Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 53 (6) (2024), 1712 – 1714
DOI : 10.15672/hujms.1269057

Erratum

Discussion on (k, s)-Riemann Liouville fractional integral and applications

Bouharket Benaissa [1], Mehmet Zeki Sarikaya*[2]

[1] *Faculty of Material Sciences, Laboratory informatics and Mathematics, University of Tiaret. Algeria*
[2] *Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey*

Abstract

In this paper we present the correct version of Theorem 2.2 in [$(k; s)$-Riemann-Liouville fractional integral and applications, Hacet. J. Math. Stat. **45** (1), 77 - 89, 2016] and prove it.

Mathematics Subject Classification (2020). 26A33, 26A51, 26D15

Keywords. Riemann-Liouville fractional integrals, synchronous function, Tchebychev inequality, Hölder inequality

1. Introduction

In 2011, U.N. Katugompola [1] present a new fractional integration, which generalizes the Riemann - Liouville and Hadamard fractional integrals into a single form.

$$ {}^{s}\mathrm{J}_{a+}^{\alpha} f(x) = \frac{(s+1)^{1-\alpha}}{\Gamma(\alpha)} \int_a^x (x^{s+1} - t^{s+1})^{\alpha-1} t^s f(t)dt, \quad a < x \le b, \tag{1.1} $$

$$ {}^{s}\mathrm{J}_{b-}^{\alpha} f(x) = \frac{(s+1)^{1-\alpha}}{\Gamma(\alpha)} \int_x^b (t^{s+1} - x^{s+1})^{\alpha-1} t^s f(t)dt, \quad a \le x < b, \tag{1.2} $$

where $\alpha > 0$ and $s \neq -1$.
In 2016, the authors [3] introduce a new approach on fractional integration, which generalizes the Riemann-Liouville fractional integral.

$$ {}^{s}_{k}\mathrm{J}_{a+}^{\alpha} f(x) = \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\,\Gamma_k(\alpha)} \int_a^x (x^{s+1} - t^{s+1})^{\frac{\alpha}{k}-1} t^s f(t)dt, \quad a < x \le b, $$

where k, $\alpha > 0$ and $s \in \mathbb{R} - \{-1\}$.
And they give the following theorem.

Theorem 1.1 ([3], Theorem 2.2)**.** *Let $f \in L_1[a,b]$, $s \in \mathbb{R}\{-1\}$ and $k > 0$, then ${}^{s}_{k}\mathrm{J}_{a+}^{\alpha} f(x)$ and exist for any $x \in [a,b]$, $\alpha > 0$.*

---

*Corresponding Author.
Email addresses: bouharket.benaissa@univ.tiaret.dz (B. Benaissa), sarikayamz@gmail.com (M.Z. Sarikaya)
Received: 28.03.2023; Accepted: 30.12.2023

However , in 2014, U.N. Katugompola [2] gave a new approach to the generalized fractional by using the condition $s>-1$, that it is necessary for the existence of the operators ${}^{s}\mathrm{J}_{a+}^{\alpha}f(x)$ and ${}^{s}\mathrm{J}_{b-}^{\alpha}f(x)$.

1.1. Remarks on the proof of the Theorem 2.2 in [3]

(1) $P_{+}(x;t)$ exist for $\alpha>0$, $k>0$ and $s>-1$ because the base $(x^{s+1}-t^{s+1})$ is positive, however $P_{+}(x;t)$ does not exist for $s<-1$, because the base $(x^{s+1}-t^{s+1})$ is negative .
$P_{-}(x;t)$ exist for $\alpha>0$, $k>0$ and $s<-1$ because the base $(t^{s+1}-x^{s+1})$ is substantial research positive, however $P_{-}(x;t)$ does not exist for $s>-1$ because the basis $(t^{s+1}-x^{s+1})$ is negative.
We concluded that the $P(x;t)=P_{+}(x;t)+P_{-}(x;t)$ does not exist for $s\neq-1$.

(2) In the last step in the proof of Theorem 2.2, the authors said :
Hence, by Fubini's theorem $\int_a^b P(x,t)f(x)dx$ is an integrable function on $[a,b]$ as a function of $t\in[a,b]$. But, by using the Fubini theorem, we get

$$\int_a^b\int_a^b P(x,t)f(x)dxdt=\int_a^b\int_a^b P(x,t)f(x)dtdx,$$

this is different than

$$\int_a^b\int_a^b P(x,t)f(\mathbf{t})dtdx.$$

So, the existence of the operator ${}_{k}^{s}\mathrm{J}_{a+}^{\alpha}f(x)$ is not proven.

1.2. Main result

We give a correct version to Theorem 2.2 in [3] with new conditions on the function f and the parameter s.

Definition 1.2. (See [1]). The space $L_{p,s}[a,b]$ (the set of those real-valued Lebesgue measurable functions f on $[a,b]$) is defined as

$$L_{p,s}[a,b]=\left\{f:\parallel f\parallel_{p,s}=\left(\int_a^b\mid f(x)\mid^p x^s dx<\infty\right)^{\frac{1}{p}}\right\},\ p\geq1,\ s>-1. \tag{1.3}$$

For $s=0$, the space $L_{p,s}[a,b]$ reduces to the classical space $L_p[a,b]$.

Theorem 1.3. *Let $s>-1$, $\alpha>0$, $k>0$ and f be a Lebesgue measurable functions on $[a,b]$, where*

$${}_{k}^{s}\mathrm{J}_{a+}^{\alpha}f(x)=\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\,\Gamma_k(\alpha)}\int_a^x(x^{s+1}-t^{s+1})^{\frac{\alpha}{k}-1}t^s f(t)dt,\quad a<x\leq b. \tag{1.4}$$

If $f\in L_{1,s}[a,b]$, then ${}_{k}^{s}\mathrm{J}_{a+}^{\alpha}f(x)\in L_{1,s}[a,b]$ for any $x\in[a,b]$.

Proof. Let $f\in L_{1,s}[a,b]$.

- Let $\frac{\alpha}{k}=1$, it is evident.
- Let $\frac{\alpha}{k}>1$. Let $\Omega=[a,b]\times[a,b]$, we pose for all $(x,t)\in\Omega$, posing

$$F(x,t)=\begin{cases}(x^{s+1}-t^{s+1})^{\frac{\alpha}{k}-1} & ,a\leq t\leq x,\\ 0 & ,x\leq t\leq b,\end{cases}$$

we have

$$\int_a^b F(x,t)x^s dx\leq\int_a^b x^s(b^{s+1}-a^{s+1})^{\frac{\alpha}{k}-1}dx=\frac{1}{s+1}(b^{s+1}-a^{s+1})^{\frac{\alpha}{k}},$$

therefore

$$\begin{aligned}\int_a^b\int_a^b F(x,t)\left|f(t)\right|x^st^s dx\,dt &\leq \tfrac{1}{s+1}\int_a^b (b^{s+1}-a^{s+1})^{\frac{\alpha}{k}}\left|f(t)\right|t^s dt\\ &= \tfrac{1}{s+1}(b^{s+1}-a^{s+1})^{\frac{\alpha}{k}}\left\|f(t)\right\|_{L_{1,s}[a,b]}<\infty.\end{aligned}$$

We deduce that the function $F(x,t)\left|f(t)\right|x^st^s$ is integrable over Ω. Using now Fubini's theorem, we get

$$\begin{aligned}\int_a^b {}^s_k\mathrm{J}^{\alpha}_{a^+}f(x)x^s dx &= \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\,\Gamma_k(\alpha)}\int_a^b\left(\int_a^b F(x,t)\left|f(t)\right|t^s dt\right)x^s dx\\ &= \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\,\Gamma_k(\alpha)}\int_a^b\left(\int_a^b F(x,t)\left|f(t)\right|x^s dx\right)t^s dt\\ &\leq \frac{(b^{s+1}-a^{s+1})^{\frac{\alpha}{k}}}{(s+1)^{\frac{\alpha}{k}}\,k\,\Gamma_k(\alpha)}\left\|f(t)\right\|_{L_{1,s}[a,b]}<\infty,\end{aligned}$$

this gives us

$${}^s_k\mathrm{J}^{\alpha}_{a^+}f(x)\in L_{1,s}[a,b].$$

- Let $0<\frac{\alpha}{k}<1$, by using Fubini's Theorem we get

$$\begin{aligned}\int_a^b \left|{}_{a^+}\mathrm{J}^{\alpha}_{k}f(x)\right|x^s dx &= \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\,\Gamma_k(\alpha)}\int_a^b\left|\int_a^x (x^{s+1}-t^{s+1})^{\frac{\alpha}{k}-1}f(t)t^sx^s dt\right|dx\\ &\leq \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\,\Gamma_k(\alpha)}\int_a^b\int_a^x \left|f(t)\right|(x^{s+1}-t^{s+1})^{\frac{\alpha}{k}-1}t^sx^s dt\,dx\\ &= \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\,\Gamma_k(\alpha)}\int_a^b\left|f(t)\right|\left(\int_t^b (x^{s+1}-t^{s+1})^{\frac{\alpha}{k}-1}x^s dx\right)t^s dt\\ &= \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\,\Gamma_k(\alpha)}\frac{k}{\alpha\ (s+1)}\int_a^b\left|f(t)\right|(b^{s+1}-t^{s+1})^{\frac{\alpha}{k}}t^s dt\\ &\leq \frac{(b^{s+1}-a^{s+1})^{\frac{\alpha}{k}}}{(s+1)^{\frac{\alpha}{k}}\,\Gamma_k(\alpha+k)}\int_a^b\left|f(t)\right|t^s dt\\ &= \frac{(b^{s+1}-a^{s+1})^{\frac{\alpha}{k}}}{(s+1)^{\frac{\alpha}{k}}\,\Gamma_k(\alpha+k)}\left\|f(t)\right\|_{L_{1,s}[a,b]}<+\infty,\end{aligned}$$

it is equivalent to

$${}^s_k\mathrm{J}^{\alpha}_{a^+}f(x)\in L_{1,s}[a,b].$$

□

References

[1] U.N. Katugompola, *New Approach Generalized Fractional Integral*, Applied Math and Comp. **218** 860 - 865, 2011.

[2] U.N. Katugompola, *A New Approach To Generalized Fractional Derivatives*, Bul. Mat. Anal. App. **6** (4), 1 - 15, 2014.

[3] M.Z. Sarikaya, Z. Dahmani, M.E. Kiris and F. Ahmad, $(k;s)$*-Riemann-Liouville fractional integral and applications*, Hacet. J. Math. Stat. **45** (1), 77 - 89, 2016.