)$, the weight $w(u) = \sum_{e \in E(u)} f(e)$, where $E(u)$ is the set of edges incident to u. If $w(u) \neq w(v)$ for any two distinct vertices u and $v \in V(G)$, then f is called an *antimagic labeling* of G. A graph G is called *antimagic* if G has an antimagic labeling. Hartsfield and Ringel [8] conjectured that every connected graph with at least three vertices admits an antimagic labeling. They also made a weak conjecture that every tree with at least three vertices admits an antimagic labeling. These two conjectures were partially shown to be correct by several authors, but they are still

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unsolved. For a detailed and interesting review of these conjectures, one can see Chapter 6 of [7].

Arumugam et al. [3] proposed a new labeling as a relaxation of the notion of antimagic labeling. For a connected graph G, they called a bijection $f : E(G) \to \{1, 2, \dots, |E|\}$ a *local antimagic* labeling (in short, an LA-labeling) of G if for any two adjacent vertices u and v in $V(G)$, the condition $w(u) \neq w(v)$ holds. They conjectured that connected graphs with at least three vertices admits a LA-labeling. Bensmail et al. [5] solved this conjecture partially. Finally, Haslegrave proved this conjecture using probabilistic tools [9]. Based on the notion of local antimagic labeling, Arumugam et al. [3] introduced a new graph coloring parameter called *local antimagic chromatic number* (LACN). It is defined as the minimum number of colors taken over all colorings of G induced by all LA-labelings of G, denoted $\chi_{la}(G)$. Recently, several authors investigated the LACN for several families of graphs. For further study, see [11, 12, 14, 15].

In 2012, Arumugam and Kamatchi [10] introduced (a, d)-distance antimagic labeling and they obtained some basic results. For further study see [1, 2, 7, 10, 13].

The notion of local antimagic labeling motivated Divya and Devi Yamini [6] to introduce a new coloring parameter known as local distance antimagic chromatic number. Let G be a graph of order p and size q having no isolated vertices. A bijection $f : V \to \{1, 2, 3, \dots, p\}$ is called a *local distance antimagic* labeling (in short, an LDA-labeling), if $w_f(u) \neq w_f(v)$ for every edge uv of G, where $w_f(u) = \sum_{x\in N(u)} f(x)$ which is called the *weight* of u. The mapping w_f is called the *coloring* of G induced by f and f is called a k-*LDA-labeling* of G if w_f is a k-coloring of G. A graph G is called *local distance antimagic* if G has an LDA-labeling. We shall omit the subscript f if there is no ambiguity.

The *local distance antimagic chromatic number* (LDACN), denoted $\chi_{lda}(G)$, is the minimum value of k if G has a k-LDA-labelings. Divya et al. [6] obtained the LDACN for some classes of graphs. Clearly, any graph G that admits an LDA-labeling has $\chi_{lda}(G) \geq \chi(G) \geq 2$.

In [16] Priyadharshini and Nalliah studied the local distance antimagic labeling of graphs independently and they obtained the local distance antimagic chromatic number of disjoint union of m copies of complete bipartite graphs $K_{r,s}$ for some m, r, s.

Throughout this paper, we only consider simple graphs without isolated vertices unless stated otherwise.

2. Cycle related graphs

Lemma 2.1. *Suppose u and v are two non-adjacent vertices in a graph G with* $\deg(u) = \deg(v) = t \geq 2$. *If $|N(u) \cap N(v)| = t - 1$, then $w(u) \neq w(v)$ under any LDA-labeling f of G.*

Proof. Let $N(u) = \{x_i \,|\, 1 \leq i \leq t-1\} \cup \{y\}$ and $N(v) = \{x_i \,|\, 1 \leq i \leq t-1\} \cup \{z\}$, where $y \neq z$. Since $f(y) \neq f(z)$, $w(u) = f(y) + \sum_{i=1}^{t-1} f(x_i) \neq f(z) + \sum_{i=1}^{t-1} f(x_i) = w(v)$. □

Suppose G is a graph. Let $V_2 = \{v \in V(G) \mid \deg(v) = 2\}$ and let $G[V_2]$ be the subgraph of G induced by V_2. Suppose H is a component of $G[V_2]$ with order $n \geq 1$. We call H a 2-*component* of G. Clearly H is either a P_n $(n \geq 1)$ or a C_n $(n \geq 3)$.

Lemma 2.2. *Suppose G admits a 3-LDA-labeling f. If G contains a 2-component H, then $H \in \{C_3, C_4, C_{12}\} \cup \{P_n \mid 1 \leq n \leq 11\}$.*

Proof. Let a, b, c be the vertex weights.

(1) Suppose $H = P_n = u_1u_2 \cdots u_n$ with $n \geq 12$. Since $\deg_G(u_1) = 2$, we may assume $N(u_1) = \{u_0, u_2\}$. Lemma 2.1 implies that any three consecutive vertices of P_n must be of distinct weight. Without loss of generality, we assume that $w(u_1) = a$, $w(u_2) = b$ and $w(u_3) = c$. This forces vertices u_1 to u_n to have weights a, b, c

repeatedly. Let $f(u_0) = x$. Thus, we must have $f(u_2) = a - x$, $f(u_4) = c - a + x$, $f(u_6) = b - c + a - x$, $f(u_8) = c - b + x$, $f(u_{10}) = b - x$ and $f(u_{12}) = x$ which is impossible.

(2) Suppose $H = C_n = u_1u_2 \cdots u_nu_1$ with $n \geq 5$. Suppose $f(u_n) = x$. By the same argument as the previous case, we have $n \leq 12$. Since every three consecutive vertices of C_n must have distinct weights, we have $n \equiv 0 \pmod 3$. Hence, $n \in \{6, 9, 12\}$.

Suppose $n = 6$. We have

$$\begin{aligned} a &= f(u_2) + f(u_6) = f(u_3) + f(u_5), \\ b &= f(u_1) + f(u_3) = f(u_4) + f(u_6), \\ c &= f(u_2) + f(u_4) = f(u_1) + f(u_5). \end{aligned}$$

Simplifying the above three equations we have $f(u_2) = f(u_5)$, a contradiction. So $\chi_{lda}(C_6) \geq 4$.

Suppose $n = 9$. Suppose $f(u_9) = x$, then $f(u_2) = a - x$, $f(u_4) = c - a + x$, $f(u_6) = b-c+a-x$, $f(u_8) = c-b+x$, $f(u_1) = f(u_7) = b-x$ which is impossible.

We shall show in Theorem 2.3 that $\chi_{lda}(C_{12}) = 3$.

□

Theorem 2.3.

$$\chi_{lda}(C_n) = \begin{cases} 2, & n = 4; \\ 3, & n \in \{3, 12\}; \\ 4, & n \in \{6, 8, 10, 14\}; \\ 5, & n \in \{5, 7, 9\}, \end{cases}$$
$$4 \leq \chi_{lda}(C_n) \leq 5, \quad n \in \{11, 13\},$$
$$4 \leq \chi_{lda}(C_n) \leq 6, \quad n \geq 15.$$

Proof. It is easy to see that $\chi_{lda}(C_3) = 3$ and $\chi_{lda}(C_4) = 2$. For $n = 12$, by Lemma 2.1, $\chi_{lda}(C_{12}) \geq 3$. Label the vertices of C_{12} by 1, 11, 8, 2, 9, 7, 4, 10, 5, 3, 12, 6 in natural order. The vertex weights are $17, 9, 13, 17, 9, 13, 17, 9, 13, 17, 9, 13$. Thus $\chi_{lda}(C_{12}) = 3$. So we assume $n \geq 5$ with $n \neq 12$ and let $C_n = u_1u_2u_3 \cdots u_nu_1$. Lemma 2.2 implies that $\chi_{lda}(C_n) \geq 4$. We first show that $\chi_{lda}(C_n) \leq 6$.

(a) Suppose $n = 4k$ for $k \geq 2$. Define $g : V(C_{4k}) \to [1, 4k]$ by

$$g(u_{4i-3}) = i, g(u_{4i-2}) = 2k + i, g(u_{4i-1}) = 2k + 1 - i, g(u_{4i}) = 4k + 1 - i, \text{for } 1 \leq i \leq k$$

Then $w_g(u_1) = 5k + 2$ and $w_g(u_{4i-3}) = 6k + 2$ for $2 \leq i \leq k$; $w_g(u_{4i-2}) = 2k + 1$ for $1 \leq i \leq k$; $w_g(u_{4i-1}) = 6k + 1$ for $1 \leq i \leq k$; $w_g(u_{4i}) = 2k + 2$ for $1 \leq i \leq k - 1$ and $w_g(u_{4k}) = k + 2$. Clearly, when $k \geq 2$, there are 6 distinct vertex weights. Thus $\chi_{lda}(C_{4k}) \leq 6$.

(b) When $n = 5$. Label the vertices of v_1 to v_5 by 1, 2, 3, 4, 5 in order. The vertex weights of v_1 to v_5 are 7, 4, 6, 8, 5. So $\chi_{lda}(C_{4k+1}) \leq 5$. We shall show that $\chi_{lda}(C_{4k+1}) = 5$ later. Suppose $n = 4k + 1$ for $k \geq 2$. Extending the same definition of g as Case (a) and define $g(u_{4k+1}) = 4k+1$. Then $w_g(u_l)$ are the same as in Case (a) when $2 \leq l \leq 4k - 1$ and $w_g(u_{4k}) = 5k + 2$, $w_g(u_{4k+1}) = 3k + 2$ and $w_g(u_1) = 6k+2$. Thus there are 6 distinct vertex weights and hence $\chi_{lda}(C_{4k+1}) \leq 6$.

(c) Suppose $n = 4k + 2$ for $k \geq 1$. Define $g : V(C_{4k+2}) \to [1, 4k + 2]$ by $g(u_{4i-3}) = i$, $g(u_{4i-2}) = 2k + 1 + i$, $g(u_{4i-1}) = 2k + 2 - i$, $g(u_{4i}) = 4k + 3 - i$, for $1 \leq i \leq k$, $g(u_{4k+1}) = k + 1$ and $g(u_{4k+2}) = 3k + 2$.
Then $w_g(u_1) = 5k+4$ and $w_g(u_{4i-3}) = 6k+5$ for $2 \leq i \leq k+1$; $w_g(u_{4i-2}) = 2k+2$ for $1 \leq i \leq k$ and $w_g(u_{4k+2}) = k + 2$; $w_g(u_{4i-1}) = 6k + 4$ for $1 \leq i \leq k$;

$w_g(u_{4i}) = 2k+3$ for $1 \le i \le k$. Clearly, there are 6 distinct vertex weights. Thus $\chi_{lda}(C_{4k+2}) \le 6$.

(d) Suppose $n = 4k+3$ for $k \ge 1$. Extending the same definition of g as Case (c) and define $g(u_{4k+3}) = 4k+3$. Then $w_g(u_l)$ are the same as in Case (c) when $2 \le l \le 4k+1$ and $w_g(u_{4k+2}) = 5k+4$, $w_g(u_{4k+3}) = 3k+3$ and $w_g(u_1) = 6k+5$. Thus there are 6 distinct vertex weights and hence $\chi_{lda}(C_{4k+3}) \le 6$.

Following we shall find the exact value or lower bound of $\chi_{lda}(C_n)$ for some n.

(1) Suppose C_5 admits an LDA-labeling f. By Lemma 2.1, we immediately have $w(u_i) \notin \{w(u_{i+2}), w(u_{i+3}) \,|\, 1 \le i \le 5 \pmod 5\}$. Since $w(u) \ne w(v)$ for $uv \in E(C_5)$, we conclude that $w(u) \ne w(v)$ for any two vertices u, v in $V(C_5)$. Thus, $\chi_{lda}(C_5) \ge 5$. From Case (b) we have $\chi_{lda}(C_5) \le 5$. Thus, $\chi_{lda}(C_5) = 5$.

(2) Label the vertices of C_6 by 1, 3, 6, 2, 4, 5 in natural order. Then the vertex weights are 8, 7, 5, 10, 7, 5. So that $\chi_{lda}(C_6) \le 4$. Thus, $\chi_{lda}(C_6) = 4$.

(3) Label the vertices of C_7 by 1, 3, 6, 2, 4, 5, 7 in the natural order. Then the vertex weights are 10, 7, 5, 10, 7, 11, 6. So $\chi_{lda}(C_7) \le 5$. There is no 4-LDA-labeling for C_7 (please see the appendix). So $\chi_{lda}(C_7) = 5$.

(4) Label the vertices of C_8 by 1, 2, 7, 5, 4, 3, 6, 8 in natural order. The vertex weights are 10, 8, 7, 11, 8, 10, 11, 7. So, $\chi_{lda}(C_8) \le 4$. Thus, $\chi_{lda}(C_8) = 4$.

(5) For $n = 9$, there is no 4-LDA-labeling for C_9 (please see appendix). A 5-LDA-labeling is to label the vertices by 1, 2, 6, 7, 5, 3, 4, 8, 9 in natural order. The vertex weights are 11, 7, 9, 11, 10, 9, 11, 13, 9. So, $\chi_{lda}(C_9) = 5$.

(6) Label the vertices of C_{10} by 1, 2, 7, 8, 5, 6, 3, 4, 9, 10 in natural order. The vertex weights are 12, 8, 10, 12, 14, 8, 10, 12, 14, 10. So, $\chi_{lda}(C_{10}) \le 4$. Thus, $\chi_{lda}(C_{10}) = 4$.

(7) For $n = 11$, a 5-LDA-labeling is to label the vertices by 1, 3, 10, 7, 4, 6, 8, 5, 2, 9, 11 in natural order. The vertex weights are 14, 11, 10, 14, 13, 12, 11, 10, 14, 13, 10. So, $\chi_{lda}(C_{11}) \le 5$.

(8) For $n = 13$, label the vertices of C_{13} by 5, 11, 7, 2, 4, 10, 9, 1, 3, 12, 8, 6 in natural order. The vertex weights are 17, 12, 13, 11, 12, 13, 11, 12, 13, 11, 18, 13. So $\chi_{lda}(C_{13}) \le 5$.

(9) Label the vertices of C_{14} by 1, 2, 11, 12, 5, 6, 7, 8, 9, 10, 3, 4, 13, 14 in natural order. The vertex weights are 16, 12, 14, 16, 18, 12, 14, 16, 18, 12, 14, 16, 18, 14. Thus, $\chi_{lda}(C_{14}) = 4$.

□

Remark 2.4. By the proofs of Theorem 2.3 and Lemma 2.2 we have that if G contains C_5 as a component, then $\chi_{lda}(G) \ge 5$.

By computer search, we know that there is no 4-LDA-labeling for C_{11}. But we cannot find a mathematical proof at this moment. Now we propose the following problem and conjecture.

Problem 2.5. Find a mathematical proof for showing $\chi_{lda}(C_n) \ge 5$, where $n \in \{11, 13\}$.

Conjecture 2.6. *$\chi_{lda}(C_n) = 4$ for even $n \ge 16$ and $\chi_{lda}(C_n) = 5$ for odd $n \ge 13$.*

We now consider $\overline{C_n}$, the complement of $C_n, n \ge 4$.

Theorem 2.7. *For $n \ge 4$, $\chi_{lda}(\overline{C_n}) = n$.*

Proof. Let $C_n = u_1u_2\cdots u_nu_1$, $n \ge 4$. Clearly, $\overline{C_4} = 2K_2$ with $\chi_{lda}(2K_2) = 4$. We consider $n \ge 5$.

Suppose $\chi_{lda}(\overline{C_n}) \le n-1$, then there are two nonadjacent vertices with the same weight. Without loss of generality we can assume they are u_2 and u_3. Their neighborhoods intersect in $n-4$ vertices $u_5, u_6, \ldots, u_n$. Since $\deg(u_2) = \deg(u_3) = n-3$ and $|N(u_2) \cup$

$N(u_3)| = n-4$, by Lemma 2.1, we must have $w(u_2) \neq w(u_3)$, a contradiction. Thus $\chi_{lda}(\overline{C_n}) \geq n$.

We now show that $\overline{C_n}$ admits an n-LDA labeling. Suppose $n \not\equiv 0 \pmod 3$. Define $f(v_i) = i$ for $1 \leq i \leq n$. We see that f is bijective and the vertex weights of v_1 to v_n are $K-(n+3)$, $K-6$, $K-9$, $\dots$, $K-(3n-3)$, $K-2n$, respectively, where $K = n(n+1)/2$. Since $n+3$ and $2n$ are not multiples of 3, it is easy to verify that all the weights are distinct. Thus, f is an n-LDA-labeling for $\overline{C_n}$ and hence $\chi_{lda}(\overline{C_n}) = n$.

Consider $n \equiv 0 \pmod 3$. Suppose $n = 6$, label the vertices of v_1 to v_6 by 1, 2, 3, 6, 5, 4 respectively. The vertex weights of v_1 to v_6 are 14, 15, 10, 7, 6, 11. Thus, $\chi_{lda}(\overline{C_6}) = 6$. Consider $n \geq 9$. We label the vertices v_1 to v_n by 2, 1, 3, 4, 5, $\dots$, $n-3$, $n-2$, n, $n-1$ in order. We see that f is bijective and the vertex weights of v_1 to v_n are $K-(n+2)$, $K-6$, $K-8$, $K-12$, $K-15$, $\dots$, $K-(3n-9)$, $K-(3n-5)$, $K-(3n-3)$, $K-(2n+1)$, respectively, where $K = n(n+1)/2$. Since $n \geq 9$, $n+2 \not\equiv 0 \pmod 3$ and $2n+1 \not\equiv 0 \pmod 3$, it is easy to verify that all the weights are distinct. Thus, f is an n-LDA-labeling for G and hence $\chi_{lda}(\overline{C_n}) = n$. □

Suppose G and H admit LDA-labelings g and h, respectively. Let the orders of G and H be p_G and p_H, respectively. We define a labeling $f : V(G \vee H) \to [1, p_G + p_H]$ (the join graph of G with H) by

$$f(x) = \begin{cases} g(x) & \text{if } x \in V(G); \\ h(x) + p_G & \text{if } x \in V(H). \end{cases} \tag{2.1}$$

Let w_g and w_h be the colorings of G and H induced from g and h, respectively. Also let w_f be the coloring of $G \vee H$ induced from f. Thus, for $u \in V(G)$ and $v \in V(H)$,

$$\begin{aligned} w_f(u) &= \sum_{x \in N_{G \vee H}(u)} f(x) = \sum_{x \in N_G(u)} g(x) + \sum_{x \in V(H)} (h(x) + p_G) \\ &= w_g(u) + p_G p_H + \sum_{x \in V(H)} h(x) = w_g(u) + p_G p_H + \frac{1}{2} p_H(p_H + 1); \end{aligned} \tag{2.2}$$

$$\begin{aligned} w_f(v) &= \sum_{x \in N_{G \vee H}(v)} f(x) = \sum_{x \in N_H(v)} (h(x) + p_G) + \sum_{x \in V(G)} g(x) \\ &= w_h(v) + p_G \deg_H(v) + \sum_{x \in V(G)} g(x) = w_h(v) + p_G \deg_H(v) + \frac{1}{2} p_G(p_G + 1). \end{aligned} \tag{2.3}$$

Remark 2.8. Keep the notation defined above. Consider any two distinct vertices u and v.

Suppose $u, v \in V(G)$. By (2.2), $w_g(u) \neq w_g(v)$ if and only if $w_f(u) \neq w_f(v)$. Thus $|w_f(V(G))| = |w_g(V(G))|$.

Suppose $u, v \in V(H)$ and H is a regular graph. By (2.3), $w_h(u) \neq w_h(v)$ if and only if $w_f(u) \neq w_f(v)$. Thus $|w_f(V(H))| = |w_h(V(H))|$.

Theorem 2.9. *Let G be a graph of order $m \geq 2$ such that $\chi_{lda}(G)$ exists. If $n \geq m$ and $n \geq 3$, then $\chi_{lda}(G \vee C_n) \leq \chi_{lda}(G) + \chi_{lda}(C_n)$.*

Proof. Let g and h be $\chi_{lda}(G)$-LDA-labeling and $\chi_{lda}(C_n)$-LDA-labeling of G and C_n, respectively. Let f be the bijective labeling defined in (2.1). We are going to check that f is a local distance antimagic labeling of $G \vee C_n$.

Consider any two distinct vertices u and v. From Remark 2.8, it suffices to consider $u \in V(G)$ and $v \in V(C_n)$. From (2.2) and (2.3) we have

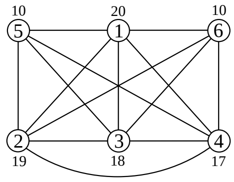
$$\begin{aligned} w_f(u) &= w_g(u) + mn + \frac{1}{2} n(n+1) \geq 1 + mn + \frac{1}{2} n(n+1); \\ w_f(v) &= w_h(v) + 2m + \frac{1}{2} m(m+1) \leq (2n-1) + 2m + \frac{1}{2} m(m+1). \end{aligned}$$

$$w_f(u) - w_f(v) \geq 2 + mn - 2n - 2m + \frac{1}{2}n(n+1) - \frac{1}{2}m(m+1).$$

1. Suppose $n > m \geq 2$. Then $w_f(u) - w_f(v) \geq 2 + mn - 2n - 2m + (m+1) = 1 + (m-2)(n-1) > 0$.
2. Suppose $m = n \geq 4$. Clearly $w_f(u) - w_f(v) \geq 2 + n^2 - 4n = (n-2)^2 - 2 > 0$.
3. Suppose $m = n = 3$. Then $w_f(u) - w_f(v) = w_g(u) - w_h(v) + 3$ and that $G \in \{C_3, P_3, \overline{C_3}, K_1 + K_2\}$. By the definition of local distance antimagic, G contains no isolated vertex. Thus G is either C_3 or P_3.

   Suppose $G = C_3$. Then $w_g(u) + 3 \geq 6$ and $w_h(v) \leq 5$. So $w_f(u) - w_f(v) > 0$.

   Suppose $G = P_3$. This is an ad hoc case. We need to redefine a labeling f for $P_3 \vee C_3$. Following is a required local distance antimagic labeling:



   Thus $\chi_{lda}(P_3 \vee C_3) \leq 5$. Since $\chi(P_3 \vee C_3) = 5$, we have $\chi_{lda}(P_3 \vee C_3) = 5$.

Thus, $w_f(V(G)) \cap w_f(V(C_n)) = \varnothing$. So f is a local distance antimagic labeling of $G \vee C_n$. Moreover, the number of distinct values of w_f is $|w_f(G \vee C_n)| = |w_f(V(G))| + |w_f(V(C_n))| = |w_g(V(G))| + |w_h(V(C_n))| = \chi_{lda}(G) + \chi_{lda}(C_n)$. Hence we have $\chi_{lda}(G \vee C_n) \leq \chi_{lda}(G) + \chi_{lda}(C_n)$. □

Corollary 2.10. *For $n \geq m \geq 3$, $\chi_{lda}(C_m \vee C_n) \leq \chi_{lda}(C_m) + \chi_{lda}(C_n)$.*

Example 2.11. By Lemma 2.1 and the argument similar to the proof of Theorem 2.3, we can conclude that $\chi_{lda}(C_5 \vee C_6) \geq 9$. By Corollary 2.10 we have $\chi_{lda}(C_5 \vee C_6) \leq \chi_{lda}(C_5) + \chi_{lda}(C_6) = 5 + 4 = 9$. Thus $\chi_{lda}(C_5 \vee C_6) = 9$.

So the bound in Theorem 2.9 (or Corollary 2.10) is sharp.

Theorem 2.12. *Let H be an r-regular graph of order n such that $\chi_{lda}(H)$ exists. If $n \geq 3$, then $\chi_{lda}(C_5 \vee H) \leq 5 + \chi_{lda}(H)$.*

Proof. Suppose $H \neq K_n$. Then $r \leq n-2$. Let g be the 5-LDA-labeling of C_5 defined in the proof of Theorem 2.3 and h be a $\chi_{lda}(H)$-LDA-labeling of H. Let f be the bijective labeling defined in (2.1). We are going to check that f is a local distance antimagic labeling of $C_5 \vee H$.

From Remark 2.8, we have $|w_f(V(C_5))| = |w_g(V(C_5))| = 5$ and $|w_f(V(H))| = |w_h(V(H))|$ and we only need to consider $u \in V(C_5)$ and $v \in V(H)$. From (2.2) and (2.3) we have

$$w_f(u) - w_f(v) = w_g(u) - w_h(v) + 5(n-r) + \frac{1}{2}n(n+1) - 15.$$

Note that $w_h(v) \leq \sum\limits_{i=n-r+1}^{n} i = \frac{r(2n-r+1)}{2}$. Thus

$$\begin{aligned} w_f(u) - w_f(v) &\geq w_g(u) - \frac{r(2n-r+1)}{2} + 5(n-r) + \frac{1}{2}n(n+1) - 15 \\ &\geq 4 + \frac{r^2 + n^2 + 11n - 11r - 2nr - 30}{2} \quad \text{(which is a decreasing function of } r) \\ &\geq \frac{(n-2)^2 + n^2 + 11n - 11(n-2) - 2n(n-2) - 22}{2} = 2. \end{aligned}$$

So f is an LDA-labeling for $C_5 \vee H$. Thus $|w_f(C_5 \vee H)| = |w_f(V(C_5))| + |w_f(V(H))| = |w_g(V(C_5))| + |w_h(V(H))| = 5 + \chi_{lda}(H)$.

Suppose $H = K_n$. Let g be the n-LDA-labeling of $H = K_n$ and h be the 5-LDA-labeling of C_5 defined in the proof of Theorem 2.3. Let f be the bijective labeling defined in (2.1). For $u \in V(H)$ and $v \in V(C_5)$, we have

$$w_f(u) = w_g(u) + 5n + 15 \geq \sum_{i=1}^{n-1} i + 5n + 15 = \frac{n^2 + 9n + 30}{2},$$
$$w_f(v) = w_h(v) + 2n + \frac{n(n+1)}{2} \leq 8 + 2n + \frac{n(n+1)}{2} = \frac{n^2 + 5n + 16}{2}.$$

Clearly $w_f(u) > w_f(v)$.

Hence we have the theorem. □

Corollary 2.13. *Let $G_1 = C_5$ and $G_k = C_5 \vee G_{k-1}$ for $k \geq 2$. Then $\chi_{lda}(G_k) = 5k$ for $k \geq 1$.*

Proof. Note that, $|V(G_k)| = 5k$ and G_k is a $(5k-3)$-regular graph for $k \geq 1$. Let f be any local antimagic labeling of G_k (it exists by the proof of Theorem 2.12). Suppose u and v are not adjacent in G_k. Since $|N(u) \cap N(v)| = 5k - 4$, by Lemma 2.1, $w_f(u) \neq w_f(v)$. Since f is a local antimagic labeling, $w_f(u) \neq w_f(v)$ for any two adjacent vertices u and v. Consequently, $w_f(u) \neq w_f(v)$ for any two distinct vertices u and v in G_k. Hence $\chi_{lda}(G_k) = 5k$. □

Theorem 2.14. *Let H be an r-regular graph of order n such that $\chi_{lda}(H)$ exists. If $n \geq 3$, then $\chi_{lda}(C_3 \vee H) \leq 3 + \chi_{lda}(H)$.*

Proof. Note that $r \leq n-1$. Let f be the labeling defined in (2.1). By a similar argument of the proof of Theorem 2.12

$$\begin{aligned} w_f(u) - w_f(v) &\geq w_g(u) - \frac{r(2n-r+1)}{2} + 3(n-r) + \frac{1}{2}n(n+1) - 6 \\ &\geq 3 + \frac{r^2 + n^2 + 7n - 7r - 2nr - 12}{2} \quad \text{(which is a decreasing function of } r) \\ &\geq \frac{(n-1)^2 + n^2 + 7n - 7(n-1) - 2n(n-1) - 6}{2} = 1. \end{aligned}$$

So f is an LDA-labeling for $C_3 \vee H$. Thus $|w_f(C_3 \vee H)| = |w_f(V(C_3))| + |w_f(V(H))| = |w_g(V(C_3))| + |w_h(V(H))| = 3 + \chi_{lda}(H)$. Hence we have the theorem. □

Corollary 2.15. *Let $H_1 = C_3$ and $H_k = H_{k-1} \vee C_5$ for $k \geq 2$. Then $\chi_{lda}(H_k) = 5k - 2$ for $k \geq 1$.*

Proof. Clearly, the corollary hold for $k = 1$. So we assume that $k \geq 2$. Note that, $H_k = C_3 \vee G_{k-1}$, where G_{k-1} is defined in Corollary 2.13 which is a $(5k-8)$-regular graph. It is known that $\chi_{lda}(G_{k-1}) = 5(k-1)$. By Theorem 2.14, we have $\chi_{lda}(H_k) \leq 3 + 5(k-1) = 5k - 2$.

Now $|V(H_k)| = 5k - 2$. For $k \geq 2$, let $V(H_k) = \{v_i \mid 1 \leq i \leq 3\} \cup \{u_{a,j} \mid 2 \leq a \leq k, 1 \leq j \leq 5\}$. Note that

(1) each v_i is adjacent to all other vertices for $1 \leq i \leq 3$;
(2) u_{a,j_1} is adjacent to u_{b,j_2} for $2 \leq a < b \leq k$, $1 \leq j_1, j_2 \leq 5$;
(3) $\deg(u_{a,j}) = 5k - 5$;
(4) if u_{a,j_1} and u_{a,j_2} are not adjacent in H_k, then $|N(u_{a,j_1}) \cap N(u_{a,j_2})| = 5k - 6$.

Together with Lemma 2.1, we conclude that all vertices of H_k are of distinct weights under any LDA-labeling. Thus, $\chi_{lda}(H_k) = 5k - 2$. □

Corollary 2.16. *For $n \geq 1$, $\chi_{lda}(C_5 \vee O_n) = 6$, where O_n is the null graph of order n.*

Proof. Let g be the LDA-labeling of C_5 defined in the proof of Theorem 2.3. So $w_g(u) \geq 4$ for any $u \in V(C_5)$. In the proof of Theorem 2.12 if we allow h is not an LDA-labeling but is a bijection between $V(O_n) \to [1, n]$ and define $w_h(v) = 0$ for all $v \in V(O_n)$ by convention, then f is still a bijection from $V(C_5 \vee O_n)$, where f is the labeling defined in (2.1). By the same argument of the proof of Theorem 2.12 we only need to check the following difference $w_f(u) - w_f(v) \neq 0$ for $u \in V(C_5)$ and $v \in V(O_n)$

$$\begin{aligned} w_f(u) - w_f(v) &= w_g(u) + 5n + \frac{1}{2}n(n+1) - 15 \\ &\geq \frac{n^2 + 11n}{2} - 11 \\ &= \frac{(n-2)(n+13)+4}{2}. \end{aligned}$$

Hence, $w_f(u) - w_f(v) > 0$ when $n \geq 2$.

When $n = 1$, $w_f(u) - w_f(v) = w_g(u) + 5 + 1 - 15 = w_g(u) - 9 < 0$, since $w_g(u) \in \{4, 5, 6, 7, 8\}$

For any LDA-labeling of $O_n \vee C_5$, by Lemma 2.1 and the requirement of LDA-labeling, all weights of vertices in C_5 are distinct. Thus $\chi_{lda}(O_n \vee C_5) \geq 6$. Consequently, $\chi_{lda}(C_5 \vee O_n) = 6$. □

Corollary 2.17. *For $n \geq 1$, $\chi_{lda}(C_3 \vee O_n) = 4$, where O_n is the null graph of order n.*

Proof. By a similar argument of the proof of Corollary 2.16, we get that

$$\begin{aligned} w_f(u) - w_f(v) &= w_g(u) + 3n + \frac{1}{2}n(n+1) - 6 \\ &\geq \frac{n^2 + 7n}{2} - 3 \\ &= \frac{(n-1)(n+8)+2}{2} \\ &> 0. \end{aligned}$$

Since $O_n \vee C_3$ contains K_4, $\chi_{lda}(O_n \vee C_3) \geq 4$ and hence $\chi_{lda}(O_n \vee C_3) = 4$. □

3. Path related graphs

Theorem 3.1.

$$\chi_{lda}(P_n) = \begin{cases} 2, & n \in \{2, 3\}; \\ 3, & n \in \{5, 11\}; \\ 4, & n \in \{4, 6, 7, 8, 9, 10\}, \end{cases}$$

$$4 \leq \chi_{lda}(P_n) \leq \begin{cases} 5, & n \geq 12, \ n \text{ is even}; \\ 6, & n \geq 13, \ n \text{ is odd}. \end{cases}$$

Proof. $\chi_{lda}(P_2) = \chi_{lda}(P_3) = 2$ is clear. Now we assume that $n \geq 4$ and let $P_n = v_1 v_2 \cdots v_n$. Suppose f is an LDA-labeling for P_n. Now $w_f(v_1) = f(v_2)$ implies that $w_f(v_3) = f(v_2) + f(v_4) > w_f(v_1)$. Since $w_f(v_2) \neq w_f(v_1)$ and $w_f(v_2) \neq w_f(v_3)$, $\chi_{lda}(P_n) \geq 3$.

Now, suppose f is a 3-LDA-labeling for P_n, $n \geq 4$. Let $w_f(v_1) = a$, $w_f(v_2) = b$ and $w_f(v_3) = c$.

Suppose $n = 3k + 1$, $k \geq 1$. Now, we have $w_f(v_{3k+1}) = f(v_{3k})$. Since $f(v_2) = a$, $f(v_{3k}) \neq a$, i.e., $w_f(v_{3k+1}) \neq a$. Since $w_f(v_{3l}) = c$, $w_f(v_{3k+1}) \neq c$. Then $w_f(v_{3k+1}) = b$. This implies $f(v_{3k}) = b$. However, $b = w_f(v_{3k-1}) = f(v_{3k-2}) + f(v_{3k})$ which is impossible. Thus $\chi_{lda}(P_{3k+1}) \geq 4$.

Suppose $n = 3k$, $k \geq 2$. Now, we have $w_f(v_{3l-2}) = a$, $w_f(v_{3l-1}) = b$ and $w_f(v_{3l}) = c$, where $1 \leq l \leq k$. We have $c = w_f(v_3) = f(v_2) + f(v_4) = a + f(v_4)$ and $a = w_f(v_{3k-2}) = f(v_{3k-3}) + f(v_{3k-1}) = f(v_{3k-3}) + w_f(v_{3k}) = f(v_{3k-3}) + c$. Thus, $f(v_4) + f(v_{3k-3}) = 0$ which is impossible. Thus $\chi_{lda}(P_{3k}) \geq 4$.

Note that the 2-component of P_n is isomorphic to P_{n-2}. By Lemma 2.2, $\chi_{lda}(P_n) \geq 4$ when $n - 2 \geq 12$.

Following we determine the value of $\chi_{lda}(P_n)$ for some $n \geq 4$.

- Clearly, $\chi_{lda}(P_4) = 4$.
- We label the vertices of P_5 by 2, 4, 3, 5, 1. Then the induced vertex weights are 4, 5, 9, 4, 5, in natural order. Hence $\chi_{lda}(P_5) = 3$.
- We label the vertices of P_6 by 6, 4, 1, 2, 3, 5. Then the induced vertex weights are 4, 7, 6, 4, 7, 3 in natural order. Hence $\chi_{lda}(P_6) = 4$.
- For P_7, a 4-LDA-labeling has vertex labels: 7, 5, 2, 3, 4, 6, 1 and the weights are 5, 9, 8, 6, 9, 5, 6. Hence $\chi_{lda}(P_7) = 4$.
- Suppose there is a 3-LDA-labeling for P_8, by the same proof of Lemma 2.2, we have $w_f(v_1) = a$, $w_f(v_2) = b$, $w_f(v_3) = c$, $w_f(v_4) = a$, $w_f(v_5) = b$, $w_f(v_6) = c$, $w_f(v_7) = a$. Now $f(v_7) + f(v_5) = w_f(v_6) = c$ and $w_f(v_8) = f(v_7) < c$. Since $w_f(v_7) = a$, $w_f(v_8) = b$.

  Suppose $f(v_1) = x$. This forces $f(v_2) = a$, $f(v_3) = b - x$, $f(v_4) = c - a$, $f(v_5) = a - b + x$, $f(v_6) = b - c + a$, $f(v_7) = c - a + b - x$. Since $f(v_7) = w_f(v_8) = b$, $c - a = x$. Now $f(v_1) = f(v_4)$ which is a contradiction. Hence $\chi_{lda}(P_8) \geq 4$. A 4-LDA-labeling has vertex labels 8, 7, 2, 1, 3, 6, 5, 4 with weights are 7, 10, 8, 5, 7, 8, 10, 5. Hence $\chi_{lda}(P_8) = 4$.
- For P_9, a 4-LDA-labeling has vertex labels 9, 8, 3, 2, 5, 4, 7, 6, 1 with weights are 8, 12, 10, 8, 6, 12, 10, 8, 6. Hence $\chi_{lda}(P_9) = 4$.
- For P_{10}, a 4-LDA-labeling has vertex labels 4, 8, 7, 1, 6, 10, 2, 3, 9, 5 with weights are 8, 11, 9, 13, 11, 8, 13, 11, 8, 9. Hence $\chi_{lda}(P_{10}) = 4$.
- For P_{11}, a 3-LDA-labeling has vertex labels 5, 10, 6, 2, 4, 9, 8, 1, 3, 11, 7 with weights are 10, 11, 12, 10, 11, 12, 10, 11, 12, 10, 11. Hence $\chi_{lda}(P_{11}) = 3$.

Now let us find the upper bound of $\chi_{lda}(P_n)$ for $n \geq 12$. Following we assume that $k \geq 3$.

(1) $n = 4k$. Define $g : V(P_{4k}) \to [1, 4k]$ by

$$g(u_{4i-3}) = 4k + 1 - i, g(u_{4i-2}) = 2k + 1 - i, g(u_{4i-1}) = 2k + i, g(u_{4i}) = i, \text{ for } 1 \leq i \leq k.$$

Then $w_g(u_{4i-3}) = 2k$; $w_g(u_{4i-2}) = 6k + 1$; $w_g(u_{4i-1}) = 2k + 1$, for $1 \leq i \leq k$; $w_g(u_{4i}) = 6k$ for $1 \leq i \leq k - 1$ and $w_g(u_{4k}) = 3k$. Clearly, there are 5 distinct vertex weights. Thus $\chi_{lda}(P_{4k}) \leq 5$.

(2) $n = 4k+1$. Extending the labeling defined in Case (1) and define $g(u_{4k+1}) = 4k+1$. Then $w_g(u_l)$ are the same as in Case (1) when $1 \leq l \leq 4k-1$, $w_g(u_{4k}) = 7k+1$ and $w_g(u_{4k+1}) = k$. Clearly, there are 6 distinct vertex weights. Thus $\chi_{lda}(P_{4k+1}) \leq 6$.

(3) $n = 4k + 2$. Define $g : V(P_{4k+2}) \to [1, 4k + 2]$ by

$$g(u_{4i-3}) = 4k + 3 - i, g(u_{4i-2}) = 2k + 2 - i, g(u_{4i-1}) = 2k + 1 + i, g(u_{4i}) = i,$$
$$\text{for } 1 \leq i \leq k \text{and } g(u_{4k+1}) = 3k + 2, g(u_{4k+2}) = k + 1.$$

Then $w_g(u_{4i-3}) = 2k+1$ for $1 \leq i \leq k+1$; $w_g(u_{4i-2}) = 6k+4$; $w_g(u_{4i-1}) = 2k+2$, for $1 \leq i \leq k$; $w_g(u_{4i}) = 6k + 3$ for $1 \leq i \leq k$ and $w_g(u_{4k+2}) = 3k + 2$. Clearly, there are 5 distinct vertex weights. Thus $\chi_{lda}(P_{4k+2}) \leq 5$.

(4) $n = 4k+3$. Extending the labeling defined in Case (3) and define $g(u_{4k+3}) = 4k+3$. Then $w_g(u_l)$ are the same as in Case (1) when $1 \leq l \leq 4k + 1$, $w_g(u_{4k+2}) = 7k + 5$ and $w_g(u_{4k+3}) = k + 1$. Clearly, there are 6 distinct vertex weights. Thus $\chi_{lda}(P_{4k+3}) \leq 6$.

So we have the theorem. □

Problem 3.2. Determine $\chi_{lda}(P_n)$ for $n \geq 12$.

Theorem 3.3. *$\chi_{lda}(2P_3) = 3$ and $\chi_{lda}(mP_3) = m$ for $m \geq 3$.*

Proof. Let the i-th copy of P_3 be $u_i x_i v_i$, $1 \leq i \leq m$. Let h be an LDA-labeling of mP_3. Since $h(x_i)$ are distinct and $h(x_i) = w_h(u_i) = w_h(v_i)$ for $1 \leq i \leq m$, $\chi_{lda}(mP_3) \geq m$.

When $m = 2$. Suppose h is a 2-LDA-labeling of $2P_3$. Since $w_h(v_1) \neq w_h(v_2)$, $w_h(x_1) = w_h(v_2)$ and $w_h(x_2) = w_h(v_1)$. Thus, $h(u_1) + h(v_1) = h(x_2)$ and $h(u_2) + h(v_2) = h(x_1)$. So $21 = \sum_{j=1}^{6} j = h(u_1) + h(v_1) + h(u_2) + h(v_2) + h(x_1) + h(x_2) = 2(h(x_1) + h(x_2))$ which is impossible. Consequently, $\chi_{lda}(2P_3) \geq 3$. Now, define $g(u_1) = 1$, $g(v_1) = 5$, $g(x_1) = 4$, $g(u_2) = 2$, $g(v_2) = 3$ and $g(x_2) = 6$. It is easy to verify that g is an LDA-labeling that induces 3 distinct vertex weights $4, 5, 6$. Therefore, $\chi_{lda}(2P_3) = 3$.

Consider $m \geq 3$. Define a bijection g such that $g(u_i) = i$ for $1 \leq i \leq m$, $g(x_1) = 2m+1$, $g(x_i) = 3m + 2 - i$ for $2 \leq i \leq m-1$, $g(x_m) = 2m$, $g(v_1) = 2m+2$, $g(v_i) = 2m+1-i$ for $2 \leq i \leq m$. Clearly, $w_g(x_1) = 2m+3$, $w_g(x_i) = 2m+1$ for $2 \leq i \leq m$, $w_g(u_i) = w_g(v_i) = g(x_i)$ for $1 \leq i \leq m$. Thus, g is an m-LDA-labeling of mP_3. This completes the proof. □

Theorem 3.4. *For $n \geq 1$, $\chi_{lda}(P_3 \vee O_n) = 3$, and $\chi_{lda}(mP_3 \vee O_n) = m+1$ for $m \geq 2$.*

Proof. Let mP_3 be defined as in the proof of Theorem 3.3. Let $V(O_n) = \{y_j \mid 1 \leq j \leq n\}$. Denote $G = mP_3 \vee O_n$. Observe that for $1 \leq i, j \leq m$, every two vertices u_i, v_j have the same degree $n+1$ such that $N(u_i) = N(v_i)$. Thus, $w(u_i) = w(v_i)$, $1 \leq i \leq m$. Moreover, $|N(u_a) \cap N(u_b)| = n$ for $1 \leq a < b \leq m$. Therefore, there are at least m distinct vertex weights contributed by $u_i, v_i, 1 \leq i \leq m$. Since each y_k, $1 \leq k \leq n$, is adjacent to all u_i, v_i, it follows that $w(y_k) \neq w(u_i), w(v_i), w(x_j)$ for $1 \leq i \leq m$ and $1 \leq j \leq n$. Consequently, $\chi_{lda}(G) \geq m+1$.

Suppose $m = 1$. Since $P_3 \vee O_n$ contains a 3-cycle, $\chi_{lda}(P_3 \vee O_n) \geq 3$. Suppose $n = 1$. We let $f(u_1) = 2$, $f(x_1) = 3$, $f(v_1) = 4$ and $f(y_1) = 1$. Then $w_f(u_1) = w_f(v_1) = 4$, $w_f(x_1) = 7$ and $w_f(y_1) = 9$. Thus $\chi_{lda}(P_3 \vee O_1) = 3$. Suppose $n \geq 2$. We let $f(u_1) = 1$, $f(x_1) = 2$, $f(v_1) = 3$ and $f(y_j) = j + 3$, $1 \leq j \leq n$. Then $w_f(u_1) = w_f(v_1) = 2 + 3n + n(n+1)/2$, $w_f(x_1) = 4 + 3n + n(n+1)/2$ and $w_f(y_j) = 6$. Clearly, all three weights are distinct and hence $\chi_{lda}(P_3 \vee O_n) = 3$ too for $n \geq 2$.

Suppose $m = 2$. We let $f(u_1) = 2$, $f(x_1) = 4$, $f(v_1) = 5$, $f(u_2) = 1$, $f(x_2) = 7$, $f(v_2) = 3$, $f(y_1) = 6$. Moreover, if $n \geq 2$ we let $f(y_j) = 6 + j$ for $2 \leq j \leq n$. Then $w_f(u_1) = w_f(v_1) = w_f(x_2) = 7 + (n^2 + 13n - 2)/2$, $w_f(u_2) = w_f(v_2) = w_f(x_1) = 4 + (n^2 + 13n - 2)/2$ and $w_f(y_j) = 22$. By using the discriminant of quadratic equation, one may check that all three weights are distinct. Hence $\chi_{lda}(2P_3 \vee O_n) = 3$.

Now, we assume that $m \geq 3$ and $n \geq 1$. Let g be the LDA-labeling of mP_3 defined in the proof of Theorem 3.3. By the same argument in the proof of Corollary 2.16, we may allow the bijection h being not an LDA-labeling. Let f be the labeling defined in (2.1). Again, we only need to consider the difference $w_f(u) - w_f(y_j)$ for each $u \in V(mP_3)$ and $y_j \in V(O_n)$. From (2.2) and (2.3) we have

(1) $w_f(u_i) = w_g(u_i) + 3mn + n(n+1)/2$,
$w_f(v_i) = w_g(v_i) + 3mn + n(n+1)/2 = w_f(u_i)$,
(2) $w_f(x_i) = w_g(x_i) + 3mn + n(n+1)/2$, and
(3) $w_f(y_j) = 3m(3m+1)/2$.

Note that, from the proof of Theorem 3.3, the set of all weights induced by g is $[2m, 3m] \setminus \{2m+2\}$ when $m \geq 3$. Thus $3m + 3mn + n(n+1)/2 \geq w_f(u) \geq 2m + 3mn + n(n+1)/2$ for $u \in V(mP_3)$. Thus

$$w_f(u) - w_f(y_j) \geq \frac{n^2 + n + m + 6mn - 9m^2}{2} \geq \frac{m^2 + 36m}{32} > 0,$$

when $4n \geq 5m$.

Now we consider $4n < 5m$. We define $f : V(mP_3 \vee O_n) \to [1, 3m+n]$ by $f(u_i) = i$ for $1 \leq i \leq m$; $f(x_1) = 2m+n+1$; $f(x_i) = 3m+n+2-i$ for $2 \leq i \leq m-1$; $f(x_m) = 2m+n$; $f(v_1) = 2m+n+2$; $f(v_i) = 2m+n+1-i$ for $2 \leq i \leq m$; $f(y_j) = m+j$ for $1 \leq j \leq n$. Then $w_f(x_1) = 2m+n+3+\frac{(2m+n+1)n}{2}$ and $w_f(x_i) = 2m+n+1+\frac{(2m+n+1)n}{2}$ for $2 \leq i \leq m$; $w_f(u_1) = w_f(v_1) = 2m+n+1+\frac{(2m+n+1)n}{2}$; $w_f(u_i) = w_f(v_i) = 3m+n+2-i+\frac{(2m+n+1)n}{2}$ for $2 \leq i \leq m$; $w_f(y_j) = \frac{m^2+m}{2} + m(4m+2n+1)$ for $1 \leq j \leq n$. For $u \in V(mP_3)$,

$$\begin{aligned} w_f(y_j) - w_f(u) &\geq \frac{m^2+m}{2} + m(4m+2n+1) - \left(3m+n+\frac{(2m+n+1)n}{2}\right) \\ &= \frac{9m^2 - 3m - 3n + 2mn - n^2}{2} \\ &> \frac{9m^2 - 3m - 3(\frac{5m}{4}) + 2mn - (\frac{5m}{4})^2}{2} \\ &= \frac{119m^2 - 108m + 32mn}{32} \\ &> 0. \end{aligned}$$

Combining the above cases, we have an $(m+1)$-LDA-labeling for $mP_3 \vee O_n$. This completes the proof. □

Theorem 3.5. *For $m, n \geq 1$, $\chi_{lda}(mP_2 \vee O_n) = 2m+1$.*

Proof. Note that $P_2 \vee O_n = K_{1,1,n}$. It is easy to verify that $\chi_{lda}(K_{1,1,n}) = 3$.

Consider $m \geq 2$. Let the i-th copy of P_2 be $u_i v_i$, $1 \leq i \leq m$. And let $G = mP_2 \vee O_n$, where $V(O_n) = \{y_j \mid 1 \leq j \leq n\}$. Observe that $\deg(u_i) = \deg(v_i) = n+1$ for $1 \leq i \leq n$. Moreover, every two vertices in $\{u_i, v_i \mid 1 \leq i \leq m\}$ have exactly n common neighbors. By Lemma 2.1, we conclude that no two vertices in $\{u_i, v_i \mid 1 \leq i \leq m\}$ have the same weights under any LDA-labeling of G. Since every vertex in $\{u_i, v_i \mid 1 \leq i \leq m\}$ is adjacent to each y_j, $1 \leq j \leq n$, we immediately have $\chi_{lda}(G) \geq 2m+1$.

Suppose $m \leq n$. Define a bijection $f : V(G) \to [1, 2m+n]$ such that $f(u_i) = 2i-1$, $f(v_i) = 2i$ and $f(y_j) = 2m+j$ for $1 \leq i \leq m, 1 \leq j \leq n$. Now, $w_f(u_i) = 2i + n(4m+n+1)/2$, $w_f(v_k) = 2k-1+n(4m+n+1)/2$, and $w_f(y_j) = m(2m+1)$ for $1 \leq i, k \leq m$ and $1 \leq j \leq n$. Clearly, $w_f(u_i)$, $w_f(v_k)$ and $w_f(y_j)$ are distinct for $1 \leq i, k \leq m$ and $1 \leq j \leq n$. Thus, f is a $(2m+1)$-LDA-labeling for G. Hence $\chi_{lda}(G) = 2m+1$.

Suppose $m \geq n+1$. Define a bijection $f : V(G) \to [1, 2m+n]$ such that $f(y_j) = j$ for $1 \leq j \leq n$, $f(u_i) = n+2i-1$ and $f(v_i) = n+2i$ for $1 \leq i \leq m$. Now, $w_f(u_i) = 2i+n+n(n+1)/2 \neq w_f(v_k) = 2k-1+n+n(n+1)/2$ for $1 \leq i, k \leq m$ and $w_f(y_j) = m(2m+2n+1)$. We have

$$\begin{aligned} w_f(y_j) - w_f(v_i) > w_f(y_j) - w_f(u_i) &= 2m^2 + 2mn + m - n - 2i - n(n+1)/2 \\ &\geq 2m^2 + 2mn + m - n - 2m - n(n+1)/2 = \frac{4m^2 + 4mn - 2m - n^2 - 3n}{2} \\ &= \frac{(2m-n)(2m+n) + n(2m-3) + 2m(n-1)}{2} > 0. \end{aligned}$$

Therefore, $w_f(u_i)$, $w_f(v_k)$ and $w_f(y_j)$ are distinct for $1 \leq i, k \leq m, 1 \leq j \leq n$. Thus, f is a $(2m+1)$-LDA-labeling for G. Hence $\chi_{lda}(G) = 2m+1$. □

4. Conclusion

The following problems arise naturally.

Problem 4.1. Characterize G such that $\chi_{lda}(G) = |V(G)|$.

Problem 4.2. Find necessary and/or sufficient condition(s) such that $G + mP_2$, the disjoint union of G and mP_2, with $\chi_{lda}(G + mP_2) = 2m$ for $m \geq 1$.

5. Appendix

Suppose there is a 4-LDA-labeling f for $C_7 = u_1u_2 \cdots u_7u_1$. For convenience, let $u_8 = u_1$ and $u_7 = u_0$. Let a, b, c, d be the induced weights. By pigeonhole principle, without loss of generality, the distribution of the weights is $w_f(u_1) = d$, $w_f(u_2) = a$, $w_f(u_3) = b$, $w_f(u_4) = c$, $w_f(u_5) = a$, $w_f(u_6) = b$, $w_f(u_7) = c$. By solving the linear system of equation $f(u_{i-1}) + f(u_{i+1}) = w_f(u_i)$, $1 \leq i \leq 7$, and $2a + 2b + 2c + d = 56$, we have

$$\left.\begin{array}{rcl} f(u_3) &=& -2b - c + 28, \\ f(u_5) &=& 2b + 2c - 28, \\ f(u_7) &=& -b - 2c + 28, \end{array}\right\} \tag{5.1}$$

$$\left.\begin{array}{rcl} f(u_2) &=& -2a - b + 28, \\ f(u_4) &=& 2a + 2b - 28, \\ f(u_6) &=& -a - 2b + 28, \end{array}\right\} \tag{5.2}$$

$$f(u_1) = a + 2b + c - 28. \tag{5.3}$$

From (5.1), since $3 \leq f(u_3) + f(u_7) = -3(b + c) + 56$, $b + c \leq 17$. Since $f(u_5)$ is even, $f(u_5) = 2(b + c) - 28 \leq 6$. Similarly, from (5.2), we have $a + b \leq 17$, $f(u_4)$ is even and $f(u_4) \leq 6$.

(a) Consider $f(u_5) = 6$. From (5.1), $b + c = 17$ and hence $f(u_3) + f(u_7) = 5$.

(a-1) Consider $f(u_4) = 4$. From (5.2), $a + b = 16$ and hence $f(u_2) + f(u_6) = 8$. From (5.3), $f(u_1) = 5$. It forces that $\{f(u_2), f(u_6)\} = \{1, 7\}$.
If $f(u_2) = 1$ and $f(u_6) = 7$, then $b = f(u_2) + f(u_4) = 5$. This implies, $12 = c = w_f(u_4) = f(u_5) + f(u_3)$ and hence $f(u_3) = 6$ which is impossible.
If $f(u_2) = 7$ and $f(u_6) = 1$, then $b = f(u_2) + f(u_4) = 11$. This implies, $5 = a = w_f(u_2) = f(u_1) + f(u_3)$ which is impossible.

(a-2) Consider $f(u_4) = 2$. From (5.2), $a + b = 15$ and hence $f(u_2) + f(u_6) = 11$. From (5.3), $f(u_1) = 4$. Now there is no solution for $f(u_2), f(u_6) \in \{1, 3, 5, 7\}$ and satisfies $f(u_2) + f(u_6) = 11$.

(b) Consider $f(u_5) = 4$. From (5.1), $b + c = 16$. It forces $f(u_3) + f(u_7) = 8$.

(b-1) Consider $f(u_4) = 6$. From (5.2), $a + b = 17$ and hence $f(u_2) + f(u_6) = 5$. From (5.3), $f(u_1) = 5$. It forces that $\{f(u_3), f(u_7)\} = \{1, 7\}$.
If $f(u_3) = 1$ and $f(u_7) = 7$, then $c = f(u_3) + f(u_5) = 5$. This implies, $11 = b = w_f(u_3) = f(u_2) + f(u_4)$ and hence $f(u_2) = 5$ which is impossible.

(b-2) Consider $f(u_4) = 2$. From (5.2), $a + b = 15$ and hence $f(u_2) + f(u_6) = 11$. From (5.3), $f(u_1) = 3$. Thus $\{f(u_2), f(u_6)\} = \{5, 6\}$.
If $f(u_2) = 5$ and $f(u_6) = 6$, then $b = f(u_2) + f(u_4) = 7$. Now $9 = c = w_f(u_4) = f(u_3) + f(u_5)$ and hence $f(u_3) = 5$ which is impossible.
If $f(u_2) = 6$, then $b = f(u_2) + f(u_4) = 8$. Now $c = 8$ which is a contradiction.

(c) Consider $f(u_5) = 2$. Then $b + c = 15$. It forces $f(u_3) + f(u_7) = 11$.

(c-1) Consider $f(u_4) = 6$. From (5.2), $a + b = 17$ and hence $f(u_2) + f(u_6) = 5$. From (5.3), $f(u_1) = 4$. Since labels 2 and 4 are occupied, there is no solution for $f(u_2) + f(u_6) = 5$.

(c-2) Consider $f(u_4) = 4$. From (5.2), $a + b = 16$ and hence $f(u_2) + f(u_6) = 8$. From (5.3), $f(u_1) = 3$. It forces that $\{f(u_2), f(u_6)\} = \{1, 7\}$.
If $f(u_2) = 1$ and $f(u_6) = 7$, then $b = f(u_2) + f(u_4) = 5$. Now $10 = c = w_f(u_4) = f(u_3) + f(u_5)$ and hence $f(u_3) = 8$ which is impossible.

If $f(u_2) = 7$ and $f(u_6) = 1$, then $b = f(u_2) + f(u_4) = 11$. Now $4 = c = w_f(u_4) = f(u_3) + f(u_5)$ and hence $f(u_3) = 2$ which is impossible.

Thus C_7 does not admit a 4-LDA-labeling.

Suppose there is a 4-LDA-labeling f for $C_9 = u_1u_2 \cdots u_9$. For convenience, we let $u_0 = u_9$ and $u_1 = u_{10}$. Let a, b, c, d be the induced weights. So at least one weight appears 3 times. Thus, the distribution of the weights is either 3-2-2-2 or 3-3-2-1.

(a) Consider 3-2-2-2 cases: Without loss of generality, we assume a appears 3 times. Then we have $3a+2b+2c+2d = 2\sum_{i=1}^{9} i = 90$. Hence $b+c+d \equiv 0 \pmod 3$ and $a \equiv 0 \pmod 2$. The distribution of the weights of $u_1, u_2, \ldots, u_9$ is *abcabdacd*, *abcadbadc*, *abcadbacd* or *abcadcabd*. The last case, if we start to read the distribution at u_7, then the distribution is *abdabcadc*. It is isomorphic to the first case (by swapping the symbols c and d).

Since $f(u_9) + f(u_2) = f(u_3) + f(u_5) = f(u_6) + f(u_8) = a$, $21 \le 3a \le 39$, i.e., $a \in \{8, 10, 12\}$. Similarly, we have $5 \le b, c, d \le 15$.

(i) Consider *abcabdacd* case. That means $w_f(u_1) = a$, $w_f(u_2) = b$, $\ldots$, $w_f(u_9) = d$. Suppose $f(u_9) = x$. We use w instead of w_f. Now $f(u_2) = w(u_1) - f(u_9)$, $f(u_4) = w(u_3) - f(u_2) = w(u_3) - w(u_1) + f(u_9)$. By a similar procedure, we have $f(u_9) = w(u_8) - w(u_6) + w(u_4) - w(u_2) + w(u_9) - w(u_7) + w(u_5) - w(u_3) + w(u_1) - f(u_9)$. Then we have $f(u_9) = a - x$. Hence we get $f(u_2) = a - x = f(u_9)$ which is impossible.

(ii) Consider *abcadbadc* case. Suppose $f(u_9) = x$. By the same procedure showed in Subcase (i) will get $f(u_3) = w(u_2) - w(u_9) + w(u_7) - w(u_5) + w(u_3) - w(u_1) + f(u_9) = b - d + x$, $f(u_5) = w(u_4) - w(u_2) + w(u_9) - w(u_7) + w(u_5) - w(u_3) + w(u_1) - f(u_9) = a - b + d - x$ and $x = f(u_9) = 2d - 2b + a - x$. But the last equality implies that $a - b + d - x = b - d + x$ which is impossible.

(iii) Consider *abcadbacd* case. By solving the linear equations $f(u_{i-1}) + f(u_{i+1}) = w_f(u_i)$ $1 \le i \le 9$ and $3a + 2b + 2c + 2d = 90$ we have

$$\begin{aligned}
&f(u_1) = -2a - 2c + 45 && f(u_2) = -a - c - 2d + 45 \\
&f(u_3) = \frac{a}{2} + c - d && f(u_4) = a + 2c + 2d - 45 \\
&f(u_5) = \frac{a}{2} - c + d && f(u_6) = -a - 2c - d + 45 \\
&f(u_7) = -2a - 2d + 45 && f(u_8) = 2a + 2c + d - 45 \\
&f(u_9) = 2a + c + 2d - 45 &&
\end{aligned} \tag{5.4}$$

Thus, $f(u_1)$, $f(u_4)$ and $f(u_7)$ are odd.

(iii-1) Consider $a = 8, 12$.

If both c and d are even, then $f(u_2)$, $f(u_6)$, $f(u_8)$ and $f(u_9)$ are odd. Thus, it is impossible.

If both c and d are odd, then all other labels are even which is impossible.

If c is odd and d is even, then $f(u_3)$, $f(u_5)$ and $f(u_6)$ are odd. Thus, it is impossible.

If c is even and d is odd, then $f(u_2)$, $f(u_3)$ and $f(u_5)$ are odd. Thus, it is impossible.

(iii-2) Consider $a = 10$. Then $b+c+d = 30$. Now, from (5.4), $f(u_4)+f(u_8)+f(u_9) = 5(a + c + d) - 135$. Since $6 \le f(u_4) + f(u_8) + f(u_9) \le 24$, $19 \le c + d \le 21$. Since $1 \le f(u_1) \le 9$, $8 \le c \le 12$. Similarly, $1 \le f(u_7) \le 9$, we have $8 \le d \le 12$.

If $c \equiv d \pmod 2$ are even, then $c + d = 20$ and hence $b = 10$ which is impossible.

If $c \not\equiv d \pmod 2$, then $c+d = 19$ or 21. Hence $b = 11$ or 9, respectively. Suppose $c+d = 19$ and $b = 11$. Since $8 \le c, d \le 12$, there is no solution with b, c, d are distinct. Suppose $c + d = 21$ and $b = 9$. Again, there is no solution with b, c, d are distinct.

(b) Consider 3-3-2-1 cases: Without loss of generality, we assume a and b appear 3 times and c appears twice. Since the distance between two same weights is at least 3, without loss of generality, we may assume that $w_f(u_1) = w_f(u_4) = w_f(u_7) = a$. Now either $w_f(u_9) = b$ or $w_f(u_2) = b$. By symmetry, we may assume that $w_f(u_2) = b$. Then $w_f(u_5) = w_f(u_8) = b$. Thus the distribution of weights is *abcabcabd*, *abcabdabc* or *abdabcabc*. These three distributions are isomorphic. Thus, there is only one case need to be deal with, which is $w_f(u_1) = w_f(u_4) = w_f(u_7) = a$, $w_f(u_2) = w_f(u_5) = w_f(u_8) = b$, $w_f(u_3) = w_f(u_6) = c$ and $w_f(u_9) = d$. Since

$$f(u_3) + f(u_5) = a = f(u_2) + f(u_9),$$
$$f(u_7) + f(u_9) = b = f(u_4) + f(u_6),$$
$$f(u_2) + f(u_4) = c = f(u_5) + f(u_7),$$

we have $f(u_3) = f(u_6)$, which is impossible.

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