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The Behavior of Solution of Fifteenth-Order Class Rational Difference Equation

E. M. Elsayed[1,2], and B. S. Alofi[1]
[1]King Abdulaziz University, Faculty of Science,
Mathematics Department, P. O. Box 80203,
Jeddah 21589, Saudi Arabia.
[2] Department of Mathematics, Faculty of Science, Mansoura University.
E-mail: emmelsayed@yahoo.com.
E-mail: badoora2al3ofi@gmail.com

Abstract

In this research, we analyze the behavior of the solutions of the differential equation

$$U_{n+1} = \frac{\alpha U_{n-14}}{\beta + \gamma U_{n-14} U_{n-9} U_{n-4}}, \quad n = 0, 1, ...,$$

where the initial values are arbitrary positive real numbers. We also provide solutions four special cases of this equation

Keywords: difference equations, stability, boundedness, solution of difference equation.
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1 Introduction

Our objective is to study the boundedness character, the steady state, the local asymptotic stability and the global behavior of the single positive steady state point of higher order differences equation given by the following formula

$$U_{n+1} = \frac{\alpha U_{n-14}}{\beta + \gamma U_{n-14} U_{n-9} U_{n-4}}, \qquad n = 0, 1, \dots , \tag{1}$$

where the initial values $U_{-14}, U_{-13}, U_{-12}, U_{-11}, U_{-10}, U_{-9}, U_{-8}, U_{-7}, U_{-6}, U_{-5}, U_{-4}, U_{-3}, U_{-2}, U_{-1}, U_0$ are arbitrary positive real numbers. We also provide solutions to some special cases of Eq. (1) where the initial values real numbers.

Difference equations or discrete dynamic systems are different fields, because several biological systems are naturally studied through discrete variables. Each dynamic system $u_{n+1} = g(u_n)$ determines a difference equation, and vice versa. Discrete dynamic system theory and difference equations have been greatly developed in the last twenty years. The application of discrete dynamic systems and difference equations has recently appeared in many fields. The theory of difference equations is extremely important in applied sciences and other fields.

The theory of difference equations will, without a doubt, continue to play an essential part in mathematics as a whole. The importance of nonlinear difference equations of order larger than one cannot be overstated.

In the literature, many applications of the theory of difference equations have been examined.

For instance, Elsayed [23-25] studied the dynamics of the solutions of recursive sequences satisfying

$$T_{n+1} = \frac{T_{n-11}}{\pm 1 \pm T_{n-3}T_{n-7}T_{n-11}},$$
$$T_{n+1} = \frac{T_{n-5}}{\pm 1 \pm T_{n-1}T_{n-3}T_{n-5}},$$
$$T_{n+1} = \frac{T_{n-8}}{\pm 1 \pm T_{n-2}T_{n-5}T_{n-8}}.$$

Abdul Khaliq et al. in [1] analyzed the asymptotic stability, global stability, periodicity of the solution of an eighth-order recursive sequences. Moreover, they obtained closed form solution of some special cases of the equation

$$P_{n+1} = aP_{n-3} + \frac{bP_{n-3}P_{n-7}}{cP_{n-3} + dP_{n-7}}.$$

Dina et al. in [12] studied the boundedness, the asymptotic behavior, the periodic character and the stability of solutions of the difference equation

$$Q_{n+1} = aQ_n + bQ_{n-1} + \beta Q_{n-1}\exp(-\gamma Q_{n-1}).$$

Chatzarakis in [11] researched the dynamics of solutions of the below difference equation

$$V_{n+1} = a + \frac{bV_n^2}{(c + dV_n)\, V_{n-1}}.$$

Alshareef et al in [3] examined some stability properties for the fixed point of the rational difference

$$S_{n+1} = aS_{n-8} + \frac{bS_{n-8}^2}{cS_{n-8} + dS_{n-17}}.$$

Alotaibi et al in [5] discussed some qualitative properties such as the boundedness, the periodicity and the global stability of the positive solutions of the nonlinear difference equation

$$U_{n+1} = aU_n + \frac{b_1U_{n-1} + b_2U_{n-2} + b_3U_{n-3} + b_4U_{n-4} + b_5U_{n-5}}{c_1U_{n-1} + c_2U_{n-2} + c_3U_{n-3} + c_4U_{n-4} + c_5U_{n-5}}.$$

2 Behavior of the Solutions of Eq. (1)

In this secion we investigated the behavior of the solution of Eq. (1) such as Local stability, global stability and boundedness character. where the initial values $U_{-14}, U_{-13}, U_{-12}, U_{-11}, U_{-10}, U_{-9}, U_{-8}, U_{-7}, U_{-6}, U_{-5}, U_{-4}, U_{-3}, U_{-2}, U_{-1}, U_0$ are arbitrary positive. Also, the Parameteα, β and γ are positive.

2.1 Local Stability

In this section we investigate the local stability character of the solutions of Eq.(1). The equilibrium point of Eq. (1) is given by

$$\bar{U} = \frac{\alpha \bar{U}}{\beta + \gamma \bar{U}^3}, \qquad n = 0, 1, ..., \tag{2}$$

therefore

$$\bar{U}\left((\beta - \alpha) + \bar{U}^3 \gamma\right) = 0, \tag{3}$$

the only unique quiliribum points is $\bar{U} = 0$, if $\beta \geq \alpha$. The equiliribum points is $\bar{U} = 0, \bar{U} = \sqrt[3]{\frac{\alpha-\beta}{\gamma}}$positive if $\beta < \alpha$. Define a function $g : (0, \infty) \to (0, \infty)$ as

$$g(x, y, z) = \frac{\alpha x}{\beta + \gamma x y z}.$$

Hence, we obtain

$$g_x(x, y, z) = \frac{\alpha \beta}{(\beta + \gamma x y z)^2},$$
$$g_y(x, y, z) = -\frac{\alpha \gamma x^2 z}{(\beta + \gamma x y z)^2},$$
$$g_z(x, y, z) = -\frac{\alpha \gamma x^2 y}{(\beta + \gamma x y z)^2},$$

it follows that

$$g_x(\bar{U}, \bar{U}, \bar{U}) = \frac{\alpha \beta}{\left(\beta + \gamma \bar{U}^3\right)^2},$$
$$g_y(\bar{U}, \bar{U}, \bar{U}) = -\frac{\alpha \gamma \bar{U}^3}{\left(\beta + \gamma \bar{U}^3\right)^2},$$
$$g_z(\bar{U}, \bar{U}, \bar{U}) = -\frac{\alpha \gamma \bar{U}^3}{\left(\beta + \gamma \bar{U}^3\right)^2}.$$

Therefore, the linearized equation about $\bar{U} = 0$ becomes

$$V_{n+1} = \frac{\alpha \beta}{\beta^2} V_{n-14},$$

and, the linearized equation about $\bar{U} = \sqrt[3]{\frac{\alpha-\beta}{\gamma}}$ becomes

$$V_{n+1} = \left(\frac{\alpha \beta}{(\beta + (\alpha - \beta))^2}\right) V_{n-14} - \frac{\alpha \gamma (\beta - \alpha)}{(\beta + (\alpha - \beta))^2} V_{n-9} - \frac{\alpha \gamma (\beta - \alpha)}{(\beta + (\alpha - \beta))^2} V_{n-4}.$$

Theorem 1 *Assume that $\beta \geq \alpha$, then the equiliribum point $\bar{U} = 0$ of Eq. (1) is locally asymptotically stable.*

Assume that $\beta + 2\gamma(\beta - \alpha) < \alpha$, then the equiliribum point $\bar{U} = \sqrt[3]{\frac{\alpha-\beta}{\gamma}}$ of Eq. (1) is locally asymptotically stable.

proof: From Theorem in [1] the the equiliribum point $\bar{U} = 0$ of Eq. (1) is locally asymptotically stable if

$$\alpha < \beta,$$

Also, the the equiliribum point $\bar{U} = \sqrt[3]{\frac{\alpha-\beta}{\gamma}}$ of Eq. (1) is locally asymptotically stable if

$$\frac{\alpha\beta}{(\beta+(\alpha-\beta))^2} + \frac{\alpha\gamma(\beta-\alpha)}{(\beta+(\alpha-\beta))^2} + \frac{\alpha\gamma(\beta-\alpha)}{(\beta+(\alpha-\beta))^2} < 1,$$

$$\alpha\beta + 2\alpha\gamma(\beta - \alpha) < \alpha^2,$$

then it follows that

$$\beta + 2\gamma(\beta - \alpha) < \alpha,$$

it means the proof is complete.

2.2 Global Attractor

In this subsection we investigate the global attractivity character of solutions of Eq. (1)

Theorem 2 *The equiliribum point of Eq. (1) is global Attractor if $\beta > \alpha$*

proof: Let p, q are real number and define $f : [p,q]^3 \to [p,q]$ a function $f(x,y,z) = \frac{\alpha x}{\beta+\gamma xyz}$, then we see that the function increasing x in and decreasing in y, z assume that (m, M) is a solution of the system

$$M = f(M,m,m) \qquad \text{and} \qquad m = f(m,M,M).$$

Hence we get

$$M = \frac{\alpha M}{\beta+\gamma M m^2}, \qquad m = \frac{\alpha m}{\beta+\gamma m M^2},$$

$$M\left(\beta + \gamma M m^2\right) = \alpha M,$$

$$m\left(\beta + \gamma m M^2\right) = \alpha m.$$

Subtracting we get

$$\beta(M - m) = \alpha(M - m),$$

$$(\beta - \alpha)(M - m) = 0.$$

then

$$M = m, \qquad \text{if} \quad \beta > \alpha,$$

we obtain by therom in [2] that he equiliribum point $\bar{U} = 0$ of Eq. (1) is global Attractor.

2.3 Boundness of solutions

In this subsection we study the boundedness of solutions of Eq. (1)

Theorem 3 *Every solution of Eq. (1) is boumded if $\alpha < 1, \beta \geq 1$,*

Proof: Sppose $\{U_n\}_{n=-14}^{\infty}$ be solution of Eq. (1). It follows from Eq. (1) that

$$U_{n+1} = \frac{\alpha U_{n-14}}{\beta + \gamma U_{n-14} U_{n-9} U_{n-4}}, \qquad \beta \geq 1 \tag{4}$$

$$< \alpha U_{n-14}, \qquad 1 > \alpha \tag{5}$$

$$< U_{n-14}, \tag{6}$$

hence

$$U_{n+1} \leq U_{n-14}, \quad \text{for} \ \ n \geq 0 \ .$$

Therefore $\{U_n\}_{-14}^{\infty}$ is bounded above by
$M = \max\{U_{-14}, U_{-13}, U_{-12}, U_{-11}, U_{-10}, U_{-9}, U_{-8}, U_{-7}, U_{-6}, U_{-5}, \ U_{-4}, U_{-3}, U_{-2}, U_{-1}, U_0\}$

2.4 Numerical result:

In this subsection, we consider a numerical examples to confirm our result.

Example 1 *Assume that $\alpha = 0.9, \beta = 1.1, \gamma = 1,$ with initial conditions*

$$IC: U_{-14} = 0.18, U_{-13} = 0.22, U_{-12} = 0.18, U_{-11} = 0.24, U_{-10} = 0.11, U_{-9} = 0.2, U_{-8} = 0.12,$$
$$U_{-7} = 0.165, U_{-6} = 0.24, U_{-5} = 0.11, U_{-4} = 0.17, U_{-3} = 0.24, U_{-2} = 0.21, U_{-1} = 0.27, U_0 = 0.17.$$

(see Fig. 1)

Example 2 *Assume that $\alpha = 0.9, \beta = 1.1, \gamma = 1$ with initial conditions*

$$\mathit{IC1}: U_{-14} = 8, U_{-13} = 12, U_{-12} = 8, U_{-11} = 14, U_{-10} = 1, U_{-9} = 10, U_{-8} = 2,$$
$$U_{-7} = 6.5, U_{-6} = 14, U_{-5} = 1, U_{-4} = 7, U_{-3} = 14, U_{-2} = 11, U_{-1} = 17, U_0 = 7.$$

$IC2 : U_{-14} = 18, U_{-13} = 22, U_{-12} = 18, U_{-11} = 24, U_{-10} = 11, U_{-9} = 20, U_{-8} = 12,$
$U_{-7} = 16.5, U_{-6} = 24, U_{-5} = 11, U_{-4} = 17, U_{-3} = 24, U_{-2} = 21, U_{-1} = 27, U_0 = 17.$

$IC3 : U_{-14} = 14, U_{-13} = 18, U_{-12} = 14, U_{-11} = 20, U_{-10} = 7, U_{-9} = 17, U_{-8} = 8,$
$U_{-7} = 12.5, U_{-6} = 20, U_{-5} = 7, U_{-4} = 13, U_{-3} = 20, U_{-2} = 17, U_{-1} = 23, U_0 = 13.$

(see Fig. 2)

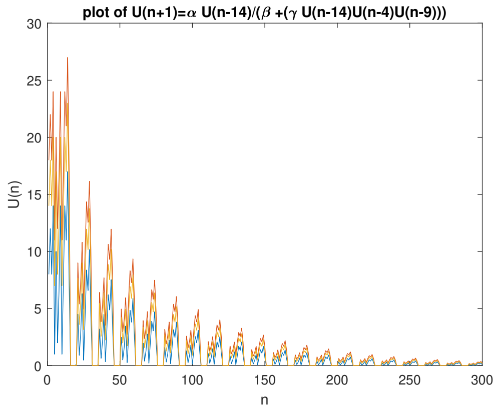

Figure 1: .

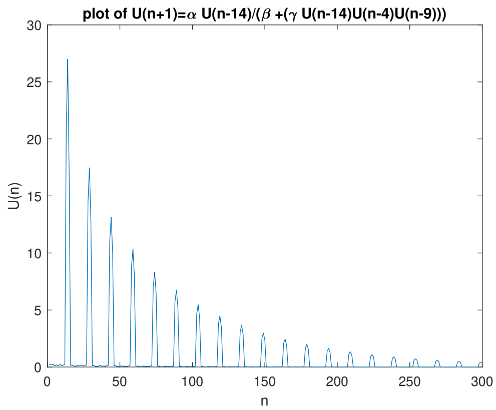

Figure 2: .

3 On The Solution of the Eq. (1)

In this section, we investigate the solutions of four special cases difference equations Eq. (1) where $n \in \mathbb{N}_0$ and the initial conditions are arbitrary nonzero real numbers $U_{-14}, U_{-13}, U_{-12}, U_{-11}, U_{-10}, U_{-9}, U_{-8}, U_{-7}, U_{-6}, U_{-5}, U_{-4}, U_{-3}, U_{-2}, U_{-1}, U_0$.

3.1 Case 1: $U_{n+1} = \frac{U_{n-14}}{1+U_{n-14}U_{n-9}U_{n-4}}$:

In this subsection, we investigate the solutions of the difference equations

$$U_{n+1} = \frac{U_{n-14}}{1+U_{n-14}U_{n-9}U_{n-4}}, \qquad n = 0, 1, ... \tag{7}$$

Theorem 4 *Assume that $\{U_n\}$ are solutions of difference equations. Then for $n = 1, 2, ...,$ we see that all solutions of Eq. (7) are given by the following formulas*

$$U_{15n-14} = \frac{A\prod_{i=0}^{n-1}(1+3iAFK)}{\prod_{i=0}^{n-1}(1+(3i+1)AFK)}, U_{15n-9} = \frac{F\prod_{i=0}^{n-1}(1+(3i+1)AFK)}{\prod_{i=0}^{n-1}(1+(3i+2)AFK)}, U_{15n-4} = \frac{K\prod_{i=0}^{n-1}(1+(3i+2)AFK)}{\prod_{i=0}^{n-1}(1+(3i+3)AFK)},$$

$$U_{15n-13} = \frac{B\prod_{i=0}^{n-1}(1+3iBGL)}{\prod_{i=0}^{n-1}(1+(3i+1)BGL)}, U_{15n-8} = \frac{G\prod_{i=0}^{n-1}(1+(3i+1)BGL)}{\prod_{i=0}^{n-1}(1+(3i+2)BGL)}, U_{15n-3} = \frac{L\prod_{i=0}^{n-1}(1+(3i+2)BGL)}{\prod_{i=0}^{n-1}(1+(3i+3)BGL)},$$

$$U_{15n-12} = \frac{C\prod_{i=0}^{n-1}(1+3iCHM)}{\prod_{i=0}^{n-1}(1+(3i+1)CHM)}, U_{15n-7} = \frac{H\prod_{i=0}^{n-1}(1+(3i+1)CHM)}{\prod_{i=0}^{n-1}(1+(3i+2)CHM)}, U_{15n-2} = \frac{M\prod_{i=0}^{n-1}(1+(3i+2)CHM)}{\prod_{i=0}^{n-1}(1+(3i+3)CHM)},$$

$$U_{15n-11} = \frac{D\prod_{i=0}^{n-1}(1+3iDIN)}{\prod_{i=0}^{n-1}(1+(3i+1)DIN)}, U_{15n-6} = \frac{I\prod_{i=0}^{n-1}(1+(3i+1)DIN)}{\prod_{i=0}^{n-1}(1+(3i+2)DIN)}, U_{15n-1} = \frac{N\prod_{i=0}^{n-1}(1+(3i+2)DIN)}{\prod_{i=0}^{n-1}(1+(3i+3)DIN)},$$

$$U_{15n-10} = \frac{E\prod_{i=0}^{n-1}(1+3iEJO)}{\prod_{i=0}^{n-1}(1+(3i+1)EJO)}, U_{15n-5} = \frac{J\prod_{i=0}^{n-1}(1+(3i+1)EJO)}{\prod_{i=0}^{n-1}(1+(3i+2)EJO)}, U_{15n} = \frac{O\prod_{i=0}^{n-1}(1+(3i+2)EJO)}{\prod_{i=0}^{n-1}(1+(3i+3)EJO)},$$

where $U_{-14} = A,\ U_{-13} = B, U_{-12} = C,\ U_{-11} = D,\ U_{-10} = E, U_{-9} = F,\ U_{-8} = G, U_{-7} = H,\ U_{-6} = I,\ U_{-5} = J, U_{-4} = K,\ U_{-3} = L, U_{-2} = M,\ U_{-1} = N$ and $U_0 = O$.

Proof: For $n = 1$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n - 1$. that is,

$$U_{15n-29}=\frac{A\prod_{i=0}^{n-2}(1+3iAFK)}{\prod_{i=0}^{n-2}(1+(3i+1)AFK)},U_{15n-24}=\frac{F\prod_{i=0}^{n-2}(1+(3i+1)AFK)}{\prod_{i=0}^{n-2}(1+(3i+2)AFK)},U_{15n-19}=\frac{K\prod_{i=0}^{n-2}(1+(3i+2)AFK)}{\prod_{i=0}^{n-2}(1+(3i+3)AFK)},$$

$$U_{15n-28}=\frac{B\prod_{i=0}^{n-2}(1+3iBGL)}{\prod_{i=0}^{n-2}(1+(3i+1)BGL)},U_{15n-23}=\frac{G\prod_{i=0}^{n-2}(1+(3i+1)BGL)}{\prod_{i=0}^{n-2}(1+(3i+2)BGL)},U_{15n-18}=\frac{L\prod_{i=0}^{n-2}(1+(3i+2)BGL)}{\prod_{i=0}^{n-2}(1+(3i+3)BGL)},$$

$$U_{15n-27}=\frac{C\prod_{i=0}^{n-2}(1+3iCHM)}{\prod_{i=0}^{n-2}(1+(3i+1)CHM)},U_{15n-22}=\frac{H\prod_{i=0}^{n-2}(1+(3i+1)CHM)}{\prod_{i=0}^{n-2}(1+(3i+2)CHM)},U_{15n-17}=\frac{M\prod_{i=0}^{n-2}(1+(3i+2)CHM)}{\prod_{i=0}^{n-2}(1+(3i+3)CHM)},$$

$$U_{15n-26}=\frac{D\prod_{i=0}^{n-2}(1+3iDIN)}{\prod_{i=0}^{n-2}(1+(3i+1)DIN)},U_{15n-21}=\frac{I\prod_{i=0}^{n-2}(1+(3i+1)DIN)}{\prod_{i=0}^{n-2}(1+(3i+2)DIN)},U_{15n-16}=\frac{N\prod_{i=0}^{n-2}(1+(3i+2)DIN)}{\prod_{i=0}^{n-2}(1+(3i+3)DIN)},$$

$$U_{15n-25}=\frac{E\prod_{i=0}^{n-2}(1+3iEJO)}{\prod_{i=0}^{n-2}(1+(3i+1)EJO)},U_{15n-20}=\frac{J\prod_{i=0}^{n-2}(1+(3i+1)EJO)}{\prod_{i=0}^{n-2}(1+(3i+2)EJO)},U_{15n-15}=\frac{O\prod_{i=0}^{n-2}(1+(3i+2)EJO)}{\prod_{i=0}^{n-2}(1+(3i+3)EJO)}.$$

Now we find from Eq. (7) that

$$
\begin{aligned}
U_{15n-14} &= \frac{U_{15n-29}}{1+U_{15n-29}U_{15n-19}U_{15n-24}} \\
&= \frac{\left(\frac{A\prod_{i=0}^{n-2}(1+3iAFK)}{\prod_{i=0}^{n-2}(1+(3i+1)AFK)}\right)}{1+\left(\frac{A\prod_{i=0}^{n-2}(1+3iAFK)}{\prod_{i=0}^{n-2}(1+(3i+1)AFK)}\right)\left(\frac{K\prod_{i=0}^{n-2}(1+(3i+2)AFK)}{\prod_{i=0}^{n-2}(1+(3i+3)AFK)}\right)\left(\frac{F\prod_{i=0}^{n-2}(1+(3i+1)AFK)}{\prod_{i=0}^{n-2}(1+(3i+2)AFK)}\right)} \\
&= \frac{A\prod_{i=0}^{n-2}(1+3iAFK)\prod_{i=0}^{n-2}(1+(3i+3)AFK)\prod_{i=0}^{n-2}(1+(3i+2)AFK)}{\left(\prod_{i=0}^{n-2}(1+(3i+2)AFK)(1+(3i+1)AFK)\right)\left(\prod_{i=0}^{n-2}(1+(3i+3)AFK)+AFK\prod_{i=0}^{n-2}(1+3iAFK)\right)} \\
&= \frac{A\prod_{i=0}^{n-2}(1+3iAFK)\prod_{i=0}^{n-2}(1+(3i+3)AFK)}{\left(\prod_{i=0}^{n-2}(1+(3i+1)AFK)\right)\left(\prod_{i=0}^{n-2}(1+(3i+3)AFK)+AFK\prod_{i=0}^{n-2}(1+3iAFK)\right)} \\
&= \frac{A\prod_{i=0}^{n-2}(1+3iAFK)\prod_{i=0}^{n-2}(1+(3i+3)AFK)}{\left(\prod_{i=0}^{n-2}(1+(3i+1)AFK)(1+3iAFK)\right)((1+(3n-3)AFK)+AFK)} \\
&= \frac{A\prod_{i=0}^{n-1}(1+3iAFK)}{\prod_{i=0}^{n-1}(1+(3i+1)AFK)},
\end{aligned}
$$

$$
\begin{aligned}
U_{15n-13} &= \frac{U_{15n-28}}{1+U_{15n-28}U_{15n-18}U_{15n-23}} \\
&= \frac{\left(\frac{B\prod_{i=0}^{n-2}(1+3iBGL)}{\prod_{i=0}^{n-2}(1+(3i+1)BGL)}\right)}{1+\left(\frac{B\prod_{i=0}^{n-2}(1+3iBGL)}{\prod_{i=0}^{n-2}(1+(3i+1)BGL)}\right)\left(\frac{L\prod_{i=0}^{n-2}(1+(3i+2)BGL)}{\prod_{i=0}^{n-2}(1+(3i+3)BGL)}\right)\left(\frac{G\prod_{i=0}^{n-2}(1+(3i+1)BGL)}{\prod_{i=0}^{n-2}(1+(3i+2)BGL)}\right)} \\
&= \frac{B\prod_{i=0}^{n-2}(1+3iBGL)\prod_{i=0}^{n-2}(1+(3i+3)BGL)\prod_{i=0}^{n-2}(1+(3i+2)BGL)}{\left(\prod_{i=0}^{n-2}(1+(3i+1)BGL)\prod_{i=0}^{n-2}(1+(3i+2)BGL)\right)\left(\prod_{i=0}^{n-2}(1+(3i+3)BGL)+BGL\prod_{i=0}^{n-2}(1+3iBGL)\right)} \\
&= \frac{B\prod_{i=0}^{n-2}(1+3iBGL)\prod_{i=0}^{n-2}(1+(3i+3)BGL)\prod_{i=0}^{n-2}(1+(3i+2)BGL)}{\left(\prod_{i=0}^{n-2}(1+(3i+1)BGL)\prod_{i=0}^{n-2}(1+(3i+2)BGL)\right)\left(\prod_{i=0}^{n-2}(1+(3i+3)BGL)+BGL\prod_{i=0}^{n-2}(1+3iBGL)\right)} \\
&= \frac{B\prod_{i=0}^{n-2}(1+3iBGL)\prod_{i=0}^{n-2}(1+(3i+3)BGL)}{\left(\prod_{i=0}^{n-2}(1+(3i+1)BGL)(1+3iBGL)\right)((1+(3n-3)BGL)+BGL)} \\
&= \frac{B\prod_{i=0}^{n-1}(1+3iBGL)}{\prod_{i=0}^{n-1}(1+(3i+1)BGL)},
\end{aligned}
$$

$$U_{15n-12} = \frac{U_{15n-27}}{1+U_{15n-27}U_{15n-17}U_{15n-22}}$$

$$= \frac{\left(\frac{C\prod_{i=0}^{n-2}(1+3iCHM)}{\prod_{i=0}^{n-2}(1+(3i+1)CHM)}\right)}{1+\left(\frac{C\prod_{i=0}^{n-2}(1+3iCHM)}{\prod_{i=0}^{n-2}(1+(3i+1)CHM)}\right)\left(\frac{M\prod_{i=0}^{n-2}(1+(3i+2)CHM)}{\prod_{i=0}^{n-2}(1+(3i+3)CHM)}\right)\left(\frac{H\prod_{i=0}^{n-2}(1+(3i+1)CHM)}{\prod_{i=0}^{n-2}(1+(3i+2)CHM)}\right)}$$

$$= \frac{C\prod_{i=0}^{n-2}(1+3iCHM)\prod_{i=0}^{n-2}(1+(3i+3)\,CHM)\prod_{i=0}^{n-2}(1+(3i+2)\,CHM)}{\left(\prod_{i=0}^{n-2}(1+(3i+1)\,CHM)\,(1+(3i+2)\,CHM)\right)\left(\prod_{i=0}^{n-2}(1+(3i+3)\,CHM)+CHM\prod_{i=0}^{n-2}(1+3iCHM)\right)}$$

$$= \frac{C\prod_{i=0}^{n-2}(1+3iCHM)\prod_{i=0}^{n-2}(1+(3i+3)\,CHM)}{\left(\prod_{i=0}^{n-2}(1+(3i+1)\,CHM)\,(1+3iCHM)\right)((1+(3n-3)\,CHM)+CHM)}$$

$$= \frac{C\prod_{i=0}^{n-1}(1+3iCHM)}{\prod_{i=0}^{n-1}(1+(3i+1)CHM)},$$

Also, we can prove the other relations. The proof is complete.

Theorem 5 *The trivial equilibrium point in Eq. (7) is unique and it is never locally asymptotically stable.*

For confirming the results of this subsection, we consider numerical examples for the Eq. (7)

Example 3 *Assume the initial conditions are*

$$IC: U_{-14}=12, U_{-13}=8, U_{-12}=2, U_{-11}=1, U_{-10}=0.5, U_{-9}=0.2, U_{-8}=1.2,$$

$$U_{-7}=5, U_{-6}=0.1, U_{-5}=3, U_{-4}=0.1, U_{-3}=3, U_{-2}=6, U_{-1}=12, U_0=8.$$

(See Fig. 3).

Example 4 *Suppose that*

$$IC: U_{-14}=2, U_{-13}=6, U_{-12}=2, U_{-11}=6, U_{-10}=-5, U_{-9}=4, U_{-8}=-4,$$
$$U_{-7}=0.5, U_{-6}=8, U_{-5}=-5, U_{-4}=1, U_{-3}=8, U_{-2}=5, U_{-1}=11, U_0=7.$$

(See Fig. 4).

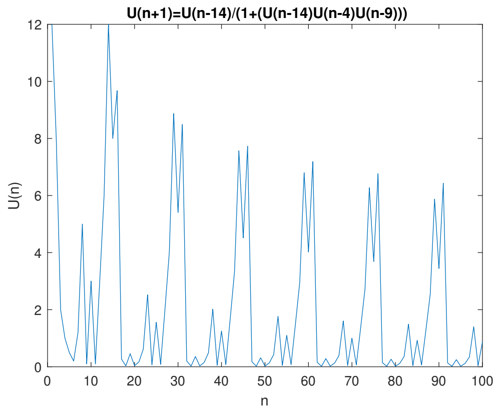

Figure 3: .

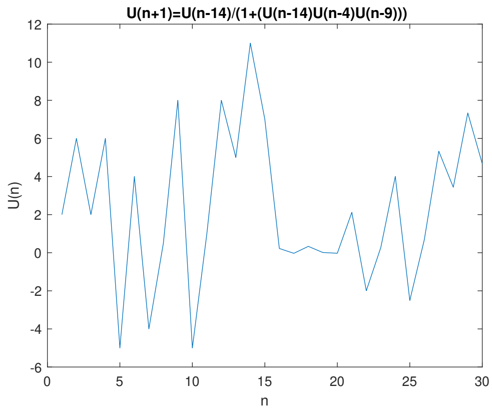

Figure 4: .

3.2 Case 2: $U_{n+1}=\frac{U_{n-14}}{1-U_{n-14}U_{n-9}U_{n-4}}$:

In this subsection, we investigate the solutions of the difference equation

$$U_{n+1}=\frac{U_{n-14}}{1-U_{n-14}U_{n-9}U_{n-4}}, \qquad n=0,1,.... \tag{8}$$

Theorem 6 *Assume that* $\{U_n\}$ *are solutions of difference equations. Then for* $n=1,2,...,$ *we see* *that all solutions of Eq. (8) are given by the following formulas*

$$U_{15n-14}=\frac{A\prod_{i=0}^{n-1}(1-3iAFK)}{\prod_{i=0}^{n-1}(1-(3i+1)AFK)},U_{15n-9}=\frac{F\prod_{i=0}^{n-1}(1-(3i+1)AFK)}{\prod_{i=0}^{n-1}(1-(3i+2)AFK)},U_{15n-4}=\frac{K\prod_{i=0}^{n-1}(1-(3i+2)AFK)}{\prod_{i=0}^{n-1}(1-(3i+3)AFK)},$$

$$U_{15n-13}=\frac{B\prod_{i=0}^{n-1}(1-3iBGL)}{\prod_{i=0}^{n-1}(1-(3i+1)BGL)},U_{15n-8}=\frac{G\prod_{i=0}^{n-1}(1-(3i+1)BGL)}{\prod_{i=0}^{n-1}(1-(3i+2)BGL)},U_{15n-3}=\frac{L\prod_{i=0}^{n-1}(1-(3i+2)BGL)}{\prod_{i=0}^{n-1}(1-(3i+3)BGL)},$$

$$U_{15n-12}=\frac{C\prod_{i=0}^{n-1}(1-3iCHM)}{\prod_{i=0}^{n-1}(1-(3i+1)CHM)},U_{15n-7}=\frac{H\prod_{i=0}^{n-1}(1-(3i+1)CHM)}{\prod_{i=0}^{n-1}(1-(3i+2)CHM)},U_{15n-2}=\frac{M\prod_{i=0}^{n-1}(1-(3i+2)CHM)}{\prod_{i=0}^{n-1}(1-(3i+3)CHM)},$$

$$U_{15n-11}=\frac{D\prod_{i=0}^{n-1}(1-3iDIN)}{\prod_{i=0}^{n-1}(1-(3i+1)DIN)},U_{15n-6}=\frac{I\prod_{i=0}^{n-1}(1-(3i+1)DIN)}{\prod_{i=0}^{n-1}(1-(3i+2)DIN)},U_{15n-1}=\frac{N\prod_{i=0}^{n-1}(1-(3i+2)DIN)}{\prod_{i=0}^{n-1}(1-(3i+3)DIN)},$$

$$U_{15n-10}=\frac{E\prod_{i=0}^{n-1}(1-3iEJO)}{\prod_{i=0}^{n-1}(1-(3i+1)EJO)},U_{15n-5}=\frac{J\prod_{i=0}^{n-1}(1-(3i+1)EJO)}{\prod_{i=0}^{n-1}(1-(3i+2)EJO)},U_{15n}=\frac{O\prod_{i=0}^{n-1}(1-(3i+2)EJO)}{\prod_{i=0}^{n-1}(1-(3i+3)EJO)},$$

where $U_{-14}=A,\ U_{-13}=B, U_{-12}=C,\ U_{-11}=D,\ U_{-10}=E, U_{-9}=F,\ U_{-8}=G, U_{-7}=H,\ U_{-6}=I,\ U_{-5}=J, U_{-4}=K,\ U_{-3}=L, U_{-2}=M,\ U_{-1}=N$ *and* $U_0=O$.

Proof:

The proof is the same as the proof of Therom 2.1.

Theorem 7 *The unique equilibrium point* $\bar{U}=0$ *in Eq. (8) is not locally asymptotically stable.*

We consider numerical examples for the Eq. (8).

Example 5 *Suppose the following initial conditions: with the initial conditions*

$$IC: U_{-14}=12, U_{-13}=8, U_{-12}=2, U_{-11}=1, U_{-10}=0.5, U_{-9}=0.2, U_{-8}=1.2,$$
$$U_{-7}=5, U_{-6}=0.1, U_{-5}=3, U_{-4}=0.1, U_{-3}=3, U_{-2}=6, U_{-1}=12, U_0=8.$$

(See Fig. 8).

Example 6 *Assume that*

$$IC: U_{-14}=2, U_{-13}=4, U_{-12}=6, U_{-11}=-3, U_{-10}=4, U_{-9}=-2, U_{-8}=0.4,$$
$$U_{-7}=4, U_{-6}=-3, U_{-5}=1, U_{-4}=1, U_{-3}=6, U_{-2}=2, U_{-1}=11, U_{0}=5.$$

(See Fig. 6).

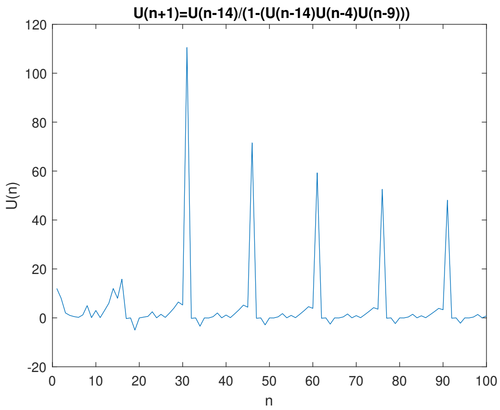

Figure 5: .

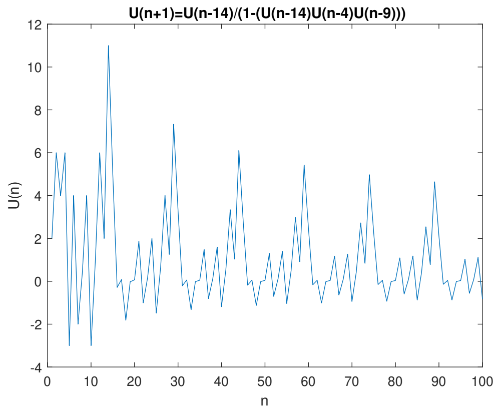

Figure 6: .

3.3 Case 3: $U_{n+1}=\frac{U_{n-14}}{-1+U_{n-14}U_{n-9}U_{n-4}}$:

In this subsection, we investigate the solutions of the difference equation

$$U_{n+1}=\frac{U_{n-14}}{-1+U_{n-14}U_{n-9}U_{n-4}}, \qquad n=0,1,..., \tag{9}$$

where $U_{n-14}U_{n-9}U_{n-4}\neq 1$.

Theorem 8 *Every solution* $\{U_n\}_{n=-14}^{\infty}$ *of Eq. (9) is periodic with period thirty and in the form*

$$\{A,B,C,D,E,F,G,H,I,J,K,L,M,N,O,$$
$$\frac{A}{AFK-1},\frac{B}{BGL-1},\frac{C}{CHM-1},\frac{D}{DIN-1},\frac{E}{EJO-1},$$
$$F(AFK-1),G(BGL-1),H(CHM-1),I(DIN-1),J(EJO-1),$$
$$\frac{K}{AFK-1},\frac{L}{BGL-1},\frac{M}{CHM-1},\frac{N}{DIN-1},\frac{O}{EJO-1},A,B,...\}$$

where $U_{-14}=A,\ U_{-13}=B, U_{-12}=C,\ U_{-11}=D,\ U_{-10}=E, U_{-9}=F,\ U_{-8}=G, U_{-7}=H,\ U_{-6}=I,\ U_{-5}=J, U_{-4}=K,\ U_{-3}=L, U_{-2}=M,\ U_{-1}=N$ *and* $U_0=O$.

Proof: We can deduce from Eq. (9) that

$$U_1=\frac{A}{AFK-1},U_2=\frac{B}{BGL-1},U_3=\frac{C}{CHM-1},U_4=\frac{D}{DIN-1},U_5=\frac{E}{EJO-1},$$
$$U_6=F(AFK-1),U_7=G(BGL-1),U_8=H(CHM-1),U_9=I(DIN-1),U_{10}=J(EJO-1),$$
$$U_{11}=\frac{K}{AFK-1},U_{12}=\frac{L}{BGL-1},U_{13}=\frac{M}{CHM-1},U_{14}=\frac{N}{DIN-1},U_{15}=\frac{O}{EJO-1},$$
$$U_{16}=A=U_{-14},U_{17}=B=U_{-13},U_{18}=C=U_{-12},U_{19}=D=U_{-11},U_{19}=E=U_{-10}.$$

Thus the proof is performed.

Theorem 9 *The equilibrium points of Eq. (9) are* 0, $\sqrt[3]{2}$ *and they are not locally asymptotically stable.*

Theorem 10 *Eq. (9) has a periodic solution of period fifteen iff* $AFK=BGL=CHM=DIN=EJO=2$.

We provide numerical examples for Eq. (9) in order to confirm the results of this subsection.

Example 7 *For confirming the results of this subsection, we consider present numerical example for Eq. (9) with the initial conditions*

$$IC: U_{-14}=1.2, U_{-13}=0.2, U_{-12}=1.4, U_{-11}=0.4, U_{-10}=1.5, U_{-9}=2, U_{-8}=1.2,$$
$$U_{-7}=0.2, U_{-6}=1.4, U_{-5}=0.4, U_{-4}=1.5, U_{-3}=2, U_{-2}=1.2, U_{-1}=0.2, U_0=1.4.$$

(See Fig. 7).

Example 8 *We provide another numerical example for Eq. (9) with initial values*

$$IC: U_{-14}=2, U_{-13}=0.5, U_{-12}=0.4, U_{-11}=0.3, U_{-10}=0.4, U_{-9}=0.2, U_{-8}=2,$$
$$U_{-7}=10, U_{-6}=5, U_{-5}=2, U_{-4}=5, U_{-3}=2, U_{-2}=0.5, U_{-1}=\frac{4}{3}, U_0=2.5.$$

. *(See Fig. 8).*

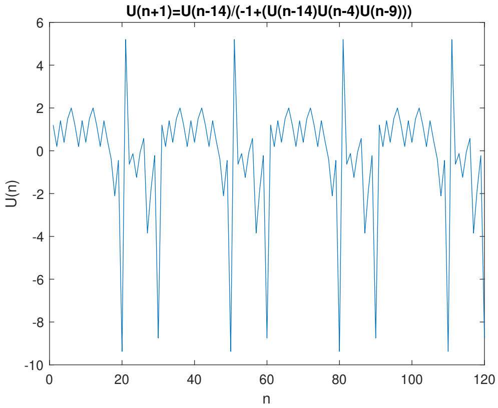

Figure 7: .

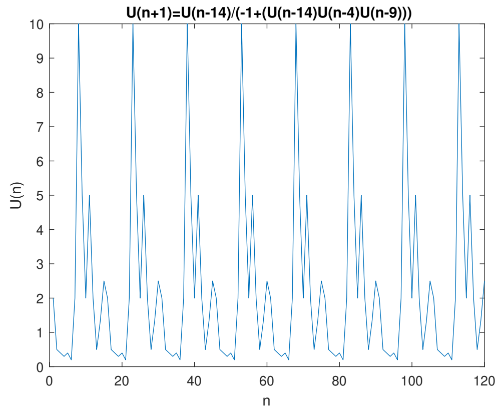

Figure 8: .

3.4 Case 4: $U_{n+1} = \frac{U_{n-14}}{-1-U_{n-14}U_{n-9}U_{n-4}}$:

In this subsection, we investigate the solutions of the difference equation

$$U_{n+1} = \frac{U_{n-14}}{-1-U_{n-14}U_{n-9}U_{n-4}}, \qquad n = 0, 1, ..., \tag{10}$$

where $U_{n-14}U_{n-9}U_{n-4} \neq -1$.

Theorem 11 *Every solution $\{U_n\}_{n=-14}^{\infty}$ of Eq. (10) is periodic with period thirty and in the form*

$$\begin{aligned}&\{A, B, C, D, E, F, G, H, I, J, K, L, M, N, O,\\ &\frac{-A}{AFK+1}, \frac{-B}{BGL+1}, \frac{-C}{CHM+1}, \frac{-D}{DIN+1}, \frac{-E}{EJO+1},\\ &-F(AFK+1), -G(BGL+1), -H(CHM+1), -I(DIN+1), -J(EJO+1),\\ &\frac{-K}{AFK+1}, \frac{-L}{BGL+1}, \frac{-M}{CHM+1}, \frac{-N}{DIN+1}, \frac{-O}{EJO+1}, A, B, ...\}\end{aligned}$$

where $U_{-14} = A,\ U_{-13} = B, U_{-12} = C,\ U_{-11} = D,\ U_{-10} = E, U_{-9} = F,\ U_{-8} = G, U_{-7} = H,\ U_{-6} = I,\ U_{-5} = J, U_{-4} = K,\ U_{-3} = L, U_{-2} = M,\ U_{-1} = N$ *and* $U_0 = O$.

Proof: The proof is similar of Therom (2.3).

Theorem 12 *Eq. (10) has only one equilibrium point, which is 0, and this equilibrium point is not locally asymptotically stable .*

Theorem 13 *Eq. (10) has a periodic solution of period fifteen iff* $AFK = BGL = CHM = DIN = EJO = 2$.

Example 9 *Assume that the starting conditions are as follows:*

$$IC : U_{-14} = 11, U_{-13} = 7, U_{-12} = 1, U_{-11} = 1, U_{-10} = 2.5, U_{-9} = 2.2, U_{-8} = 0.2,$$
$$U_{-7} = 4, U_{-6} = 2, U_{-5} = 6, U_{-4} = 4, U_{-3} = 1, U_{-2} = 2, U_{-1} = 11, U_0 = 7.$$

(See Fig. 9).

Example 10 *Suppose the initial conditions given by*

$$IC : U_{-14} = 2, U_{-13} = 0.5, U_{-12} = -0.4, U_{-11} = 0.3, U_{-10} = 0.4, U_{-9} = 0.2, U_{-8} = -2,$$
$$U_{-7} = 10, U_{-6} = 5, U_{-5} = -2, U_{-4} = -5, U_{-3} = 2, U_{-2} = 0.5, U_{-1} = -\frac{4}{3}, U_0 = 2.5.$$

(See Fig. 10.

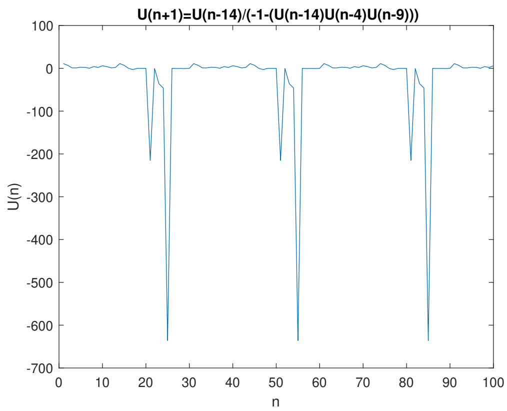

Figure 9: .

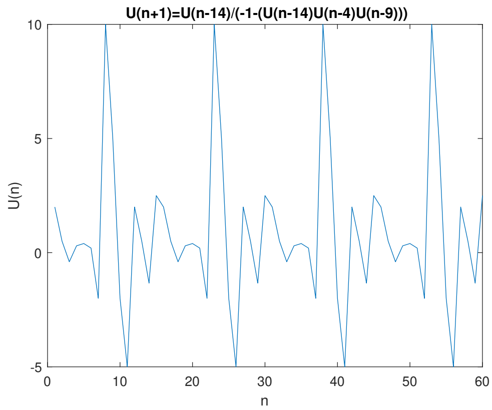

Figure 10: .

Data Availability

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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