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On Pointwise k-Slant Submanifolds of Almost Contact Metric Manifolds

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(Dedicated to the memory of Prof. Dr. Krishan Lal DUGGAL (1929 - 2022))

ABSTRACT

We establish some properties of the k-slant and pointwise k-slant submanifolds of an almost contact metric manifold with a special view towards the integrability of the component distributions. We prove some results for totally geodesic pointwise k-slant submanifolds. Furthermore, we obtain some nonexistence results for pointwise k-slant submanifolds in the almost contact metric setting.

Keywords: *Pointwise k-slant submanifold, almost contact metric manifold, integrable distribution, totally geodesic submanifold.*
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1. Preliminaries

The theory of *slant submanifolds* has been initiated by Chen [3] and lately studied by many authors in different contexts: almost complex (Hermitian, Kähler), almost contact (K-contact, cosymplectic, β-Sasakian, α-Kenmotsu), almost product, almost paracontact, metallic geometry. Generalizing the concept of slant submanifold, the notions of *semi-slant* [11], *hemi-slant* [14], and *bi-slant* [1] *submanifold* have been treated in the last years. In [6], Etayo has introduced the notion of *pointwise slant submanifold* (see also [4]) by letting the slant angle to depend on the points of the submanifold. More general, for submanifolds whose tangent bundle can be decomposed into an arbitrary number of orthogonal slant distributions, there have been defined the concepts of k-*slant* and *pointwise* k-*slant submanifold* [8], containing all the above mentioned cases.

In the present paper we derive some algebraic and geometric properties of a pointwise k-slant submanifold of an almost contact metric manifold, with a special view towards the integrability of the component distributions in the (α, β)-contact metric case. We would like to mention that the integrability problem for the underlying distributions of pseudo-slant submanifolds of trans-Sasakian manifolds has been discussed in [5].

An *almost contact metric structure* [13] on a $(2n+1)$-dimensional smooth manifold $\tilde{M}$ consists of a $(1,1)$-tensor field ϕ, a vector field ξ, a 1-form η, and a Riemannian metric g satisfying:

$$\phi^2 = -(I - \eta \otimes \xi), \quad \eta(\xi) = 1, \quad g(\phi\cdot, \phi\cdot) = g - \eta \otimes \eta,$$

which further imply:

$$\phi\xi = 0, \quad \eta \circ \phi = 0, \quad i_\xi g = \eta, \quad g(\phi\cdot, \cdot) = -g(\cdot, \phi\cdot).$$

If there exist two smooth real functions α and β on the almost contact metric manifold $(\tilde{M}, \phi, \xi, \eta, g)$ such that the Levi-Civita connection $\tilde{\nabla}$ of g satisfies:

$$(\tilde{\nabla}_X \phi)Y = \alpha[g(\phi X, Y)\xi - \eta(Y)\phi X] + \beta[g(X,Y)\xi - \eta(Y)X] \tag{1.1}$$

for any $X, Y \in \Gamma(T\tilde{M})$, then we call $\tilde{M}$ an (α, β)-*contact metric manifold*.

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For different values of (α,β) in (1.1), we obtain the following particular cases. An (α,β)-contact metric manifold (called also *trans-Sasakian manifold* [10]) is:
(i) *cosymplectic* for $(0,0)$;
(ii) *β-Sasakian* for $(0,\beta)$ with β a nonzero constant, in particular, Sasakian if $\beta=1$;
(iii) *α-Kenmotsu* for $(\alpha,0)$ with α a nonzero constant, in particular, Kenmotsu if $\alpha=1$.
Marrero proved in [9] that a connected trans-Sasakian manifold of dimension ≥ 5 is either cosymplectic or β-Sasakian or α-Kenmotsu.

By a direct computation, from (1.1) we obtain

$$\tilde{\nabla}_X\xi=\alpha[X-\eta(X)\xi]-\beta\phi X \tag{1.2}$$

for any $X\in\Gamma(T\tilde{M})$.

If we denote by $F_{\alpha,\beta}:=\alpha\phi+\beta I$, relations (1.1) and (1.2) can be written as:

$$\tilde{\nabla}_X\phi=(F_{\alpha,\beta}X)^\flat\otimes\xi-\xi^\flat\otimes(F_{\alpha,\beta}X),\ \ \tilde{\nabla}_X\xi=-F_{\alpha,\beta}\phi X$$

for any $X\in\Gamma(T\tilde{M})$, where we denoted by $(\cdot)^\flat$ the dual 1-form of a vector field with respect to g, and we also get $(\tilde{\nabla}_X F_{\alpha,\beta})Y=\alpha(\tilde{\nabla}_X\phi)Y+F_{X(\alpha),X(\beta)}Y$ for any $X,Y\in\Gamma(T\tilde{M})$.

Let M be an immersed submanifold of an almost contact metric manifold $(\tilde{M},\phi,\xi,\eta,g)$. We denote also by g the induced metric on M and by ∇ and $\tilde{\nabla}$ the Levi-Civita connections on M and $\tilde{M}$, respectively. The Gauss and Weingarten equations are:

$$\tilde{\nabla}_XY=\nabla_XY+h(X,Y),\ \ \tilde{\nabla}_XU=-A_UX+\nabla^\perp_XU$$

for any $X,Y\in\Gamma(TM)$ and $U\in\Gamma(T^\perp M)$, where h is the second fundamental form and A is the shape operator, related by $g(h(X,Y),U)=g(A_UX,Y)$.
For any $x\in M$, the tangent space $T_x\tilde{M}$ decomposes into

$$T_x\tilde{M}=T_xM\oplus T_x^\perp M,$$

and, for any $X\in\Gamma(TM)$ and $U\in\Gamma(T^\perp M)$, we write:

$$\phi X=(\phi X)^\top+(\phi X)^\perp=:TX+NX,\ \ \phi U=(\phi U)^\top+(\phi U)^\perp=:tU+nU,$$

where TX, NX and tU, nU denote the tangent and the normal component of ϕX and ϕU, respectively.
If the contact vector field ξ is tangent to M, using (1.2) and Gauss equation, by identifying the tangent and the normal components of $\tilde{\nabla}_X\xi$, we immediately deduce that:

$$\nabla_X\xi=\alpha[X-\eta(X)\xi]-\beta TX,\ \ h(X,\xi)=-\beta NX \tag{1.3}$$

for any $X\in\Gamma(TM)$.

2. Definition and basic properties

Recently, in [8], the notion of pointwise k-slant submanifold, which generalizes the notion of k-slant submanifold defined in the same paper, has been introduced.

We recall that a distribution $D\subset TM$ is called a *pointwise slant distribution* if, at each point $x\in M$, the angle $\theta(x)$ between ϕX_x and D_x is nonzero and independent of the choice of the tangent vector $X_x\in D_x\setminus\{0\}$, but it depends on $x\in M$. In this case, the function θ is called the *slant function*.

Let M be an immersed submanifold of an almost contact metric manifold $(\tilde{M},\phi,\xi,\eta,g)$ such that $\xi\in\Gamma(TM)$.

Definition 2.1. [8] M is called a *pointwise k-slant submanifold of $\tilde{M}$* ($k\in\mathbb{N}^*$) if there exist mutually orthogonal smooth regular distributions $D_0,\dots,D_k$ such that:
(i) $T_xM=(D_0)_x\oplus(D_1)_x\oplus\cdots\oplus(D_k)_x\oplus\langle\xi\rangle_x$ for any $x\in M$;
(ii) $T(D_i)\subseteq D_i$ for any $i\in\{1,\dots,k\}$;
(iii) D_0 is invariant (even trivial) and D_i, $i\in\{1,\dots,k\}$, are nontrivial pointwise slant distributions with slant functions θ_i, $\theta_i(x)\in(0,\frac{\pi}{2}]$ for $x\in M$ and $i\in\{1,\dots,k\}$, which are pointwise distinct (i.e., $\theta_i(x)\neq\theta_j(x)$ for any $x\in M$ and $i\neq j$).

For a more compact notation, we will also denote by θ_0 the null angle, i.e., the "slant" angle of the invariant distribution D_0 (when D_0 is not trivial).

We notice [8] that the condition (ii) in the Definition 2.1 is equivalent to the following condition:
(ii)′ $\phi(D_i) \perp D_j$ for any $i \neq j$, $i, j \in \{1, \ldots, k\}$.
Hence, (i), (ii)′, and (iii) are alternative defining conditions for a pointwise k-slant submanifold of an almost contact metric manifold with tangent contact vector field.

We also remark [7] that the slant functions θ_i are continuous and [8] that, for any $X \in \Gamma(D_i) \setminus \{0\}$ and $x \in M$, the angle $\theta_i(x)$ between ϕX_x and T_xM agrees with the angle between ϕX_x and $(D_i)_x$, and it satisfies:

$$\cos\theta_i(x) \cdot \|\phi X_x\| = \|TX_x\|.$$

If θ_i is constant for any $i \in \{1, \ldots, k\}$, then the submanifold M is called a *k-slant submanifold* [8], and all the results for pointwise k-slant submanifolds are thereby true for k-slant submanifolds.

We notice that under a certain assumption [7], the slant function is smooth. We will need the differentiability condition in Theorem 3.20.

Here we construct new examples of pointwise k-slant submanifold and k-slant submanifold of an almost contact metric manifold.

Example 1. Let $\tilde{M} := \{(x_1, \ldots, x_{4k+3}) \in \mathbb{R}^{4k+3} \ : \ \sum_{i=1}^{4k+3} x_i^2 < 1\}$, $k \geq 2$, where we denote by $(x_1, \ldots, x_{4k+3})$ the canonical coordinates in $\mathbb{R}^{4k+3}$. Consider the natural basis $\{e_1 = \frac{\partial}{\partial x_1}, \ldots, e_{4k+3} = \frac{\partial}{\partial x_{4k+3}}\}$ of $T\tilde{M}$ and define the following elements: a vector field ξ, a 1-form η, and a $(1,1)$-tensor field ϕ by

$$\xi = \frac{\partial}{\partial x_{4k+3}}, \quad \eta = dx_{4k+3},$$

$$\phi e_1 = e_2, \quad \phi e_2 = -e_1,$$

$$\phi e_{4j-1} = \sqrt{\frac{1}{k}\Big(j-1+\sum_{i=4j-1}^{4j+2} x_i^2\Big)} \cdot e_{4j} - \sqrt{1-\frac{1}{k}\Big(j-1+\sum_{i=4j-1}^{4j+2} x_i^2\Big)} \cdot e_{4j+2},$$

$$\phi e_{4j} = -\sqrt{\frac{1}{k}\Big(j-1+\sum_{i=4j-1}^{4j+2} x_i^2\Big)} \cdot e_{4j-1} - \sqrt{1-\frac{1}{k}\Big(j-1+\sum_{i=4j-1}^{4j+2} x_i^2\Big)} \cdot e_{4j+1},$$

$$\phi e_{4j+1} = \sqrt{1-\frac{1}{k}\Big(j-1+\sum_{i=4j-1}^{4j+2} x_i^2\Big)} \cdot e_{4j} + \sqrt{\frac{1}{k}\Big(j-1+\sum_{i=4j-1}^{4j+2} x_i^2\Big)} \cdot e_{4j+2},$$

$$\phi e_{4j+2} = \sqrt{1-\frac{1}{k}\Big(j-1+\sum_{i=4j-1}^{4j+2} x_i^2\Big)} \cdot e_{4j-1} - \sqrt{\frac{1}{k}\Big(j-1+\sum_{i=4j-1}^{4j+2} x_i^2\Big)} \cdot e_{4j+1},$$

$$\phi e_{4k+3} = 0$$

for any $j \in \{1, \ldots, k\}$.

Then, $\tilde{M}$ is an almost contact metric manifold, with the almost contact structure (ϕ, ξ, η) and with the compatible metric g induced by the canonical metric g from $\mathbb{R}^{4k+3}$ given by $g(e_i, e_j) = \delta_{ij}$, $i, j \in \{1, \ldots, 4k+3\}$. The submanifold

$$M := \{(x_1, \ldots, x_{4k+3}) \in \tilde{M} : \ x_{4j+1} = x_{4j+2} = 0, \ j \in \{1, \ldots, k\}\}$$

of $\tilde{M}$ is a pointwise k-slant submanifold, with

$$D_0 = \langle e_1, e_2 \rangle, \qquad D_j = \langle e_{4j-1}, e_{4j} \rangle, \ j \in \{1, \ldots, k\},$$

D_0 an invariant distribution and every D_j, $j \in \{1, \ldots, k\}$, a pointwise slant distribution of slant function

$$\theta_j(x) = \arccos\sqrt{\frac{1}{k}\Big(j-1+\sum_{i=4j-1}^{4j+2} x_i^2\Big)}.$$

Example 2. If in Example 1 we take $\sqrt{\frac{j-1}{k}} \in [0,1)$ instead of $\sqrt{\frac{1}{k}\left(j-1+\sum_{i=4j-1}^{4j+2} x_i^2\right)}$ for each $j \in \{1,\dots,k\}$, then M is a k-slant submanifold of $\tilde{M}$ with the slant angles $\theta_j = \arccos\sqrt{\frac{j-1}{k}}$, $j \in \{1,\dots,k\}$.

If M is a pointwise k-slant submanifold of an almost contact metric manifold $(\tilde{M},\phi,\xi,\eta,g)$ such that ξ is tangent to M, then we have the following decompositions [8] of the tangent and normal bundles of M:

$$TM = \oplus_{i=0}^{k} D_i \oplus \langle\xi\rangle, \;\; T^{\perp}M = \oplus_{i=1}^{k} N(D_i) \oplus H,$$

where $\phi(H) = H$. Denoting by P_i the projection from TM onto D_i, $i \in \{0,\dots,k\}$, by Q_i the projection from $T^{\perp}M$ onto $N(D_i)$, $i \in \{1,\dots,k\}$, and by Q_0 the projection from $T^{\perp}M$ onto H, for any $X \in \Gamma(TM)$ and $U \in \Gamma(T^{\perp}M)$, we have:

$$X = \sum_{i=0}^{k} P_i X + \eta(X)\xi, \;\; U = \sum_{i=0}^{k} Q_i U,$$

and, from the definition, we immediately get (see also [8]) the following two lemmas which we shall later use.

Lemma 2.2. *If M is a pointwise k-slant submanifold of an almost contact metric manifold $(\tilde{M},\phi,\xi,\eta,g)$ with $\xi \in \Gamma(TM)$, then the four operators T, N, t, and n satisfy:*
(i) T and n are g-skew-symmetric, and tN and Nt are g-symmetric, i.e.,

$$g(TX,Y) = -g(X,TY), \;\; g(nU,V) = -g(U,nV),$$

$$g(tNX,Y) = g(X,tNY), \;\; g(NtU,V) = g(U,NtV)$$

for any $X,Y \in \Gamma(TM)$ and $U,V \in \Gamma(T^{\perp}M)$;
(ii)

$$g(tU,tV) = -g(U,NtV), \;\; g(NTX,U) = g(X,TtU),$$

$$g(NX,V) = -g(X,tV), \;\; g(nNX,U) = g(X,tnU)$$

for any $X,Y \in \Gamma(TM)$ and $U,V \in \Gamma(T^{\perp}M)$;
(iii)

$$g(TX,TY) = \sum_{i=0}^{k} \cos^2\theta_i \cdot g(P_iX,P_iY), \;\; g(NX,NY) = \sum_{i=1}^{k} \sin^2\theta_i \cdot g(P_iX,P_iY),$$

$$g(tU,tV) = \sum_{i=0}^{k} \sin^2\theta_i \cdot g(Q_iU,Q_iV), \;\; g(nU,nV) = \sum_{i=1}^{k} \cos^2\theta_i \cdot g(Q_iU,Q_iV)$$

for any $X,Y \in \Gamma(TM)$ and $U,V \in \Gamma(T^{\perp}M)$.

Lemma 2.3. *If M is a pointwise k-slant submanifold of an almost contact metric manifold $(\tilde{M},\phi,\xi,\eta,g)$ with $\xi \in \Gamma(TM)$, then:*

$$T^2 = -\sum_{i=0}^{k} \cos^2\theta_i \cdot P_i, \;\; tN = -\sum_{i=1}^{k} \sin^2\theta_i \cdot P_i, \tag{2.1}$$

$$n^2 = -\sum_{i=0}^{k} \cos^2\theta_i \cdot Q_i, \;\; Nt = -\sum_{i=1}^{k} \sin^2\theta_i \cdot Q_i. \tag{2.2}$$

3. Integrability of the distributions

In the entire section, we will assume that the contact vector field ξ is tangent to the immersed submanifold M of the contact metric manifold $\tilde{M}$. From (1.1), by using the Gauss and Weingarten equations, after identifying the tangent and normal components, we get the following lemma.

Lemma 3.1. *If M is an immersed submanifold of an (α,β)-contact metric manifold $(\tilde{M},\phi,\xi,\eta,g)$, then, for any $X,Y\in\Gamma(TM)$, we have:*
(i) $(\nabla_X T)Y := \nabla_X TY - T(\nabla_X Y) = A_{NY}X + th(X,Y) - \eta(Y)(\alpha TX + \beta X) + g(\alpha TX + \beta X, Y)\xi$;
(ii) $(\nabla_X N)Y := \nabla^{\perp}_X NY - N(\nabla_X Y) = -h(X,TY) + nh(X,Y) - \alpha\eta(Y)NX$.

We shall further consider M a pointwise k-slant submanifold of an almost contact metric manifold $(\tilde{M},\phi,\xi,\eta,g)$ with $TM = \oplus_{i=0}^{k} D_i \oplus \langle\xi\rangle$.

A necessary and sufficient condition for the integrability of the distribution D_i, $i\in\{0,\dots,k\}$, is given in the next theorem.

Theorem 3.2. *Let M be a pointwise k-slant submanifold of an (α,β)-contact metric manifold $(\tilde{M},\phi,\xi,\eta,g)$. Then, the distribution D_i, $i\in\{0,\dots,k\}$, is integrable if and only if we have:*
(i) $g(X,\nabla_Y Z) = g(Y,\nabla_X Z)$ for any $X,Y\in\Gamma(D_i)$ and $Z\in\Gamma(D_j)$, $j\in\{0,\dots,k\}$ with $j\neq i$;
(ii) $\beta(\frac{\pi}{2}-\theta_i)=0$ or D_i is trivial for $i=0$.

Proof. D_i is integrable if and only if $g([X,Y],\xi)=0$ and $g([X,Y],Z)=0$ for any $X,Y\in\Gamma(D_i)$ and $Z\in\Gamma(D_j)$, $j\in\{0,\dots,k\}$ with $j\neq i$. By a direct computation and using (1.3), we get:

$$g([X,Y],\xi) = -g(Y,\nabla_X\xi) + g(X,\nabla_Y\xi) = 2\beta g(TX,Y), \quad g([X,Y],Z) = g(X,\nabla_Y Z) - g(Y,\nabla_X Z).$$

Then, $g([X,Y],\xi)=0$ if and only if $\beta(\frac{\pi}{2}-\theta_i)=0$ or D_i is trivial for $i=0$, and we get the conclusion. □

We can further deduce

Corollary 3.3. *In a pointwise k-slant submanifold of an (α,β)-contact metric manifold, if the invariant distribution D_0 is nontrivial, then D_0 is integrable if and only if $\beta=0$ and (i) from Theorem 3.2 holds for any $X,Y\in\Gamma(D_0)$ and $Z\in\Gamma(D_j)$, $j\in\{1,\dots,k\}$.*

Moreover, if $\beta(x)\neq 0$ for any $x\in M$ (in particular, if $\tilde{M}$ is a β-Sasakian manifold), then the pointwise slant distributions which are not anti-invariant are not integrable.

In particular, we have

Corollary 3.4. *If M is a k-slant submanifold of an (α,β)-contact metric manifold, then the distribution D_i, $i\in\{0,\dots,k\}$, is integrable if and only if we have:*
(i) $g(X,\nabla_Y Z) = g(Y,\nabla_X Z)$ for any $X,Y\in\Gamma(D_i)$ and $Z\in\Gamma(D_j)$, $j\in\{0,\dots,k\}$ with $j\neq i$;
(ii) $\beta=0$ or D_i is an anti-invariant distribution (i.e., $\theta_i=\frac{\pi}{2}$) for $i>0$ or trivial for $i=0$.

Corollary 3.5. *In a k-slant submanifold of an (α,β)-contact metric manifold with β a nonidentically zero function, the nontrivial distributions which are not anti-invariant are not integrable.*

We shall further characterize the integrability of the distributions in terms of the second fundamental form and of the shape operator.

Theorem 3.6. *Let M be a pointwise k-slant submanifold of an (α,β)-contact metric manifold $(\tilde{M},\phi,\xi,\eta,g)$. Then:*
(i) the distribution $D_0\oplus\langle\xi\rangle$ is integrable if and only if

$$h(X,TY) = h(TX,Y)$$

for any $X,Y\in\Gamma(D_0\oplus\langle\xi\rangle)$;
(ii) for $i\in\{0,\dots,k\}$ with $\theta_j(x)\neq\frac{\pi}{2}$ for any $x\in M$ and any $j\neq i$, the distribution $D_i\oplus\langle\xi\rangle$ is integrable if and only if

$$P_j(\nabla_X TY - \nabla_Y TX + A_{NX}Y - A_{NY}X) = 0$$

for any $X,Y\in\Gamma(D_i\oplus\langle\xi\rangle)$ and any $j\in\{0,\dots,k\}$, $j\neq i$.

Proof. From (1.1), we have

$$\tilde{\nabla}_X\phi Y = \phi(\tilde{\nabla}_X Y) + \alpha[g(\phi X,Y)\xi - \eta(Y)\phi X] + \beta[g(X,Y)\xi - \eta(Y)X]$$

for any $X,Y\in\Gamma(TM)$, and, using Gauss and Weingarten formulae, we obtain:

$$\begin{aligned}\nabla_X TY + h(X,TY) &= T(\nabla_X Y) + N(\nabla_X Y) + th(X,Y) + nh(X,Y)\\ &+\alpha[g(TX,Y)\xi - \eta(Y)(TX+NX)] + \beta[g(X,Y)\xi - \eta(Y)X] + A_{NY}X - \nabla^\perp_X NY\\ &= \sum_{i=0}^{k} TP_i(\nabla_X Y) + \sum_{i=1}^{k} NP_i(\nabla_X Y) + th(X,Y) + nh(X,Y)\\ &+\alpha[g(TX,Y)\xi - \eta(Y)(TX+NX)] + \beta[g(X,Y)\xi - \eta(Y)X] + A_{NY}X - \nabla^\perp_X NY.\end{aligned}$$

Identifying the tangent and the normal components in the previous relation, we get:

$$\begin{aligned}\nabla_X TY &= \sum_{i=0}^{k} TP_i(\nabla_X Y) + th(X,Y) + A_{NY}X + \alpha[g(TX,Y)\xi - \eta(Y)TX] + \beta[g(X,Y)\xi - \eta(Y)X],\\ h(X,TY) &= \sum_{i=1}^{k} NP_i(\nabla_X Y) + nh(X,Y) - \nabla^\perp_X NY - \alpha\eta(Y)NX\end{aligned}$$

for any $X, Y \in \Gamma(TM)$.

(i) For any $X, Y \in \Gamma(D_0 \oplus \langle\xi\rangle)$ we have $NX = NY = 0$, and, since h is symmetric, we get

$$h(X,TY) - h(TX,Y) = \sum_{i=1}^{k} NP_i([X,Y]).$$

If the distribution $D_0 \oplus \langle\xi\rangle$ is integrable, then $[X,Y] \in \Gamma(D_0 \oplus \langle\xi\rangle)$, hence $P_i([X,Y]) = 0$ for any $i \in \{1,\dots,k\}$, and we get the conclusion. Conversely, if $h(X,TY) = h(TX,Y)$, then $\sum_{i=1}^{k} NP_i([X,Y]) = 0$, hence $P_i([X,Y]) = 0$ for any $i \in \{1,\dots,k\}$, since $\theta_i \neq 0$, and we get the conclusion.

(ii) For any $X, Y \in \Gamma(D_i \oplus \langle\xi\rangle)$, from Lemma 3.1 and since h is symmetric, we get:

$$\begin{aligned}T[X,Y] &= T(\nabla_X Y) - T(\nabla_Y X)\\ &= \nabla_X TY - A_{NY}X - th(X,Y) + \eta(Y)(\alpha TX + \beta X) - g(\alpha TX + \beta X, Y)\xi\\ &-\nabla_Y TX + A_{NX}Y + th(Y,X) - \eta(X)(\alpha TY + \beta Y) + g(\alpha TY + \beta Y, X)\xi\\ &= \nabla_X TY - \nabla_Y TX + A_{NX}Y - A_{NY}X + \eta(Y)(\alpha TX + \beta X) - \eta(X)(\alpha TY + \beta Y) - 2\alpha g(TX,Y)\xi;\end{aligned}$$

therefore, for any $j \neq i$, we obtain

$$P_j(T[X,Y]) = P_j(\nabla_X TY - \nabla_Y TX + A_{NX}Y - A_{NY}X).$$

If the distribution $D_i \oplus \langle\xi\rangle$ is integrable, then $[X,Y] \in \Gamma(D_i \oplus \langle\xi\rangle)$, hence $T[X,Y] \in \Gamma(D_i)$, and we get the conclusion. Conversely, if $P_j(T[X,Y]) = 0$ for any $j \in \{0,\dots,k\}$, $j \neq i$, then $T[X,Y] \in \Gamma(D_i)$, and since $\theta_j(x) \neq \frac{\pi}{2}$ for any $x \in M$ and any $j \neq i$, we get the conclusion. □

As a consequence, for totally geodesic submanifolds (i.e., for $h = 0$), we have

Corollary 3.7. *If M is a totally geodesic pointwise k-slant submanifold of an (α,β)-contact metric manifold $(\tilde{M}, \phi, \xi, \eta, g)$, then:*

(i) the distribution $D_0 \oplus \langle\xi\rangle$ is integrable;

(ii) for $i \in \{0,\dots,k\}$ with $\theta_j(x) \neq \frac{\pi}{2}$ for any $x \in M$ and any $j \neq i$, the distribution $D_i \oplus \langle\xi\rangle$ is integrable if and only if

$$P_j(\nabla_X TY) = P_j(\nabla_Y TX)$$

for any $X, Y \in \Gamma(D_i \oplus \langle\xi\rangle)$ and any $j \in \{0,\dots,k\}$, $j \neq i$.

From (1.3), we deduce

Proposition 3.8. *If M is a totally geodesic pointwise k-slant submanifold (in particular, k-slant submanifold) of an (α,β)-contact metric manifold, then $\beta = 0$.*

We can further deduce

Corollary 3.9. *There do not exist totally geodesic pointwise k-slant submanifolds of a β-Sasakian manifold with tangent contact vector field.*

Characterization results for a cosymplectic manifold are provided by the next two propositions.

Proposition 3.10. *Let $(\tilde{M}, \phi, \xi, \eta, g)$ be an (α, β)-contact metric manifold. Then, $\tilde{M}$ is a cosymplectic manifold if and only if $\tilde{\nabla}\phi$ is a Codazzi tensor field.*

Proof. From (1.1), we deduce that $(\tilde{\nabla}_X \phi)Y = (\tilde{\nabla}_Y \phi)X$ if and only if

$$\alpha[\eta(Y)\phi X - \eta(X)\phi Y] + \beta[\eta(Y)X - \eta(X)Y] = 2\alpha g(\phi X, Y)\xi$$

for any $X, Y \in \Gamma(T\tilde{M})$. We take $Y = \xi$, and we get $\alpha\phi X + \beta[X - \eta(X)\xi] = 0$ for any $X \in \Gamma(T\tilde{M})$. By applying ϕ, we get $(\alpha^2 + \beta^2)[X - \eta(X)\xi] = 0$ for any $X \in \Gamma(T\tilde{M})$, hence $\alpha = \beta = 0$ (so $\tilde{M}$ is a cosymplectic manifold). The converse implication is trivial. □

Proposition 3.11. *Let $(\tilde{M}, \phi, \xi, \eta, g)$ be an (α, β)-contact metric manifold. Then, the following assertions are equivalent:*
(i) $\tilde{M}$ is a cosymplectic manifold;
(ii) $\tilde{\nabla}\phi^2 = 0$;
(iii) $\tilde{\nabla}\phi^2$ is a Codazzi tensor field.

Proof. We remark that for any $(1, 1)$-tensor field ϕ on $\tilde{M}$, we have

$$(\tilde{\nabla}_X \phi^2)Y = (\tilde{\nabla}_X \phi)\phi Y + \phi((\tilde{\nabla}_X \phi)Y) \tag{3.1}$$

for any $X, Y \in \Gamma(T\tilde{M})$, hence (i) $\Longrightarrow$ (ii) is trivial. Also, (ii) $\Longrightarrow$ (iii) is trivial; therefore, we just have to prove (iii) $\Longrightarrow$ (i).

From (1.1), we obtain

$$(\tilde{\nabla}_X \phi^2)Y = \{\alpha[g(X, Y) - 2\eta(X)\eta(Y)] + \beta g(X, \phi Y)\}\xi + \eta(Y)(\alpha X - \beta\phi X)$$

for any $X, Y \in \Gamma(T\tilde{M})$, hence $(\tilde{\nabla}_X \phi^2)Y = (\tilde{\nabla}_Y \phi^2)X$ if and only if

$$\alpha[\eta(Y)X - \eta(X)Y] - \beta[\eta(Y)\phi X - \eta(X)\phi Y] = 2\beta g(\phi X, Y)\xi$$

for any $X, Y \in \Gamma(T\tilde{M})$. We take $Y = \xi$, and we get $\alpha[X - \eta(X)\xi] - \beta\phi X = 0$ for any $X \in \Gamma(T\tilde{M})$. By applying ϕ, we get $(\alpha^2 + \beta^2)[X - \eta(X)\xi] = 0$ for any $X \in \Gamma(T\tilde{M})$, hence $\alpha = \beta = 0$ (so $\tilde{M}$ is a cosymplectic manifold). □

We can further deduce

Corollary 3.12. *There do not exist α-Kenmotsu or β-Sasakian manifolds with $\tilde{\nabla}\phi$ or $\tilde{\nabla}\phi^2$ Codazzi tensor fields.*

A characterization result for totally geodesic pointwise k-slant submanifolds with T or N parallel tensor w.r.t. ∇ is provided by the next theorem.

Theorem 3.13. *Let M be a totally geodesic pointwise k-slant submanifold of an (α, β)-contact metric manifold $(\tilde{M}, \phi, \xi, \eta, g)$. Then:*
(i) the following assertions are equivalent:
(a) $\tilde{M}$ is a cosymplectic manifold, or, $k = 1$, M is a pointwise slant submanifold, $\alpha(\frac{\pi}{2} - \theta_1) = 0$ and $\beta = 0$;
(b) $\nabla T = 0$;
(c) ∇T is a Codazzi tensor field;
(ii) $\nabla N = 0$ if and only if $\alpha = 0$. In particular, for α, β constants, $\nabla N = 0$ if and only if $\tilde{M}$ is a β-Sasakian or a cosymplectic manifold.

Proof. From Lemma 3.1, if M is totally geodesic, for any $X, Y \in \Gamma(TM)$, we get:

$$\begin{aligned}(\nabla_X T)Y &= -\eta(Y)(\alpha TX + \beta X) + g(\alpha TX + \beta X, Y)\xi, \\ (\nabla_X N)Y &= -\alpha\eta(Y)NX.\end{aligned}$$

(i) If $\tilde{M}$ is cosymplectic, then (a) $\Longrightarrow$ (b) is trivial. If M is a pointwise slant submanifold with slant function θ_1 and $\alpha(\frac{\pi}{2} - \theta_1) = 0$, then $\alpha TX = 0$ for any $X \in \Gamma(TM)$, hence (a) $\Longrightarrow$ (b). Implication (b) $\Longrightarrow$ (c) is trivial, so we just have to prove (c) $\Longrightarrow$ (a).

We notice that $(\nabla_X T)Y = (\nabla_Y T)X$ if and only if

$$\alpha[\eta(Y)TX - \eta(X)TY] + \beta[\eta(Y)X - \eta(X)Y] = 2\alpha g(TX, Y)\xi$$

for any $X, Y \in \Gamma(TM)$. We take $Y = \xi$, and we get $\alpha TX + \beta[X - \eta(X)\xi] = 0$ for any $X \in \Gamma(TM)$. By applying T, we get $(\alpha^2 \cos^2\theta_i + \beta^2)X = 0$ for any $X \in \Gamma(D_i)$, $i > 0$, hence $\beta = 0$ and $\alpha(\frac{\pi}{2} - \theta_i) = 0$; therefore, $\beta = 0$, which implies $\alpha TX = 0$ for any $X \in \Gamma(TM)$. If $\alpha \neq 0$ then D_0 is trivial, $k = 1$ and M is a pointwise slant submanifold. Otherwise, $\alpha = 0$; hence the conclusion.

(ii) If $(\nabla_X N)Y$=0 for any $X, Y \in \Gamma(TM)$, we take $Y = \xi$, and we get $\alpha = 0$, hence the conclusion. Again, the converse implication is trivial. □

We can further deduce

Corollary 3.14. *(i) There do not exist totally geodesic pointwise k-slant submanifolds of an (α, β)-contact metric manifold with $\beta \neq 0$ (in particular, of a β-Sasakian manifold), with ∇T Codazzi tensor field and with tangent contact vector field.*

(ii) There do not exist totally geodesic pointwise k-slant submanifolds of an (α, β)-contact metric manifold with $\alpha \neq 0$ (in particular, of an α-Kenmotsu manifold) satisfying $\nabla N = 0$ with tangent contact vector field.

A characterization result for pointwise k-slant submanifolds with T^2 parallel tensor w.r.t. ∇ is provided by the next theorem.

Theorem 3.15. *Let M be a connected pointwise k-slant submanifold of an (α, β)-contact metric manifold $(\tilde{M}, \phi, \xi, \eta, g)$. Then, $\nabla T^2 = 0$ if and only if either M is an anti-invariant submanifold for $k = 1$, or M is a k-slant submanifold and $\tilde{M}$ is a cosymplectic manifold satisfying*

$$\sum_{j=0}^{k} (\cos^2\theta_j - \cos^2\theta_i) P_i(\nabla_X P_j Y) = 0 \tag{3.2}$$

for any $X, Y \in \Gamma(TM)$ and any $i \in \{0, \ldots, k\}$.

Proof. By using (2.1), we get:

$$\begin{aligned}
(\nabla_X T^2)Y &:= \nabla_X T^2 Y - T^2(\nabla_X Y) \\
&= -\sum_{i=0}^{k} \nabla_X(\cos^2\theta_i \cdot P_i Y) + \sum_{i=0}^{k} \cos^2\theta_i \cdot P_i(\nabla_X Y) \\
&= -\sum_{i=0}^{k} X(\cos^2\theta_i) P_i Y - \sum_{i=0}^{k} \cos^2\theta_i [\nabla_X P_i Y - P_i(\nabla_X Y)].
\end{aligned}$$

Now, from (1.3), we obtain:

$$\begin{aligned}
\nabla_X P_i Y &= \sum_{j=0}^{k} P_j(\nabla_X P_i Y) + \eta(\nabla_X P_i Y)\xi \\
&= \sum_{j=0}^{k} P_j(\nabla_X P_i Y) - g(P_i Y, \nabla_X \xi)\xi \\
&= \sum_{j=0}^{k} P_j(\nabla_X P_i Y) - [\alpha g(X, P_i Y) + \beta g(X, T P_i Y)]\xi
\end{aligned}$$

and

$$\begin{aligned}
P_i(\nabla_X Y) &= P_i\Big(\nabla_X\big(\sum_{j=0}^{k} P_j Y + \eta(Y)\xi\big)\Big) \\
&= \sum_{j=0}^{k} P_i(\nabla_X P_j Y) + X(\eta(Y)) P_i \xi + \eta(Y) P_i(\nabla_X \xi) \\
&= \sum_{j=0}^{k} P_i(\nabla_X P_j Y) + \eta(Y)[\alpha P_i X - \beta P_i(TX)];
\end{aligned}$$

hence,

$$\begin{aligned}(\nabla_X T^2)Y &= -\sum_{i=0}^{k} X(\cos^2\theta_i)P_iY + \sum_{0\le i,j\le k}(\cos^2\theta_i - \cos^2\theta_j)P_i(\nabla_X P_jY)\\ &+\alpha\sum_{i=0}^{k}\cos^2\theta_i\eta(Y)P_iX - \beta\sum_{i=0}^{k}\cos^2\theta_i\eta(Y)P_i(TX) + \sum_{i=0}^{k}\cos^2\theta_i[\alpha g(X,P_iY)+\beta g(X,TP_iY)]\xi.\end{aligned}$$

Taking into account the orthogonality of the distributions, the condition $\nabla T^2 = 0$ is equivalent to:

$$\begin{cases}\sum_{j=0}^{k}(\cos^2\theta_i - \cos^2\theta_j)P_i(\nabla_X P_jY) + \cos^2\theta_i\eta(Y)[\alpha P_iX - \beta P_i(TX)] - X(\cos^2\theta_i)P_iY = 0\\ \sum_{j=0}^{k}\cos^2\theta_j(\alpha P_jY + \beta TP_jY) = 0\end{cases}$$

for any $X, Y \in \Gamma(TM)$ and any $i \in \{0,\dots,k\}$. We take $Y \in \Gamma(D_i)$, and we get:

$$\begin{cases} X(\cos^2\theta_i)Y = 0\\ \cos^2\theta_i(\alpha Y + \beta TY) = 0\end{cases}$$

for any $X \in \Gamma(TM)$ and $Y \in \Gamma(D_i)$, hence θ_i is a constant for any $i \in \{0,\dots,k\}$ (so M is a k-slant submanifold), and, from the second equation, we deduce that either $\theta_i = \frac{\pi}{2}$ or, for D_i nontrivial, $\alpha Y + \beta TY = 0$ for any $Y \in \Gamma(D_i)$. By applying T, we get $(\alpha^2 + \beta^2\cos^2\theta_i)Y = 0$ for any $Y \in \Gamma(D_i)$, hence $\alpha = 0$ and $\beta(\frac{\pi}{2} - \theta_i) = 0$; therefore, if $\theta_i \neq \frac{\pi}{2}$, then $\alpha = \beta = 0$ (so $\tilde{M}$ is a cosymplectic manifold). The converse implication follows immediately. □

We can further deduce

Corollary 3.16. *There do not exist connected pointwise k-slant submanifolds of a non-cosymplectic (α,β)-contact metric manifold satisfying $\nabla T^2 = 0$, with tangent contact vector field, which are not anti-invariant.*

In particular, we have

Corollary 3.17. *There do not exist pointwise k-slant submanifolds of an α-Kenmotsu or of a β-Sasakian manifold satisfying $\nabla T^2 = 0$, with tangent contact vector field, which are not anti-invariant.*

Hence, we recover the analogue result proved by Chen [2] for pointwise slant submanifolds of almost Hermitian manifolds and by Laţcu [8] for pointwise k-slant distributions in the almost Hermitian, almost product, almost contact and almost paracontact metric settings, namely, if T^2 is parallel w.r.t. ∇, then the slant functions of a pointwise k-slant submanifold are constant.

Proposition 3.18. *Let M be a pointwise k-slant submanifold of an (α,β)-contact metric manifold $(\tilde{M},\phi,\xi,\eta,g)$. Then, ∇N is a Codazzi tensor field if and only if, for any $X, Y \in \Gamma(TM)$,*

$$h(TX,Y) - h(X,TY) = \alpha[\eta(Y)NX - \eta(X)NY],$$

or equivalent

$$T(A_UX) + A_U(TX) = \alpha[\eta(X)tU - g(X,tU)\xi] \tag{3.3}$$

for any $X \in \Gamma(TM)$ and $U \in \Gamma(T^\perp M)$. In this case:

(i) $h(TX,Y) = h(X,TY)$ for any $X, Y \in \Gamma(D_i)$, $i \in \{0,\dots,k\}$;

(ii) $h(TX,\xi) = \alpha NX$ and $T(A_U\xi) = \alpha tU$ for any $X \in \Gamma(TM)$ and $U \in \Gamma(T^\perp M)$;

(iii) if, for $i \in \{1,\dots,k\}$, $\theta_i(x) \neq \frac{\pi}{2}$ for any $x \in M$, then $(-\cos^2\theta_i)$ is an eigenfunction of T^2 corresponding to all the eigenvectors $A_UX \neq 0$ for $X \in \Gamma(D_i)$ and $U \in \Gamma(T^\perp M)$ if and only if $\alpha = 0$.

Proof. From Lemma 3.1 and since h is symmetric, we notice that $(\nabla_X N)Y = (\nabla_Y N)X$ if and only if

$$\alpha[\eta(Y)NX - \eta(X)NY] = h(TX,Y) - h(X,TY)$$

for any $X, Y \in \Gamma(TM)$, and we get:

$$\begin{aligned} g(A_U(TX) + T(A_UX), Y) &= g(h(TX,Y) - h(X,TY), U)\\ &= \alpha[\eta(Y)g(NX,U) - \eta(X)g(NY,U)]\\ &= \alpha[-g(\xi,Y)g(X,tU) + \eta(X)g(Y,tU)]\\ &= g(\alpha[-g(X,tU)\xi + \eta(X)tU], Y)\end{aligned}$$

for any $X, Y \in \Gamma(TM)$ and $U \in \Gamma(T^{\perp}M)$. For $X, Y \in \Gamma(D_i)$, we immediately get (i). Now, we take $X = \xi$, and we obtain

$$g(T(A_U\xi), Y) = \alpha g(tU, Y),$$

equivalent to

$$-g(h(TY, \xi), U) = -\alpha g(NY, U);$$

hence, we get (ii).

If we take TX instead of X in (3.3) and we apply T also to (3.3), then we obtain

$$T^2(A_UX) - A_U(T^2X) = \alpha[\eta(X)TtU - g(X, TtU)\xi]$$

for any $X \in \Gamma(TM)$ and $U \in \Gamma(T^{\perp}M)$; hence,

$$\sum_{j=0}^{k} \Big[T^2(A_U(P_jX)) + \cos^2\theta_j \cdot A_U(P_jX)\Big] + \eta(X)T^2(A_U\xi) = \alpha[\eta(X)TtU - g(X, TtU)\xi]$$

for any $X \in \Gamma(TM)$ and $U \in \Gamma(T^{\perp}M)$; therefore,

$$T^2(A_UX) + \cos^2\theta_i \cdot A_UX = -\alpha g(NTX, U)\xi$$

for any $X \in \Gamma(D_i)$ and $U \in \Gamma(T^{\perp}M)$, hence the conclusion. □

Now, we shall characterize the integrability of the distributions in terms of ∇T and ∇T^2, respectively.

Proposition 3.19. *Let M be a pointwise k-slant submanifold of an (α, β)-contact metric manifold $(\tilde{M}, \phi, \xi, \eta, g)$ such that ∇T is a Codazzi tensor field. Then:*

(i) for any $X, Y \in \Gamma(D_i)$, we have $T(A_{NX}Y) = T(A_{NY}X)$ and $A_{NX}\xi \in \Gamma(D_i)$;

(ii) if $\beta(x) \neq 0$ for any $x \in M$, the distribution $D_i \oplus \langle\xi\rangle$ is integrable if and only if $A_{N[X,Y]}\xi \in \Gamma(D_i)$ for any $X, Y \in \Gamma(D_i \oplus \langle\xi\rangle)$.

Proof. From Lemma 3.1, we notice that, for any $X, Y \in \Gamma(TM)$, $(\nabla_X T)Y = (\nabla_Y T)X$ if and only if

$$\alpha[\eta(Y)TX - \eta(X)TY] + \beta[\eta(Y)X - \eta(X)Y] - 2\alpha g(TX, Y)\xi = A_{NY}X - A_{NX}Y,$$

which implies $A_{NX}\xi = -\alpha TX - \beta X + \beta\eta(X)\xi$. Also, for any $X, Y \in \Gamma(D_i)$, we have $A_{NX}Y - A_{NY}X = 2\alpha g(TX, Y)\xi$; hence, we get (i).

Now, for any $X, Y \in \Gamma(TM)$, we get $A_{N[X,Y]}\xi = -\alpha T[X, Y] - \beta[X, Y] + \beta\eta([X, Y])\xi$; therefore, if $D_i \oplus \langle\xi\rangle$ is integrable, then, for any $X, Y \in \Gamma(D_i \oplus \langle\xi\rangle)$, we have $[X, Y] \in \Gamma(D_i \oplus \langle\xi\rangle)$, $T[X, Y] \in \Gamma(D_i)$, hence $A_{N[X,Y]}\xi \in \Gamma(D_i)$. Conversely, if $A_{N[X,Y]}\xi \in \Gamma(D_i)$ for any $X, Y \in \Gamma(D_i \oplus \langle\xi\rangle)$, then

$$\alpha \sum_{j=0}^{k} TP_j([X, Y]) + \beta \sum_{j=0}^{k} P_j([X, Y]) \in \Gamma(D_i);$$

hence, $\alpha TP_j([X, Y]) + \beta P_j([X, Y]) = 0$ for any $j \neq i$. By applying T, we get $(\alpha^2\cos^2\theta_j + \beta^2)P_j([X, Y]) = 0$ for any $j \neq i$, hence $[X, Y] \in \Gamma(D_i \oplus \langle\xi\rangle)$, and we get the conclusion. □

Theorem 3.20. *Let M be a pointwise k-slant submanifold of an almost contact metric manifold $(\tilde{M}, \phi, \xi, \eta, g)$. If $\theta_i(x) \neq \frac{\pi}{2}$ for any $x \in M$, then the distribution D_i is integrable and $X(\theta_i)Y = Y(\theta_i)X$ for any X, $Y \in \Gamma(D_i)$ if and only if ∇T^2 is a Codazzi tensor field on D_i.*

Proof. Since $\theta_i(x) \neq \frac{\pi}{2}$ for any $x \in M$, the slant function θ_i is smooth (see [7]). For any X, $Y \in \Gamma(TM)$, we have

$$\begin{aligned}(\nabla_X T^2)Y - (\nabla_Y T^2)X &= \sum_{j=0}^{k} \sin(2\theta_j)[X(\theta_j)P_jY - Y(\theta_j)P_jX] \\ &\quad - \sum_{j=0}^{k} \cos^2\theta_j[(\nabla_X P_jY - \nabla_Y P_jX) - P_j(\nabla_X Y - \nabla_Y X)].\end{aligned}$$

Taking into account that

$$\nabla_X P_l Y = \sum_{j=0}^{k} P_j(\nabla_X P_l Y) + \eta(\nabla_X P_l Y)\xi,$$

the above relation becomes

$$\begin{aligned}(\nabla_X T^2)Y - (\nabla_Y T^2)X &= \sum_{j=0}^{k} P_j\Big(\sin(2\theta_j)[X(\theta_j)Y - Y(\theta_j)X]\Big)\\ &\quad - \sum_{j=0}^{k} P_j\Big(\sum_{l=0}^{k}\cos^2\theta_l(\nabla_X P_l Y - \nabla_Y P_l X)\Big) + \sum_{j=0}^{k} P_j\Big(\cos^2\theta_j(\nabla_X Y - \nabla_Y X)\Big)\\ &\quad - \sum_{j=0}^{k}\cos^2\theta_j \eta(\nabla_X P_j Y - \nabla_Y P_j X)\xi.\end{aligned}$$

If we take $X, Y \in \Gamma(D_i)$, we obtain

$$\begin{aligned}(\nabla_X T^2)Y - (\nabla_Y T^2)X &= \sin(2\theta_i)[X(\theta_i)Y - Y(\theta_i)X]\\ &\quad - \sum_{j=0}^{k} P_j\Big(\cos^2\theta_i(\nabla_X Y - \nabla_Y X)\Big) + \sum_{j=0}^{k} P_j\Big(\cos^2\theta_j(\nabla_X Y - \nabla_Y X)\Big)\\ &\quad - \cos^2\theta_i \eta(\nabla_X Y - \nabla_Y X)\xi,\end{aligned}$$

and we deduce that ∇T^2 is a Codazzi tensor field on D_i if and only if

$$\begin{cases}\sin(2\theta_i)[X(\theta_i)Y - Y(\theta_i)X] = 0\\ (\cos^2\theta_j - \cos^2\theta_i)P_j([X,Y]) = 0 \text{ for any } j \neq i\\ \cos^2\theta_i \eta([X,Y]) = 0\end{cases};$$

hence, under the hypotheses and because θ_i and θ_j are pointwise distinct for $i \neq j$, we get the conclusion. □

Remark 3.21. A geometric interpretation of the condition from Theorem 3.20 is the following: if the slant function θ_i satisfies $X(\theta_i)Y = Y(\theta_i)X$ for any X, $Y \in \Gamma(D_i)$, and there exists $X_0 \in \Gamma(D_i)$ orthogonal to $\nabla\theta_i$ such that $X_0(x) \neq 0$ for any $x \in M$, then the gradient of θ_i is orthogonal to D_i.

In particular, we have

Corollary 3.22. *If M is a k-slant submanifold of an almost contact metric manifold, then, for $\theta_i \neq \frac{\pi}{2}$, the distribution D_i is integrable if and only if ∇T^2 is a Codazzi tensor field on D_i.*

Also, from Theorem 3.20, for the invariant distribution of TM, we deduce

Proposition 3.23. *Let M be a pointwise k-slant submanifold of an almost contact metric manifold. Then, the invariant distribution D_0 is integrable if and only if ∇T^2 is a Codazzi tensor field on D_0.*

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