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Inextensible Flows of Space Curves According to a New Orthogonal Frame with Curvature in $\mathbb{E}_1^3$

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ABSTRACT

The aim of this paper is to calculate inextensible curve flow using a new orthogonal frame in three-dimensional Minkowski space. First, the history of the subject the main geometric results are presented. Then we obtain inextensible flow of Frenet frame and curvatures. Last, the necessary and sufficient conditions for inextensible curve flow are given by a partial differential equation (PDE) which is involve curvature.

Keywords: *Inextensible flow, Frenet-Serret frame, modified orthogonal frame, Minkowski 3-Space.*
AMS Subject Classification (2020): *Primary:53C44 ; Secondary: 53A35; 51B20.*

1. Introduction

Numerous applications of curve theory can be found in mathematics, physics, and engineering [3, 7]. External influences were largely disregarded in most of these investigations. However, this atypical temporal parameter, or external influences, play a significant role in more recent investigations. This has increased the significance of curve flow, or the way that curves change over time [1, 10, 14, 16]. If the arclength of a curve is protected, the flow is inextensible. First, Kwon and Park investigated developable surfaces and inextensible flows of space curves in E^3 [8, 9]. They are followed by a study of the inextensible flow of space curves and surfaces [1, 5, 10, 13, 14, 15]. Sasai introduced the modified orthogonal frame in 1984 [12]. In this work, new formulation for the flows of inextensible curves in 3-dimensional minkowski space are presented. For an inextensible curve flow, we provide the required and sufficient conditions, which are denoted by a PDE that includes the curvatures and torsion. Our anticipate that mathematicians who specialize in mathematical modeling will find these results useful.

2. Preliminaries

We offer a brief overview of the modified orthogonal frame with curvature (MOFC) in $\mathbb{E}_1^3$ in this section. The three dimensional Minkowskian space $\mathbb{E}_1^3$ is the real vector space R^3 provided with the standard flat metric given by

$$\langle , \rangle = -dx_1^2 + dx_2^2 + dx_3^2$$

where (x_1, x_2, x_3) cartesian coordinate system of three dimensional Minkowski space $\mathbb{E}_1^3$. A vector v in $\mathbb{E}_1^3$ is called spacelike vector, timelike vector or null vector if $\langle v, v\rangle > 0$ or $v = 0$, if $\langle v, v\rangle < 0$ and if $\langle v, v\rangle = 0$ and $v \neq 0$, respectively. The same holds is true for curves in Minkowski 3-space. When $F(s)$ is taken as a curve in three-dimensional Minkowski space, which can locally be timelike, spacelike, or null (lightlike), if each of its vector of velocity $F'(s) = \overrightarrow{t}$ spacelike, timelike, or null (lightlike). Let's assume that the moving Frenet frame along the

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curve $F(s)$ in three-dimensional Minkowski space is $\left\{\vec{t}, \vec{n}, \vec{b}\right\}$. The Frenet formulas for an arbitrary curve $F(s)$ Minkowskian 3-space are presented in detail in [4, 16]. If F is a timelike curve, then Frenet derivative formula is given by

$$\begin{bmatrix} \vec{t}' \\ \vec{n}' \\ \vec{b}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} . \begin{bmatrix} \vec{t} \\ \vec{n} \\ \vec{b} \end{bmatrix} .$$

Unless otherwise stated, κ and τ will be taken as the curvature and torsion of the space curve F in $\mathbb{E}_1^3$, respectively. The arclength parameter for F is s. The relationships listed below are particularly valid:

$$\langle \vec{t}, \vec{t} \rangle = -1,\ \langle \vec{n}, \vec{n} \rangle = \langle \vec{b}, \vec{b} \rangle = 1,\ \langle \vec{t}, \vec{n} \rangle = \langle \vec{t}, \vec{b} \rangle = \langle \vec{n}, \vec{b} \rangle = 0.$$

If F is a spacelike curve with a spacelike principal normal, then Frenet derivative formula is given by $\vec{n}$,

$$\begin{bmatrix} \vec{t}' \\ \vec{n}' \\ \vec{b}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix} . \begin{bmatrix} \vec{t} \\ \vec{n} \\ \vec{b} \end{bmatrix} ,$$

where

$$\langle \vec{t}, \vec{t} \rangle = \langle \vec{n}, \vec{n} \rangle = 1,\ \langle \vec{b}, \vec{b} \rangle = -1,\ \langle \vec{t}, \vec{n} \rangle = \langle \vec{t}, \vec{b} \rangle = \langle \vec{n}, \vec{b} \rangle = 0.$$

If F is a spacelike curve with a spacelike principal binormal, then Frenet derivative formula is given by $\vec{b}$,

$$\begin{bmatrix} \vec{t}' \\ \vec{n}' \\ \vec{b}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix} . \begin{bmatrix} \vec{t} \\ \vec{n} \\ \vec{b} \end{bmatrix} ,$$

where

$$\langle \vec{t}, \vec{t} \rangle = \langle \vec{b}, \vec{b} \rangle = 1,\ \langle \vec{n}, \vec{n} \rangle = -1,\ \langle \vec{t}, \vec{n} \rangle = \langle \vec{t}, \vec{b} \rangle = \langle \vec{n}, \vec{b} \rangle = 0.$$

If F is a pseudo null curve, then Frenet derivative formulas have the following form .

$$\begin{bmatrix} \vec{t}' \\ \vec{n}' \\ \vec{b}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ 0 & \tau & 0 \\ -\kappa & 0 & -\tau \end{bmatrix} . \begin{bmatrix} \vec{t} \\ \vec{n} \\ \vec{b} \end{bmatrix} . \tag{2.1}$$

In (2.1) the first curvature $\kappa = 0$ if and only if F is straight line, or $\kappa = 1$ in all other circumstances. In this instance, the following criteria are met.

$$\langle \vec{t}, \vec{t} \rangle = \langle \vec{n}, \vec{b} \rangle = 1,\ \langle \vec{n}, \vec{n} \rangle = \langle \vec{t}, \vec{n} \rangle = \langle \vec{t}, \vec{b} \rangle = \langle \vec{b}, \vec{b} \rangle = 0.$$

in [16]. Let F be a unit-speed curve in $\mathbb{E}_1^3$ and let F $\left\{\vec{t}, \vec{n}, \vec{b}\right\}$ denote the Frenet frame along the curve. The relations between the well-known classical Frenet frame $\left\{\vec{t}, \vec{n}, \vec{b}\right\}$ and orthogonal frame $\left\{\vec{T}, \vec{N}, \vec{B}\right\}$ at $\kappa \neq 0$ are

$$\vec{T} = \vec{t}\ , \vec{N} = \kappa\, \vec{n}\ ,\ \vec{B} = \kappa\, \vec{b}$$

The subsequent modified orthogonal frames are valid in this situation: If F is timelike curve, then the derivative formulas of orthogonal Frenet frame is

$$\begin{bmatrix} \vec{T}' \\ \vec{N}' \\ \vec{B}' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \kappa^2 & \frac{\kappa'}{\kappa} & \tau \\ 0 & -\tau & \frac{\kappa'}{\kappa} \end{bmatrix} \begin{bmatrix} \vec{T} \\ \vec{N} \\ \vec{B} \end{bmatrix} . \tag{2.2}$$

The derivative formulas are written as in the following form if F is a spacelike curve and F have a spacelike principal normal $\vec{n}$ then the derivative formulas of orthogonal Frenet frame is

$$\begin{bmatrix} \vec{T}' \\ \vec{N}' \\ \vec{B}' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\kappa^2 & \frac{\kappa'}{\kappa} & \tau \\ 0 & \tau & \frac{\kappa'}{\kappa} \end{bmatrix} \begin{bmatrix} \vec{T} \\ \vec{N} \\ \vec{B} \end{bmatrix} . \tag{2.3}$$

If F is a spacelike curve and F have a spacelike binormal $\overrightarrow{b}$, then the following derivative formulas are given by;

$$\begin{bmatrix} \overrightarrow{T}' \\ \overrightarrow{N}' \\ \overrightarrow{B}' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \kappa^2 & \frac{\kappa'}{\kappa} & \tau \\ 0 & \tau & \frac{\kappa'}{\kappa} \end{bmatrix} \begin{bmatrix} \overrightarrow{T} \\ \overrightarrow{N} \\ \overrightarrow{B} \end{bmatrix}. \tag{2.4}$$

If F is pseudo null curve, the derivative formulas of orthogonal Frenet frame are given by

$$\begin{bmatrix} \overrightarrow{T}' \\ \overrightarrow{N}' \\ \overrightarrow{B}' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & \frac{\kappa'}{\kappa} + \tau & 0 \\ -\kappa^2 & 0 & \frac{\kappa'}{\kappa} - \tau \end{bmatrix} \begin{bmatrix} \overrightarrow{T} \\ \overrightarrow{N} \\ \overrightarrow{B} \end{bmatrix} \tag{2.5}$$

in [2, 6, 11, 12].

3. Inextensible flows of curve in modified orthogonal frame with curvature

Throughout this work, unless otherwise stated, we will give the $F(u,t)$ transformation as follows;

$$F : [0,l] \times [0,\omega) \to E^3$$
$$(u,t) \to F(u,t),$$

is a class of differentiable curves with one parameter on the MOFC in $\mathbb{E}_1^3$, where l is the arc-length of the initial curve. Now assume that u be parameter ($0 \leq u \leq l$) of a space curve. If velocity vector of F is given by $v = \left|\frac{\partial F}{\partial u}\right|$, then the arc-length of curve F is

$$S(u) = \int\limits_0^u \left\| \frac{\partial F}{\partial u} \right\| du.$$

where $\left|\frac{\partial F}{\partial u}\right| = \left|\langle \frac{\partial F}{\partial u}, \frac{\partial F}{\partial u} \rangle\right|^{1/2}$. The operator $\frac{\partial}{\partial s}$ is given in the form below, depending on u

$$\frac{\partial}{\partial s} = \frac{1}{v} \frac{\partial}{\partial u}.$$

The parameter of arc-length is $ds = vdu$. A flow of F with MOF$\left\{\overrightarrow{T}, \overrightarrow{N}, \overrightarrow{B}\right\}$ in $\mathbb{E}_1^3$ can be express as

$$\frac{\partial F}{\partial t} = f_1 \overrightarrow{T} + f_2 \overrightarrow{N} + f_3 \overrightarrow{B}.$$

Let $S(u,t) = \int\limits_0^u v du$ be the arc-length variation of curve F in $\mathbb{E}_1^3$. The requirement that the curve not be dependent to any extension or compression can be phrased by the condition $\int\limits_0^u S(u,t) = \int\limits_0^u \frac{\partial v}{\partial s} du = 0$, for all $u \in [0,l]$. The necessary and sufficient conditions for inextensible flow according to timelike, spacelike or ligthlike of curve in $\mathbb{E}_1^3$ have been given the following theorems.

3.1. *Inextensible flows of timelike curve in a new orthogonal frame with curvature*

Definition 3.1. A timelike curve flow $\frac{\partial F}{\partial t}$ on the MOFC in $\mathbb{E}_1^3$ are called as inextensible if

$$\frac{\partial}{\partial t} \left| \frac{\partial F}{\partial u} \right| = 0.$$

Next theorem states the necessary and enough situations for the inextensible flow of timelike curves in $\mathbb{E}_1^3$.

Theorem 3.1. *Suppose that $\frac{\partial F}{\partial t} = f_1 \overrightarrow{T} + f_2 \overrightarrow{N} + f_3 \overrightarrow{B}$ be a regular and differentiable flow of the timelike curve F with curvature on the MOF $\left\{\overrightarrow{T}, \overrightarrow{N}, \overrightarrow{B}\right\}$in 3-dimensional Minkowski space.The flow is inextensible necessary and sufficient condition*

$$\frac{\partial f_1}{\partial s} = -\kappa^2 f_2.$$

Proof. We have equation (2.2) and $\langle \frac{\partial F}{\partial u}, \frac{\partial F}{\partial u} \rangle = v^2$. Because parameters u and t are independent of coordinates, the operators $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial t}$ are commute and we can write

$$\begin{aligned}
2v\frac{\partial v}{\partial t} &= \frac{\partial}{\partial t}\langle \frac{\partial F}{\partial u}, \frac{\partial F}{\partial u} \rangle \\
&= 2\langle \frac{\partial F}{\partial u}, \frac{\partial}{\partial u}(\frac{\partial F}{\partial t}) \rangle \\
&= 2\langle \frac{\partial F}{\partial u}, \frac{\partial}{\partial u}(f_1\overrightarrow{T} + f_2\overrightarrow{N} + f_3\overrightarrow{B}) \rangle \\
&= 2v\langle \overrightarrow{T}, \frac{\partial}{\partial u}(f_1)\overrightarrow{T} + vf_1\overrightarrow{N} + \frac{\partial}{\partial u}(f_2)\overrightarrow{N} + vf_2(\kappa^2\overrightarrow{T} + \frac{\kappa'}{\kappa}\overrightarrow{N} + \tau\overrightarrow{B}) \rangle \\
&\quad + 2v\langle \overrightarrow{T}, \frac{\partial}{\partial u}(f_3)\overrightarrow{B} + vf_3(-\tau\overrightarrow{N} + \frac{\kappa'}{\kappa}\overrightarrow{B}) \rangle \\
&= 2v\ (\frac{\partial}{\partial u}(f_1) + \kappa^2 f_2 v).
\end{aligned}$$

Thus we hold

$$\frac{\partial v}{\partial t} = (\frac{\partial}{\partial u}(f_1) + \kappa^2 f_2 v). \tag{3.1}$$

Now assume that $\frac{\partial F}{\partial t}$ be extensible. From the eq.(3.1) we get

$$\begin{aligned}
\frac{\partial}{\partial t}S(u,t) &= \int_0^u \frac{\partial v}{\partial t}du \\
&= \int_0^u (\frac{\partial}{\partial u}(f_1) + \kappa^2 f_2 v)du \\
&= 0.
\end{aligned}$$

□

For all $u \in [0, l]$ This mean that $\frac{\partial}{\partial u}(f_1) = -\kappa^2 f_2 v$ or $\frac{\partial}{\partial s}(f_1) = -\kappa^2 f_2$. The other part of the proof of theorem can be completed in a similar steps. Now, we are limited to curves with arclength parameters. Lets we take $v = 1$ and the local coordinate u corresponds to s. By a parameter change, the following theorem can be given

Theorem 3.2. *Let $\frac{\partial F}{\partial t} = f_1\overrightarrow{T} + f_2\overrightarrow{N} + f_3\overrightarrow{B}$ be the smooth flow of the 3-dimensional Minkowski space timelike curve F on MOF $\left\{\overrightarrow{T}, \overrightarrow{N}, \overrightarrow{B}\right\}$ .The form of derivative formulas are given as in the following:*

$$\begin{aligned}
\frac{\partial \overrightarrow{T}}{\partial t} &= (f_1 + \frac{\partial}{\partial s}(f_2) + \frac{\kappa'}{\kappa}f_2 - \tau f_3)\overrightarrow{N} + (\tau f_2 + \frac{\partial}{\partial s}(f_3) + \frac{\kappa'}{\kappa}f_3)\overrightarrow{B}, \\
\frac{\partial \overrightarrow{N}}{\partial t} &= \kappa^2(f_1 + \frac{\partial}{\partial s}(f_2) + \frac{\kappa'}{\kappa}f_2 - \tau f_3)\overrightarrow{T} + \frac{1}{\kappa^2}\lambda\overrightarrow{B}, \\
\frac{\partial \overrightarrow{B}}{\partial t} &= \kappa^2(\tau f_2 + \frac{\partial}{\partial s}(f_3) + \frac{\kappa'}{\kappa}f_3)\overrightarrow{T} - \frac{1}{\kappa^2}\lambda\overrightarrow{N},
\end{aligned}$$

where $\langle \frac{\partial \overrightarrow{N}}{\partial t}, \overrightarrow{B} \rangle = \lambda$.

Proof. Using Theorem 3.1 and the MOFC in $\mathbb{E}_1^3$, we calculate

$$\begin{aligned}
\frac{\partial \overrightarrow{T}}{\partial t} =& \frac{\partial}{\partial t}\frac{\partial F}{\partial s}\\
=& \frac{\partial}{\partial s}(f_1\overrightarrow{T} + f_2\overrightarrow{N} + f_3\overrightarrow{B})\\
=& \frac{\partial}{\partial s}(f_1)\overrightarrow{T} + f_1\overrightarrow{N} + \frac{\partial}{\partial s}(f_2)\overrightarrow{N} + f_2(\kappa^2\overrightarrow{T} + \frac{\kappa^{'}}{\kappa}\overrightarrow{N} + \tau\overrightarrow{B})\frac{\partial}{\partial s}(f_3)\overrightarrow{B}\\
&+ f_3(-\tau\overrightarrow{N} + \frac{\kappa^{'}}{\kappa}\overrightarrow{B})\\
=& (f_1 + \frac{\partial}{\partial s}(f_2) + \frac{\kappa^{'}}{\kappa}f_2 - \tau f_3)\overrightarrow{N} + (\tau f_2 + \frac{\partial}{\partial s}(f_3) + \frac{\kappa^{'}}{\kappa}f_3)\overrightarrow{B}.
\end{aligned}$$

Now differentiating the MOFC according to t;

$$\begin{cases}
\frac{\partial}{\partial t}\langle\overrightarrow{T},\overrightarrow{N}\rangle & = 0 = \kappa^2(f_1 + \frac{\partial}{\partial s}(f_2) + \frac{\kappa^{'}}{\kappa}f_2 - \tau f_3) + \langle\overrightarrow{T}, \frac{\partial\overrightarrow{N}}{\partial t}\rangle,\\
\frac{\partial}{\partial t}\langle\overrightarrow{T},\overrightarrow{B}\rangle & = 0 = \kappa^2(\tau f_2 + \frac{\partial}{\partial s}(f_3) + \frac{\kappa^{'}}{\kappa}f_3) + \langle\overrightarrow{T}, \frac{\partial\overrightarrow{B}}{\partial t}\rangle,\\
\frac{\partial}{\partial t}\langle\overrightarrow{N},\overrightarrow{B}\rangle & = 0 = \lambda + \langle\overrightarrow{N}, \frac{\partial\overrightarrow{B}}{\partial t}\rangle.
\end{cases} \tag{3.2}$$

From the equation (3.2) we obtain

$$\begin{aligned}
\frac{\partial\overrightarrow{N}}{\partial t} =& \kappa^2(f_1 + \frac{\partial}{\partial s}(f_2) + \frac{\kappa^{'}}{\kappa}f_2 - \tau f_3)\overrightarrow{T} + \frac{1}{\kappa^2}\lambda\overrightarrow{B},\\
\frac{\partial\overrightarrow{B}}{\partial t} =& \kappa^2(\tau f_2 + \frac{\partial}{\partial s}(f_3) + \frac{\kappa^{'}}{\kappa}f_3)\overrightarrow{T} - \frac{1}{\kappa^2}\lambda\overrightarrow{N},
\end{aligned}$$

where $\langle\frac{\partial\overrightarrow{N}}{\partial t}, \overrightarrow{B}\rangle = \lambda$. □

The next theorem states the circumstances on PDE involving the (κ) and (τ) for flow $F(s,t)$ to be inextensible.

Theorem 3.3. *Assume that the curve flow $\frac{\partial F}{\partial t} = f_1\overrightarrow{T} + f_2\overrightarrow{N} + f_3\overrightarrow{B}$ on the MOFC in 3-dimensional Minkowski space is inextensible. Then the next equations obtained:*

$$\begin{aligned}
\frac{\partial\kappa^2}{\partial t} =& \frac{\partial}{\partial s}(\kappa^2 f_1) + \frac{\partial}{\partial s}(\kappa^2\frac{\partial}{\partial s}(f_2)) + \frac{\partial}{\partial s}(\kappa\kappa^{'}(f_2)) - \frac{\partial}{\partial s}(\kappa^2\tau f_3)\\
&- \kappa\kappa^{'}(f_1 + \frac{\partial}{\partial s}(f_2) + \frac{\kappa^{'}}{\kappa}f_2 - \tau f_3) - \kappa^2\tau(\tau f_2 + \frac{\partial}{\partial s}(f_3) + \frac{\kappa^{'}}{\kappa}f_3),\\
\frac{\partial\tau}{\partial t} =& \lambda\frac{\partial}{\partial s}(\frac{1}{\kappa^2}) + \frac{1}{\kappa^2}\frac{\partial}{\partial s}(\lambda) - \kappa^2(\tau f_2 + \frac{\partial}{\partial s}(f_3) + \frac{\kappa^{'}}{\kappa}f_3),
\end{aligned}$$

where

$$\begin{aligned}
\lambda =& \kappa^2\tau(f_1 + \frac{\partial}{\partial s}(f_2) + \frac{\kappa^{'}}{\kappa}f_2 - \tau f_3) + \kappa^2(\frac{\partial}{\partial s}(\tau f_2) + \frac{\partial^2}{\partial s^2}(f_3) + \frac{\partial}{\partial s}(\frac{\kappa^{'}}{\kappa}f_3))\\
&+ \kappa\kappa^{'}(\tau f_2 + \frac{\partial}{\partial s}(f_3) + \frac{\kappa^{'}}{\kappa}f_3)
\end{aligned}$$

Proof. Noting that $\frac{\partial}{\partial s}\frac{\partial \overrightarrow{T}}{\partial t} = \frac{\partial}{\partial t}\frac{\partial \overrightarrow{T}}{\partial s}$, Differentiating $\frac{\partial \overrightarrow{T}}{\partial t}$ according to s, we get

$$\begin{aligned}
\frac{\partial}{\partial s}\frac{\partial \overrightarrow{T}}{\partial t} =& \frac{\partial}{\partial s}[(f_1 + \frac{\partial}{\partial s}(f_2) + \frac{\kappa^{'}}{\kappa}f_2 - \tau f_3)\overrightarrow{N} + (\tau f_2 + \frac{\partial}{\partial s}(f_3) + \frac{\kappa^{'}}{\kappa}f_3)\overrightarrow{B}] \\
=& (\frac{\partial}{\partial s}(f_1) + \frac{\partial^2}{\partial s^2}(f_2) + \frac{\partial}{\partial s}(\frac{\kappa^{'}}{\kappa}f_2) - \frac{\partial}{\partial s}(\tau f_3))\overrightarrow{N} \\
&+ (f_1 + \frac{\partial}{\partial s}(f_2) + \frac{\kappa^{'}}{\kappa}f_2 - \tau f_3)(\kappa^2\overrightarrow{T} + \frac{\kappa^{'}}{\kappa}\overrightarrow{N} + \tau\overrightarrow{B}) \\
&+ (\frac{\partial}{\partial s}(\tau f_2) + \frac{\partial^2}{\partial s^2}(f_3) + \frac{\partial}{\partial s}(\frac{\kappa^{'}}{\kappa}f_3))\overrightarrow{B} \\
&+ (\tau f_2 + \frac{\partial}{\partial s}(f_3) + \frac{\kappa^{'}}{\kappa}f_3)(-\tau\overrightarrow{N} + \frac{\kappa^{'}}{\kappa}\overrightarrow{B}) \\
=& ((\kappa^2)(f_1 + \frac{\partial}{\partial s}(f_2) + \frac{\kappa^{'}}{\kappa}f_2 - \tau f_3))\overrightarrow{T} \\
&+ (\frac{\partial}{\partial s}(f_1) + \frac{\partial^2}{\partial s^2}(f_2) + \frac{\partial}{\partial s}(\frac{\kappa^{'}}{\kappa}f_2) - \frac{\partial}{\partial s}(\tau f_3)\overrightarrow{N} \\
&+ (\frac{\kappa^{'}}{\kappa}(f_1 + \frac{\partial}{\partial s}(f_2) + \frac{\kappa^{'}}{\kappa}f_2 - \tau f_3) - \tau(\tau f_2 + \frac{\partial}{\partial s}(f_3) + \frac{\kappa^{'}}{\kappa}f_3))\overrightarrow{N} \\
&+ (\tau(f_1 + \frac{\partial}{\partial s}(f_2) + \frac{\kappa^{'}}{\kappa}f_2 - \tau f_3))\overrightarrow{B} \\
&+ (\frac{\partial}{\partial s}(\tau f_2) + \frac{\partial^2}{\partial s^2}(f_3) + \frac{\partial}{\partial s}(\frac{\kappa^{'}}{\kappa}f_3))\overrightarrow{B} + \frac{\kappa^{'}}{\kappa}(\tau f_2 + \frac{\partial}{\partial s}(f_3) + \frac{\kappa^{'}}{\kappa}f_3))\overrightarrow{B}
\end{aligned}$$

and while

$$\frac{\partial}{\partial t}\frac{\partial \overrightarrow{T}}{\partial s} = \frac{\partial}{\partial t}\overrightarrow{N} = \kappa^2(f_1 + \frac{\partial}{\partial s}(f_2) + \frac{\kappa^{'}}{\kappa}f_2 - \tau f_3)\overrightarrow{T} + \frac{1}{\kappa^2}\lambda\overrightarrow{B},$$

thus

$$\begin{aligned}
\lambda =& \kappa^2\tau(f_1 + \frac{\partial}{\partial s}(f_2) + \frac{\kappa^{'}}{\kappa}f_2 - \tau f_3) + \kappa^2(\frac{\partial}{\partial s}(\tau f_2) + \frac{\partial^2}{\partial s^2}(f_3) + \frac{\partial}{\partial s}(\frac{\kappa^{'}}{\kappa}f_3)) \\
&+ \kappa\kappa^{'}(\tau f_2 + \frac{\partial}{\partial s}(f_3) + \frac{\kappa^{'}}{\kappa}f_3)
\end{aligned}$$

Since $\frac{\partial}{\partial s}\frac{\partial \overrightarrow{N}}{\partial t} = \frac{\partial}{\partial t}\frac{\partial \overrightarrow{N}}{\partial s}$, we have

$$\begin{aligned}
\frac{\partial}{\partial s}\frac{\partial \overrightarrow{N}}{\partial t} =& \frac{\partial}{\partial s}[(\kappa^2 f_1 + \kappa^2\frac{\partial}{\partial s}(f_2) + \kappa\kappa^{'}f_2 - \kappa^2\tau f_3)\overrightarrow{T} + \frac{1}{\kappa^2}\lambda\overrightarrow{B}] \\
=& (\frac{\partial}{\partial s}(\kappa^2 f_1) + \frac{\partial}{\partial s}(\kappa^2\frac{\partial}{\partial s}(f_2)) + \frac{\partial}{\partial s}(\kappa\kappa^{'}f_2) - \frac{\partial}{\partial s}(\kappa^2\tau f_3))\overrightarrow{T} \\
&+ \kappa^2(f_1 + \frac{\partial}{\partial s}(f_2) + \frac{\kappa^{'}}{\kappa}f_2 - \tau f_3)\overrightarrow{N} + \lambda\frac{\partial}{\partial s}(\frac{1}{\kappa^2})\overrightarrow{B} \\
&+ \frac{1}{\kappa^2}\lambda(-\tau\overrightarrow{N} + \frac{\kappa^{'}}{\kappa}\overrightarrow{B}) + \frac{1}{\kappa^2}\frac{\partial}{\partial s}(\lambda)\overrightarrow{B},
\end{aligned}$$

while

$$\begin{aligned}\frac{\partial}{\partial t}\frac{\partial \overrightarrow{N}}{\partial s} =&\frac{\partial}{\partial t}[(\kappa^2\overrightarrow{T}+\frac{\kappa^{'}}{\kappa}\overrightarrow{N}+\tau\overrightarrow{B})]\\ =&\frac{\partial}{\partial t}(\kappa^2)\overrightarrow{T}+\kappa^2(f_1+\frac{\partial}{\partial s}(f_2)+\frac{\kappa^{'}}{\kappa}f_2-\tau f_3)\overrightarrow{N}\\ &+\kappa^2((\tau f_2+\frac{\partial}{\partial s}(f_3)+\frac{\kappa^{'}}{\kappa}f_3)\overrightarrow{B}+\frac{\partial}{\partial t}(\frac{\kappa^{'}}{\kappa})\overrightarrow{N}\\ &+\kappa\kappa^{'}(f_1+\frac{\partial}{\partial s}(f_2)+\frac{\kappa^{'}}{\kappa}f_2-\tau f_3)\overrightarrow{T}+\frac{\kappa^{'}}{\kappa^3}\lambda\overrightarrow{B}+\frac{\partial}{\partial t}(\tau)\overrightarrow{B}\\ &+\kappa^2\tau(\tau f_2+\frac{\partial}{\partial s}(f_3)+\frac{\kappa^{'}}{\kappa}f_3)\overrightarrow{T}-\frac{\tau}{\kappa^2}\lambda\overrightarrow{N}.\end{aligned}$$

thus we get

$$\begin{aligned}\frac{\partial}{\partial t}\frac{\partial \overrightarrow{N}}{\partial s} =&(\frac{\partial}{\partial t}(\kappa^2)+\kappa\kappa^{'}(f_1+\frac{\partial}{\partial s}(f_2)+\frac{\kappa^{'}}{\kappa}f_2-\tau f_3)+\kappa^2\tau(\tau f_2+\frac{\partial}{\partial s}(f_3)+\frac{\kappa^{'}}{\kappa}f_3))\overrightarrow{T}\\ &+(\kappa^2(f_1+\frac{\partial}{\partial s}(f_2)+\frac{\kappa^{'}}{\kappa}f_2-\tau f_3)+\frac{\partial}{\partial t}(\frac{\kappa^{'}}{\kappa})-\frac{\tau}{\kappa^2}\lambda)\overrightarrow{N}+(\frac{\kappa^{'}}{\kappa^3}\lambda+\frac{\partial}{\partial t}(\tau))\overrightarrow{B}\end{aligned}$$

Hence we see that

$$\begin{aligned}\frac{\partial \kappa^2}{\partial t} =&\frac{\partial}{\partial s}(\kappa^2 f_1)+\frac{\partial}{\partial s}(\kappa^2\frac{\partial}{\partial s}(f_2))+\frac{\partial}{\partial s}(\kappa\kappa^{'}(f_2))-\frac{\partial}{\partial s}(\kappa^2\tau f_3)-\kappa\kappa^{'}(f_1+\frac{\partial}{\partial s}(f_2)+\frac{\kappa^{'}}{\kappa}f_2-\tau f_3)-\kappa^2\tau(\tau f_2+\frac{\partial}{\partial s}(f_3)+\frac{\kappa^{'}}{\kappa}f\\ \frac{\partial \tau}{\partial t} =&\lambda\frac{\partial}{\partial s}(\frac{1}{\kappa^2})+\frac{1}{\kappa^2}\frac{\partial}{\partial s}(\lambda)-\kappa^2(\tau f_2+\frac{\partial}{\partial s}(f_3)+\frac{\kappa^{'}}{\kappa}f_3),\end{aligned}$$

The proof of the equality $\frac{\partial}{\partial s}\frac{\partial \overrightarrow{B}}{\partial t}=\frac{\partial}{\partial t}\frac{\partial \overrightarrow{B}}{\partial s}$ can be completed following similar steps as in above. □

3.2. *Inextensible flows of spacelike curve with a spacelike principal normal $\overrightarrow{n}$ on MOFC*

In this section we suppose that a spacelike curve and it have a spacelike principal normal curve.

Definition 3.2. A spacelike curve flow $\frac{\partial F}{\partial t}$ on the MOFC in Minkowski 3-space is defined as be inextensible if

$$\frac{\partial}{\partial t}\left|\frac{\partial F}{\partial u}\right|=0.$$

The next theorem states the requisite and adequate circumstances for the inextensible flow of a spacelike curve with a spacelike principal normal $\overrightarrow{n}$ in $\mathbb{E}^3_1$.

Theorem 3.4. *Let $\frac{\partial F}{\partial t}=f_1\overrightarrow{T}+f_2\overrightarrow{N}+f_3\overrightarrow{B}$ be a regular and differentiable flow of the spacelike curve F curvature on MOFC $\overrightarrow{T},\overrightarrow{N},\overrightarrow{B}$ in 3-dimensional Minkowski space.The flow of curve F said to be inextensible if and only if*

$$\frac{\partial f_1}{\partial s}=\kappa^2 f_2.$$

Proof. We have equation (2.3) and $\langle \frac{\partial F}{\partial u}, \frac{\partial F}{\partial u} \rangle = v^2$. Because parameters u and t are independent of coordinates, the operators $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial t}$ are commute and so we have

$$\begin{aligned}
2v\frac{\partial v}{\partial t} =& \frac{\partial}{\partial t}\langle \frac{\partial F}{\partial u}, \frac{\partial F}{\partial u} t\rangle \\
=& 2\langle \frac{\partial F}{\partial u}, \frac{\partial}{\partial u}(\frac{\partial F}{\partial t})\rangle \\
=& 2\langle \frac{\partial F}{\partial u}, \frac{\partial}{\partial u}(f_1\overrightarrow{T} + f_2\overrightarrow{N} + f_3\overrightarrow{B})\rangle \\
=& 2v\langle \overrightarrow{T}, \frac{\partial}{\partial u}(f_1)\overrightarrow{T} + vf_1\overrightarrow{N} + \frac{\partial}{\partial u}(f_2)\overrightarrow{N} + vf_2(-\kappa^2\overrightarrow{T} + \frac{\kappa^{'}}{\kappa}\overrightarrow{N} + \tau\overrightarrow{B})\rangle \\
&+ 2v\langle \overrightarrow{T}, \frac{\partial}{\partial u}(f_3)\overrightarrow{B} + vf_3(\tau\overrightarrow{N} + \frac{\kappa^{'}}{\kappa}\overrightarrow{B})\rangle \\
=& 2v(\frac{\partial}{\partial u}(f_1) - \kappa^2 f_2 v).
\end{aligned}$$

Thus we get

$$\frac{\partial v}{\partial t} = (\frac{\partial}{\partial u}(f_1) - \kappa^2 f_2 v). \tag{3.3}$$

Now $\frac{\partial F}{\partial t}$ be extensible. From eq.(3.3) we get

$$\begin{aligned}
\frac{\partial}{\partial t}S(u,t) &= \int_0^u \frac{\partial v}{\partial t}du \\
&= \int_0^u (\frac{\partial}{\partial u}(f_1) + \kappa^2 f_2 v)du \\
&= 0.
\end{aligned}$$

for all $\forall u \in [0, l]$ This means that $\frac{\partial}{\partial u}(f_1) = \kappa^2 f_2 v$ or $\frac{\partial}{\partial s}(f_1) = \kappa^2 f_2$. The other part of the proof can be completed by similar steps as in above. □

Now, we're limited to curves with arclength parameterization. Accordingly, $v = 1$ and the local coordinate u relates to the curve's arclength s. By a parameter change, we may give that following theorem.

Theorem 3.5. *Let $\frac{\partial F}{\partial t} = f_1\overrightarrow{T} + f_2\overrightarrow{N} + f_3\overrightarrow{B}$ be a regular and differentiable flow of the spacelike curve F on MOFC $\left\{\overrightarrow{T}, \overrightarrow{N}, \overrightarrow{B}\right\}$in 3-dimensional Minkowski space. In this case*

$$\begin{aligned}
\frac{\partial \overrightarrow{T}}{\partial t} &= (f_1 + \frac{\partial}{\partial s}(f_2) + \frac{\kappa^{'}}{\kappa}f_2 + \tau f_3)\overrightarrow{N} + (\tau f_2 + \frac{\partial}{\partial s}(f_3) + \frac{\kappa^{'}}{\kappa}f_3)\overrightarrow{B}, \\
\frac{\partial \overrightarrow{N}}{\partial t} &= -\kappa^2(f_1 + \frac{\partial}{\partial s}(f_2) + \frac{\kappa^{'}}{\kappa}f_2 + \tau f_3)\overrightarrow{T} - \frac{1}{\kappa^2}\lambda\overrightarrow{B}, \\
\frac{\partial \overrightarrow{B}}{\partial t} &= \kappa^2(\tau f_2 + \frac{\partial}{\partial s}(f_3) + \frac{\kappa^{'}}{\kappa}f_3)\overrightarrow{T} - \frac{1}{\kappa^2}\lambda\overrightarrow{N},
\end{aligned}$$

where $\langle \frac{\partial \overrightarrow{N}}{\partial t}, \overrightarrow{B} \rangle = \lambda$.

Proof. Using Theorem 3.4 and the MOFC in $\mathbb{E}_1^3$, we calculate

$$\begin{aligned}\frac{\partial \overrightarrow{T}}{\partial t} =& \frac{\partial}{\partial t}\frac{\partial F}{\partial s}\\ =&\frac{\partial}{\partial s}(f_1\overrightarrow{T}+f_2\overrightarrow{N}+f_3\overrightarrow{B})\\ =&\frac{\partial}{\partial s}(f_1)\overrightarrow{T}+f_1\overrightarrow{N}+\frac{\partial}{\partial s}(f_2)\overrightarrow{N}+f_2(-\kappa^2\overrightarrow{T}+\frac{\kappa'}{\kappa}\overrightarrow{N}+\tau\overrightarrow{B})+\frac{\partial}{\partial s}(f_3)\overrightarrow{B}+f_3(\tau\overrightarrow{N}+\frac{\kappa'}{\kappa}\overrightarrow{B})\\ =&(f_1+\frac{\partial}{\partial s}(f_2)+\frac{\kappa'}{\kappa}f_2+\tau f_3)\overrightarrow{N}+(\tau f_2+\frac{\partial}{\partial s}(f_3)+\frac{\kappa'}{\kappa}f_3)\overrightarrow{B}.\end{aligned}$$

Now differentiating the modified orthogonal frame according to t

$$\begin{cases}\frac{\partial}{\partial t}\langle\overrightarrow{T},\overrightarrow{N}\rangle &= 0=\kappa^2(f_1+\frac{\partial}{\partial s}(f_2)+\frac{\kappa'}{\kappa}f_2+\tau f_3)+\langle\overrightarrow{T},\frac{\partial\overrightarrow{N}}{\partial t}\rangle,\\ \frac{\partial}{\partial t}\langle\overrightarrow{T},\overrightarrow{B}\rangle &= 0=-\kappa^2(\tau f_2+\frac{\partial}{\partial s}(f_3)+\frac{\kappa'}{\kappa}f_3)+\langle\overrightarrow{T},\frac{\partial\overrightarrow{B}}{\partial t}\rangle,\\ \frac{\partial}{\partial t}\langle\overrightarrow{N},\overrightarrow{B}\rangle &= 0=\lambda+\langle\overrightarrow{N},\frac{\partial\overrightarrow{B}}{\partial t}\rangle.\end{cases} \tag{3.4}$$

From the previous equation (3.4) we get

$$\begin{aligned}\frac{\partial\overrightarrow{N}}{\partial t} &= -\kappa^2(f_1+\frac{\partial}{\partial s}(f_2)+\frac{\kappa'}{\kappa}f_2+\tau f_3)\overrightarrow{T}-\frac{1}{\kappa^2}\lambda\overrightarrow{B},\\ \frac{\partial\overrightarrow{B}}{\partial t} &= \kappa^2(\tau f_2+\frac{\partial}{\partial s}(f_3)+\frac{\kappa'}{\kappa}f_3)\overrightarrow{T}-\frac{1}{\kappa^2}\lambda\overrightarrow{N},\end{aligned}$$

where $\langle\frac{\partial\overrightarrow{N}}{\partial t},\overrightarrow{B}\rangle=\lambda$ □

The following theorem presufficient for the curve flow $F(s,t)$ to be inextensible for a PDE involving curvatures and torsion.

Theorem 3.6. *Assume that curve flow* $\frac{\partial F}{\partial t}=f_1\overrightarrow{T}+f_2\overrightarrow{N}+f_3\overrightarrow{B}$ *on the MOFC in 3-dimensional Minkowski space is inextensible. Then the next equations of partial PDE holds:*

$$\begin{aligned}\frac{\partial\kappa^2}{\partial t} =&\frac{\partial}{\partial s}(\kappa^2 f_1)+\frac{\partial}{\partial s}(\kappa^2\frac{\partial}{\partial s}(f_2))+\frac{\partial}{\partial s}(\kappa\kappa' f_2)+\frac{\partial}{\partial s}(\kappa^2\tau f_3)\\ &+\kappa\kappa'(f_1+\frac{\partial}{\partial s}(f_2)+\frac{\kappa'}{\kappa}f_2+\tau f_3)+\kappa^2\tau(\tau f_2+\frac{\partial}{\partial s}(f_3)+\frac{\kappa'}{\kappa}f_3)\\ \frac{\partial\tau}{\partial t} =&\kappa^2(\tau f_2+\frac{\partial}{\partial s}(f_3)+\frac{\kappa'}{\kappa}f_3)-\frac{1}{\kappa^2}\frac{\partial}{\partial s}(\lambda)-\lambda\frac{\partial}{\partial s}(\frac{1}{\kappa^2})\end{aligned}$$

where

$$\lambda=-\kappa^2\tau(f_1+\frac{\partial}{\partial s}f_2+\frac{\kappa'}{\kappa}f_2+\tau f_3)-\kappa^2(\frac{\partial}{\partial s}(\tau f_2)+\frac{\partial^2}{\partial s^2}(f_3)+\frac{\partial}{\partial s}(\frac{\kappa'}{\kappa}(f_3))-\kappa\kappa'(\tau f_2+\frac{\partial}{\partial s}(f_3)+\frac{\kappa'}{\kappa}(f_3))$$

Proof. Noting that $\frac{\partial}{\partial s}\frac{\partial\overrightarrow{T}}{\partial t}=\frac{\partial}{\partial t}\frac{\partial\overrightarrow{T}}{\partial s}$. Differentiating $\frac{\partial\overrightarrow{T}}{\partial t}$ according to s, we hold

$$\begin{aligned}\frac{\partial}{\partial s}\frac{\partial\overrightarrow{T}}{\partial t} =&\frac{\partial}{\partial s}[(f_1+\frac{\partial}{\partial s}(f_2)+\frac{\kappa'}{\kappa}f_2+\tau f_3)\overrightarrow{N}+(\tau f_2+\frac{\partial}{\partial s}(f_3)+\frac{\kappa'}{\kappa}f_3)\overrightarrow{B}]\\ =&(\frac{\partial}{\partial s}(f_1)+\frac{\partial^2}{\partial s^2}(f_2)+\frac{\partial}{\partial s}(\frac{\kappa'}{\kappa}f_2)+\frac{\partial}{\partial s}(\tau f_3))\overrightarrow{N}\\ &+(f_1+\frac{\partial}{\partial s}(f_2)+\frac{\kappa'}{\kappa}f_2+\tau f_3)(-\kappa^2\overrightarrow{T}+\frac{\kappa'}{\kappa}\overrightarrow{N}+\tau\overrightarrow{B})\\ &+(\frac{\partial}{\partial s}(\tau f_2)+\frac{\partial^2}{\partial s^2}(f_3)+\frac{\partial}{\partial s}(\frac{\kappa'}{\kappa}f_3))\overrightarrow{B}\\ &+(\tau f_2+\frac{\partial}{\partial s}(f_3)+\frac{\kappa'}{\kappa}f_3)(\tau\overrightarrow{N}+\frac{\kappa'}{\kappa}\overrightarrow{B}),\end{aligned}$$

and while

$$\frac{\partial}{\partial t}\frac{\partial \overrightarrow{T}}{\partial s} = \frac{\partial}{\partial t}\overrightarrow{N} = -\kappa^2(f_1 + \frac{\partial}{\partial s}(f_2) + \frac{\kappa^{'}}{\kappa}f_2 + \tau f_3)\overrightarrow{T} - \frac{1}{\kappa^2}\lambda\overrightarrow{B},$$

thus

$$\lambda = -\kappa^2\tau(f_1 + \frac{\partial}{\partial s}f_2 + \frac{\kappa^{'}}{\kappa}f_2 + \tau f_3) - \kappa^2(\frac{\partial}{\partial s}(\tau f_2) + \frac{\partial^2}{\partial s^2}(f_3) + \frac{\partial}{\partial s}(\frac{\kappa^{'}}{\kappa}(f_3)) - \kappa\kappa^{'}(\tau f_2 + \frac{\partial}{\partial s}(f_3) + \frac{\kappa^{'}}{\kappa}(f_3))$$

Since $\frac{\partial}{\partial s}\frac{\partial \overrightarrow{N}}{\partial t} = \frac{\partial}{\partial t}\frac{\partial \overrightarrow{N}}{\partial s}$, we have

$$\begin{aligned}
\frac{\partial}{\partial s}\frac{\partial \overrightarrow{N}}{\partial t} =& \frac{\partial}{\partial s}[-\kappa^2(f_1 + \frac{\partial}{\partial s}(f_2) + \frac{\kappa^{'}}{\kappa}f_2 + \tau f_3)\overrightarrow{T} - \frac{1}{\kappa^2}\lambda\overrightarrow{B}] \\
=&(-\frac{\partial}{\partial s}(\kappa^2 f_1) - \frac{\partial}{\partial s}(\kappa^2\frac{\partial}{\partial s}(f_2)) - \frac{\partial}{\partial s}(\kappa\kappa^{'} f_2) - \frac{\partial}{\partial s}(\kappa^2\tau f_3))\overrightarrow{T} \\
&- \kappa^2(f_1 + \frac{\partial}{\partial s}(f_2) + \frac{\kappa^{'}}{\kappa}f_2 + \tau f_3)\overrightarrow{N} \\
&- \lambda\frac{\partial}{\partial s}(\frac{1}{\kappa^2})\overrightarrow{B} - \frac{1}{\kappa^2}\frac{\partial}{\partial s}(\lambda)\overrightarrow{B} - \frac{1}{\kappa^2}\lambda(\tau\overrightarrow{N} + \frac{\kappa^{'}}{\kappa}\overrightarrow{B}),
\end{aligned}$$

while

$$\begin{aligned}
\frac{\partial}{\partial t}\frac{\partial \overrightarrow{N}}{\partial s} =& \frac{\partial}{\partial t}[(-\kappa^2\overrightarrow{T} + \frac{\kappa^{'}}{\kappa}\overrightarrow{N} + \tau\overrightarrow{B})] \\
=& -\frac{\partial}{\partial t}(\kappa^2)\overrightarrow{T} - \kappa^2(f_1 + \frac{\partial}{\partial s}(f_2) + \frac{\kappa^{'}}{\kappa}f_2 + \tau f_3)\overrightarrow{N} \\
&- \kappa^2((\tau f_2 + \frac{\partial}{\partial s}(f_3) + \frac{\kappa^{'}}{\kappa}f_3)\overrightarrow{B} \\
&+ \frac{\partial}{\partial t}(\frac{\kappa^{'}}{\kappa})\overrightarrow{N} - \frac{\kappa^{'}}{\kappa}((\kappa^2 f_1 + \kappa^2\frac{\partial}{\partial s}(f_2) + \kappa\kappa^{'} f_2 + \kappa^2\tau f_3))\overrightarrow{T} - \frac{\kappa^{'}}{\kappa^3}\lambda\overrightarrow{B} \\
&+ \frac{\partial}{\partial t}(\tau)\overrightarrow{B} + \tau((\kappa^2\tau f_2 + \kappa^2\frac{\partial}{\partial s}(f_3) + \kappa\kappa^{'} f_3))\overrightarrow{T} - \frac{\tau}{\kappa^2}\lambda\overrightarrow{N}.
\end{aligned}$$

thus we get

$$\begin{aligned}
\frac{\partial}{\partial t}\frac{\partial \overrightarrow{N}}{\partial s} =&(-\frac{\partial}{\partial t}(\kappa^2) - \frac{\kappa^{'}}{\kappa}(\kappa^2 f_1 + \kappa^2\frac{\partial}{\partial s}(f_2) + \kappa\kappa^{'} f_2 + \kappa^2\tau f_3) + \tau(\kappa^2\tau f_2 + \kappa^2\frac{\partial}{\partial s}(f_3) + \kappa\kappa^{'} f_3))\overrightarrow{T} \\
&(-\kappa^2(f_1 + \frac{\partial}{\partial s}(f_2) + \frac{\kappa^{'}}{\kappa}f_2 + \tau f_3) + \frac{\partial}{\partial t}(\frac{\kappa^{'}}{\kappa}) - \frac{\tau}{\kappa^2}\lambda)\overrightarrow{N} + (-\kappa^2(\tau f_2 + \frac{\partial}{\partial s}(f_3) + \frac{\kappa^{'}}{\kappa}f_3) - \frac{\kappa^{'}}{\kappa^3}\lambda + \frac{\partial}{\partial t}(\tau))\overrightarrow{B}
\end{aligned}$$

Hence we see that

$$\begin{aligned}
\frac{\partial \kappa^2}{\partial t} =& \frac{\partial}{\partial s}(\kappa^2 f_1) + \frac{\partial}{\partial s}(\kappa^2\frac{\partial}{\partial s}(f_2)) + \frac{\partial}{\partial s}(\kappa\kappa^{'} f_2) + \frac{\partial}{\partial s}(\kappa^2\tau f_3) \\
&+ \kappa\kappa^{'}(f_1 + \frac{\partial}{\partial s}(f_2) + \frac{\kappa^{'}}{\kappa}f_2 + \tau f_3) + \kappa^2\tau(\tau f_2 + \frac{\partial}{\partial s}(f_3) + \frac{\kappa^{'}}{\kappa}f_3), \\
\frac{\partial \tau}{\partial t} =& \kappa^2(\tau f_2 + \frac{\partial}{\partial s}(f_3) + \frac{\kappa^{'}}{\kappa}f_3) - \frac{1}{\kappa^2}\frac{\partial}{\partial s}(\lambda) - \lambda\frac{\partial}{\partial s}(\frac{1}{\kappa^2}).
\end{aligned}$$

The proof of the equality $\frac{\partial}{\partial s}\frac{\partial \overrightarrow{B}}{\partial t} = \frac{\partial}{\partial t}\frac{\partial \overrightarrow{B}}{\partial s}$ can be completed in above similar way. □

3.3. *Inextensible flows of spacelike curve with a spacelike binormal $\overrightarrow{b}$ on MOFC*

In this section we suppose that a spacelike curve and it has spacelike binormal $\overrightarrow{b}$.

Definition 3.3. A spacelike curve flow $\frac{\partial F}{\partial t}$ on the MOFC in Minkowski 3-space is defined as be inextensible if

$$\frac{\partial}{\partial t}\left|\frac{\partial F}{\partial u}\right| = 0.$$

Theorem 3.7. *Assume that $\frac{\partial F}{\partial t} = f_1\overrightarrow{T} + f_2\overrightarrow{N} + f_3\overrightarrow{B}$ be a differentiable and regular flow of the spacelike curve F with binormal $\overrightarrow{b}$ MOFC $\overrightarrow{T}, \overrightarrow{N}, \overrightarrow{B}$ in 3-dimensional Minkowski space.The flow is inextensible if and only if*

$$\frac{\partial f_1}{\partial s} = -\kappa^2 f_2.$$

Proof. We have equation (2.4) and $\langle \frac{\partial F}{\partial u}, \frac{\partial F}{\partial u}\rangle = v^2$. The operators $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial t}$commute because the parametters u and t are independent coordinates. So we get

$$\begin{aligned}
2v\frac{\partial v}{\partial t} =& \frac{\partial}{\partial t}\langle \frac{\partial F}{\partial u}, \frac{\partial F}{\partial u}\rangle \\
=& 2\langle \frac{\partial F}{\partial u}, \frac{\partial}{\partial u}(\frac{\partial F}{\partial t})t\rangle \\
=& 2\langle \frac{\partial F}{\partial u}, \frac{\partial}{\partial u}(f_1\overrightarrow{T} + f_2\overrightarrow{N} + f_3\overrightarrow{B})\rangle \\
=& 2v\langle \overrightarrow{T}, \frac{\partial}{\partial u}(f_1)\overrightarrow{T} + vf_1\overrightarrow{N} + \frac{\partial}{\partial u}(f_2)\overrightarrow{N} + vf_2(\kappa^2\overrightarrow{T} + \frac{\kappa'}{\kappa}\overrightarrow{N} + \tau\overrightarrow{B})\rangle \\
&+ 2v\langle \overrightarrow{T}, \frac{\partial}{\partial u}(f_3)\overrightarrow{B} + vf_3(\tau\overrightarrow{N} + \frac{\kappa'}{\kappa}\overrightarrow{B})\rangle \\
=& 2v(\frac{\partial}{\partial u}(f_1) + \kappa^2 f_2 v).
\end{aligned}$$

Thus we get

$$\frac{\partial v}{\partial t} = (\frac{\partial}{\partial u}(f_1) + \kappa^2 f_2 v). \tag{3.5}$$

Now let the flow $\frac{\partial F}{\partial t}$ is extensible. From eq.(3.5) we hold

$$\begin{aligned}
\frac{\partial}{\partial t}S(u,t) &= \int_0^u \frac{\partial v}{\partial t}du \\
&= \int_0^u \Big(\frac{\partial}{\partial u}(f_1) + \kappa^2 f_2 v\Big)du \\
&= 0,
\end{aligned}$$

for all $\forall u \in [0,l]$. That is; $\frac{\partial}{\partial u}(f_1) = -\kappa^2 f_2 v$ or $\frac{\partial}{\partial s}(f_1) = -\kappa^2 f_2$. The other part of the proof can be completed by similar steps as in above. □

We now circumscribe ourselves to arc-length parameterized curves. That is, $v = 1$, and the local coordinate u relates to the curve arclength s. By a parameter change, we may give that following theorem.

Theorem 3.8. *Let $\frac{\partial F}{\partial t} = f_1\overrightarrow{T} + f_2\overrightarrow{N} + f_3\overrightarrow{B}$ be a differentiable and regular flow of the spacelike curve F with binormal $\overrightarrow{b}$ on MOF $\left\{\overrightarrow{T}, \overrightarrow{N}, \overrightarrow{B}\right\}$in 3-dimensional Minkowski space.In this case*

$$\frac{\partial \overrightarrow{T}}{\partial t} = (f_1 + \frac{\partial}{\partial s}(f_2) + \frac{\kappa^{'}}{\kappa} f_2 + \tau f_3)\overrightarrow{N} + (\tau f_2 + \frac{\partial}{\partial s}(f_3) + \frac{\kappa^{'}}{\kappa} f_3)\overrightarrow{B},$$
$$\frac{\partial \overrightarrow{N}}{\partial t} = \kappa^2(f_1 + \frac{\partial}{\partial s}(f_2) + \frac{\kappa^{'}}{\kappa} f_2 + \tau f_3)\overrightarrow{T} + \frac{1}{\kappa^2}\lambda\overrightarrow{B},$$
$$\frac{\partial \overrightarrow{B}}{\partial t} = -\kappa^2(\tau f_2 + \frac{\partial}{\partial s}(f_3) + \frac{\kappa^{'}}{\kappa} f_3)\overrightarrow{T} + \frac{1}{\kappa^2}\lambda\overrightarrow{N},$$

where $\langle \frac{\partial \overrightarrow{N}}{\partial t}, \overrightarrow{B}\rangle = \lambda$.

Proof. Using Theorem 3.7 and the MOFC in 3-dimensional Minkowski space, we calculate

$$\begin{aligned}\frac{\partial \overrightarrow{T}}{\partial t} =& \frac{\partial}{\partial t}\frac{\partial F}{\partial s}\\ =& \frac{\partial}{\partial s}(f_1\overrightarrow{T} + f_2\overrightarrow{N} + f_3\overrightarrow{B})\\ =& \frac{\partial}{\partial s}(f_1)\overrightarrow{T} + f_1\overrightarrow{N} + \frac{\partial}{\partial s}(f_2)\overrightarrow{N} + f_2(\kappa^2\overrightarrow{T} + \frac{\kappa^{'}}{\kappa}\overrightarrow{N} + \tau\overrightarrow{B}) + \frac{\partial}{\partial s}(f_3)\overrightarrow{B}\\ &+ f_3(\tau\overrightarrow{N} + \frac{\kappa^{'}}{\kappa}\overrightarrow{B})\\ =& (f_1 + \frac{\partial}{\partial s}(f_2) + \frac{\kappa^{'}}{\kappa} f_2 + \tau f_3)\overrightarrow{N} + (\tau f_2 + \frac{\partial}{\partial s}(f_3) + \frac{\kappa^{'}}{\kappa} f_3)\overrightarrow{B}.\end{aligned}$$

Now differentiating the MOF according to t

$$\begin{cases}\frac{\partial}{\partial t}\langle \overrightarrow{T}, \overrightarrow{N}\rangle &= 0 = -\kappa^2(f_1 + \frac{\partial}{\partial s}(f_2) + \frac{\kappa^{'}}{\kappa} f_2 + \kappa^2\tau f_3) + \langle \overrightarrow{T}, \frac{\partial \overrightarrow{N}}{\partial t}\rangle,\\ \frac{\partial}{\partial t}\langle \overrightarrow{T}, \overrightarrow{B}\rangle &= 0 = \kappa^2(\tau f_2 + \frac{\partial}{\partial s}(f_3) + \frac{\kappa^{'}}{\kappa} f_3) + \langle \overrightarrow{T}, \frac{\partial \overrightarrow{B}}{\partial t}\rangle,\\ \frac{\partial}{\partial t}\langle \overrightarrow{N}, \overrightarrow{B}\rangle &= 0 = \lambda + \langle \overrightarrow{N}, \frac{\partial \overrightarrow{B}}{\partial t}\rangle.\end{cases} \tag{3.6}$$

From the previous equation (3.6) we obtain

$$\frac{\partial \overrightarrow{N}}{\partial t} = \kappa^2(f_1 + \frac{\partial}{\partial s}(f_2) + \frac{\kappa^{'}}{\kappa} f_2 + \tau f_3)\overrightarrow{T} + \frac{1}{\kappa^2}\lambda\overrightarrow{B},$$
$$\frac{\partial \overrightarrow{B}}{\partial t} = -\kappa^2(\tau f_2 + \frac{\partial}{\partial s}(f_3) + \frac{\kappa^{'}}{\kappa} f_3)\overrightarrow{T} + \frac{1}{\kappa^2}\lambda\overrightarrow{N},$$

respectively, where $\langle \frac{\partial \overrightarrow{N}}{\partial t}, \overrightarrow{B}\rangle = \lambda$. □

The next theorem states the circumstances on PDE involving the curvatures(κ and τ for the curve flow $F(s,t)$ to be inextensible.

Theorem 3.9. *Assume that curve flow* $\frac{\partial F}{\partial t} = f_1\overrightarrow{T} + f_2\overrightarrow{N} + f_3\overrightarrow{B}$ *on the MOFC in 3-dimensional Minkowski space is inextensible. Consequently, the following PDE hold.*

$$\begin{aligned}\frac{\partial \kappa^2}{\partial t} =& \frac{\partial}{\partial s}(\kappa^2 f_1) + \frac{\partial}{\partial s}(\kappa^2\frac{\partial}{\partial s}(f_2)) + \frac{\partial}{\partial s}(\kappa\kappa^{'} f_2) + \frac{\partial}{\partial s}(\kappa^2\tau f_3) - \kappa\kappa^{'}(f_1 + \frac{\partial}{\partial s}(f_2) + \frac{\kappa^{'}}{\kappa} f_2 + \tau f_3) + \kappa^2\tau(\tau f_2 + \frac{\partial}{\partial s} f_3 + \frac{\kappa^{'}}{\kappa} f_3)\\ \frac{\partial \tau}{\partial t} =& \lambda\frac{\partial}{\partial s}(\frac{1}{\kappa^2}) + \frac{1}{\kappa^2}\frac{\partial}{\partial s}(\lambda) - \kappa^2(\tau f_2 + \frac{\partial}{\partial s}(f_3) + \frac{\kappa^{'}}{\kappa} f_3),\end{aligned}$$

where

$$\lambda = \kappa^2\tau(f_1 + \frac{\partial}{\partial s} f_2 + \frac{\kappa^{'}}{\kappa}(f_2) + \tau f_3) + \kappa^2(\frac{\partial}{\partial s}(\tau f_2) + \frac{\partial^2}{\partial s^2}(f_3) + \frac{\partial}{\partial s}(\frac{\kappa^{'}}{\kappa} f_3) + \frac{\partial}{\partial s}(\tau f_3)) + \kappa\kappa^{'}(\tau f_2 + \frac{\partial}{\partial s}(f_3) + \frac{\kappa^{'}}{\kappa} f_3).$$

Proof. $\frac{\partial}{\partial s}\frac{\partial \overrightarrow{T}}{\partial t}=\frac{\partial}{\partial t}\frac{\partial \overrightarrow{T}}{\partial s}$.Differentiating $\frac{\partial \overrightarrow{T}}{\partial t}$ with respect to parameter s, we hold

$$\begin{aligned}\frac{\partial}{\partial s}\frac{\partial \overrightarrow{T}}{\partial t}=&\frac{\partial}{\partial s}[(f_1+\frac{\partial}{\partial s}(f_2)+\frac{\kappa^{'}}{\kappa}f_2+\tau f_3)\overrightarrow{N}+(\tau f_2+\frac{\partial}{\partial s}(f_3)+\frac{\kappa^{'}}{\kappa}f_3)\overrightarrow{B}]\\ =&(\frac{\partial}{\partial s}(f_1)+\frac{\partial^2}{\partial s^2}(f_2)+\frac{\partial}{\partial s}(\frac{\kappa^{'}}{\kappa}f_2)+\frac{\partial}{\partial s}(\tau f_3))\overrightarrow{N}\\ &+(f_1+\frac{\partial}{\partial s}(f_2)+\frac{\kappa^{'}}{\kappa}f_2+\tau f_3)(\kappa^2\overrightarrow{T}+\frac{\kappa^{'}}{\kappa}\overrightarrow{N}+\tau\overrightarrow{B})\\ &+(\frac{\partial}{\partial s}(\tau f_2)+\frac{\partial^2}{\partial s^2}(f_3)+\frac{\partial}{\partial s}(\frac{\kappa^{'}}{\kappa}f_3))\overrightarrow{B}\\ &+(\tau f_2+\frac{\partial}{\partial s}(f_3)+\frac{\kappa^{'}}{\kappa}f_3)(\tau\overrightarrow{N}+\frac{\kappa^{'}}{\kappa}\overrightarrow{B}),\end{aligned}$$

and while

$$\frac{\partial}{\partial t}\frac{\partial \overrightarrow{T}}{\partial s}=\frac{\partial}{\partial t}\overrightarrow{N}=\kappa^2(f_1+\frac{\partial}{\partial s}(f_2)+\frac{\kappa^{'}}{\kappa}f_2+\tau f_3)\overrightarrow{T}+\frac{1}{\kappa^2}\lambda\overrightarrow{B},$$

thus

$$\lambda=\kappa^2\tau(f_1+\frac{\partial}{\partial s}f_2+\frac{\kappa^{'}}{\kappa}(f_2)+\tau f_3)+\kappa^2(\frac{\partial}{\partial s}(\tau f_2)+\frac{\partial^2}{\partial s^2}(f_3)+\frac{\partial}{\partial s}(\frac{\kappa^{'}}{\kappa}f_3)+\frac{\partial}{\partial s}(\tau f_3))+\kappa\kappa^{'}(\tau f_2+\frac{\partial}{\partial s}(f_3)+\frac{\kappa^{'}}{\kappa}f_3)$$

Since $\frac{\partial}{\partial s}\frac{\partial \overrightarrow{N}}{\partial t}=\frac{\partial}{\partial t}\frac{\partial \overrightarrow{N}}{\partial s}$, we have

$$\begin{aligned}\frac{\partial}{\partial s}\frac{\partial \overrightarrow{N}}{\partial t}=&\frac{\partial}{\partial s}[(\kappa^2f_1+\kappa^2\frac{\partial}{\partial s}(f_2)+\kappa\kappa^{'}f_2+\kappa^2\tau f_3)\overrightarrow{T}+\frac{1}{\kappa^2}\lambda\overrightarrow{B}\\ =&(\frac{\partial}{\partial s}(\kappa^2f_1)+\frac{\partial}{\partial s}(\kappa^2\frac{\partial}{\partial s}f_2)+\frac{\partial}{\partial s}(\kappa\kappa^{'}f_2)+\frac{\partial}{\partial s}(\kappa^2\tau f_3))\overrightarrow{T}\\ &+(\kappa^2f_1+\kappa^2\frac{\partial}{\partial s}(f_2)+\kappa\kappa^{'}f_2+\kappa^2\tau f_3)\overrightarrow{N}\\ &+\lambda\frac{\partial}{\partial s}(\frac{1}{\kappa^2})\overrightarrow{B}+\frac{1}{\kappa^2}\frac{\partial}{\partial s}(\lambda)+\frac{1}{\kappa^2}\lambda(\tau\overrightarrow{N}+\frac{\kappa^{'}}{\kappa}\overrightarrow{B}),\end{aligned}$$

while

$$\begin{aligned}\frac{\partial}{\partial t}\frac{\partial \overrightarrow{N}}{\partial s}=&\frac{\partial}{\partial t}(\kappa^2\overrightarrow{T}+\frac{\kappa^{'}}{\kappa}\overrightarrow{N}+\tau\overrightarrow{B})\\ =&\frac{\partial}{\partial t}(\kappa^2)\overrightarrow{T}+\kappa^2(f_1+\frac{\partial}{\partial s}(f_2)+\frac{\kappa^{'}}{\kappa}f_2+\tau f_3)\overrightarrow{N}\\ &+\kappa^2(\tau f_2+\frac{\partial}{\partial s}(f_3)+\frac{\kappa^{'}}{\kappa}f_3)\overrightarrow{B}+\frac{\partial}{\partial t}(\frac{\kappa^{'}}{\kappa})\overrightarrow{N}\\ &+\frac{\kappa^{'}}{\kappa}((\kappa^2f_1+\kappa^2\frac{\partial}{\partial s}(f_2)+\kappa\kappa^{'}f_2+\kappa^2\tau f_3))\overrightarrow{T}+\frac{\kappa^{'}}{\kappa^3}\lambda\overrightarrow{B}\\ &+\frac{\partial}{\partial t}(\tau)\overrightarrow{B}-\tau(\kappa^2\tau f_2+\kappa^2\frac{\partial}{\partial s}(f_3)+\kappa\kappa^{'}f_3)\overrightarrow{T}+\frac{\tau}{\kappa^2}\lambda\overrightarrow{N}.\end{aligned}$$

Thus we get

$$\begin{aligned}\frac{\partial}{\partial t}\frac{\partial \overrightarrow{N}}{\partial s}=&(\frac{\partial}{\partial t}(\kappa^2)+\frac{\kappa^{'}}{\kappa}(\kappa^2f_1+\kappa^2\frac{\partial}{\partial s}(f_2)+\kappa\kappa^{'}f_2+\kappa^2\tau f_3)-\tau(\kappa^2\tau f_2+\kappa^2\frac{\partial}{\partial s}(f_3)+\kappa\kappa^{'}f_3))\overrightarrow{T}\\ &+(\kappa^2(f_1+\frac{\partial}{\partial s}(f_2)+\frac{\kappa^{'}}{\kappa}f_2+\tau f_3)+\frac{\partial}{\partial t}(\frac{\kappa^{'}}{\kappa})+\frac{\tau}{\kappa^2}\lambda)\overrightarrow{N}+(\kappa^2(\tau f_2+\frac{\partial}{\partial s}(f_3)+\frac{\kappa^{'}}{\kappa}f_3)+\frac{\kappa^{'}}{\kappa^3}\lambda+\frac{\partial}{\partial t}(\tau))\overrightarrow{B}\end{aligned}$$

Hence we see that

$$\frac{\partial \kappa^2}{\partial t} =\frac{\partial}{\partial s}(\kappa^2 f_1) + \frac{\partial}{\partial s}(\kappa^2 \frac{\partial}{\partial s}(f_2)) + \frac{\partial}{\partial s}(\kappa^{'} \kappa f_2) + \frac{\partial}{\partial s}(\kappa^2 \tau f_3) - \kappa\kappa^{'}(f_1 + \frac{\partial}{\partial s}(f_2) + \frac{\kappa^{'}}{\kappa} f_2 + \tau f_3) + \kappa^2 \tau(\tau f_2 + \frac{\partial}{\partial s} f_3 + \frac{\kappa^{'}}{\kappa} f_3)$$

and

$$\frac{\partial \tau}{\partial t} =\lambda \frac{\partial}{\partial s}(\frac{1}{\kappa^2}) + \frac{1}{\kappa^2}\frac{\partial}{\partial s}(\lambda) - \kappa^2(\tau f_2 + \frac{\partial}{\partial s}(f_3) + \frac{\kappa^{'}}{\kappa} f_3),$$

The proof of the equality $\frac{\partial}{\partial s}\frac{\partial \overrightarrow{B}}{\partial t} = \frac{\partial}{\partial t}\frac{\partial \overrightarrow{B}}{\partial s}$ may be obtained using a similar argument to that in above. □

3.4. *Inextensible flows of pseudo null curve on MOFC with curvature*

Definition 3.4. A pseudo null curve flow $\frac{\partial F}{\partial t}$ on the MOFC in $\mathbb{E}_1^3$ is defined as inextensible if

$$\frac{\partial}{\partial t}\left|\frac{\partial F}{\partial u}\right| = 0.$$

Theorem 3.10. *Let $\frac{\partial F}{\partial t} = f_1 \overrightarrow{T} + f_2 \overrightarrow{N} + f_3 \overrightarrow{B}$ be a regular and differentiable flow of the pseudo null curve on MOFC $\left\{\overrightarrow{T}, \overrightarrow{N}, \overrightarrow{B}\right\}$in 3-dimensional Minkowski space.The flow is inextensible if and only if*

$$\frac{\partial f_1}{\partial s} = \kappa^2 f_3\ . \tag{3.7}$$

Proof. We have equation (2.5) and $\langle \frac{\partial F}{\partial u}, \frac{\partial F}{\partial u}\rangle = v^2$. The operators $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial t}$commute since u and t are independent coordinates. We have

$$\begin{aligned}
2v\frac{\partial v}{\partial t} =& \frac{\partial}{\partial t}\langle \frac{\partial F}{\partial u}, \frac{\partial F}{\partial u}\rangle \\
=&2\langle \frac{\partial F}{\partial u}, \frac{\partial}{\partial u}(\frac{\partial F}{\partial t})\rangle \\
=&2\langle \frac{\partial F}{\partial u}, \frac{\partial}{\partial u}(f_1 \overrightarrow{T} + f_2 \overrightarrow{N} + f_3 \overrightarrow{B})\rangle \\
=&2v\langle \overrightarrow{T}, \frac{\partial}{\partial u}(f_1)\overrightarrow{T} + v f_1 \overrightarrow{N} + 2\frac{\partial}{\partial u}(f_2)\overrightarrow{N} + v f_2(\frac{\kappa^{'}}{\kappa}\overrightarrow{N} + \tau \overrightarrow{N})\rangle \\
&+ 2v\langle \overrightarrow{T}, \frac{\partial}{\partial u}(f_3)\overrightarrow{B} + v f_3(-\kappa^2 \overrightarrow{T} + \frac{\kappa^{'}}{\kappa}\overrightarrow{B} - \tau \overrightarrow{B})\rangle \\
=&2v(\frac{\partial}{\partial u}(f_1) - \kappa^2 f_3 v).
\end{aligned}$$

Now let $\frac{\partial F}{\partial t}$ be extensible. From (3.7) we get

$$\frac{\partial}{\partial t} S(u,t) = \int_0^u \frac{\partial v}{\partial t} du = \int_0^u (\frac{\partial}{\partial u}(f_1) - \kappa^2 f_3 v) du = 0.$$

for all $\forall u \in [0,l]$ This means that $\frac{\partial}{\partial u}(f_1) = \kappa^2 f_3 v$ or $\frac{\partial}{\partial s}(f_1) = \kappa^2 f_3$. The other part of the proof can be completed by similar steps as in above. □

Theorem 3.11. *Let $\frac{\partial F}{\partial t} = f_1 \overrightarrow{T} + f_2 \overrightarrow{N} + f_3 \overrightarrow{B}$ be a differentiable and regular flow of the spacelike curve F with binormal $\overrightarrow{b}$ on MOFC $\left\{\overrightarrow{T}, \overrightarrow{N}, \overrightarrow{B}\right\}$in 3-dimensional Minkowski space. In this case*

$$\begin{aligned}
\frac{\partial \overrightarrow{T}}{\partial t} &= (f_1 + \frac{\partial}{\partial s}(f_2) + \frac{\kappa^{'}}{\kappa} f_2 + \tau f_2)\overrightarrow{N} + (\frac{\partial}{\partial s}(f_3) + \frac{\kappa^{'}}{\kappa} f_3 - \tau f_3)\overrightarrow{B}, \\
\frac{\partial \overrightarrow{N}}{\partial t} &= -\kappa^2(\frac{\partial}{\partial s}(f_3) + \frac{\kappa^{'}}{\kappa} f_3 - \tau f_3)\overrightarrow{T}, \\
\frac{\partial \overrightarrow{B}}{\partial t} &= -\kappa^2(f_1 + \frac{\partial}{\partial s}(f_2) + \frac{\kappa^{'}}{\kappa} f_2 + \tau f_2)\overrightarrow{T}.
\end{aligned}$$

Proof. Using Theorem 3.7 and the MOFC in $\mathbb{E}_1^3$, we calculate

$$\begin{aligned}\frac{\partial \overrightarrow{T}}{\partial t} =& \frac{\partial}{\partial t}\frac{\partial F}{\partial s}\\ =& \frac{\partial}{\partial s}(f_1\overrightarrow{T}+f_2\overrightarrow{N}+f_3\overrightarrow{B})\\ =& \frac{\partial}{\partial s}(f_1)\overrightarrow{T}+f_1\overrightarrow{N}+\frac{\partial}{\partial s}(f_2)\overrightarrow{N}+f_2(\frac{\kappa^{'}}{\kappa}\overrightarrow{N}+\tau\overrightarrow{N})+\frac{\partial}{\partial s}(f_3)\overrightarrow{B}\\ &+f_3(-\kappa^2\overrightarrow{T}+\frac{\kappa^{'}}{\kappa}\overrightarrow{B}-\tau\overrightarrow{B})\\ =& (f_1+\frac{\partial}{\partial s}(f_2)+\frac{\kappa^{'}}{\kappa}f_2+\tau f_2)\overrightarrow{N}+(-\tau f_3+\frac{\partial}{\partial s}(f_3)+\frac{\kappa^{'}}{\kappa}f_3)\overrightarrow{B}.\end{aligned}$$

Now, differentiating the MOFC according to t

$$\begin{aligned}0 = \frac{\partial}{\partial t}\langle\overrightarrow{T},\overrightarrow{N}\rangle &= \langle\frac{\partial \overrightarrow{T}}{\partial t},\overrightarrow{N}\rangle+\langle\overrightarrow{T},\frac{\partial \overrightarrow{N}}{\partial t}\rangle\\ &= (-\kappa^2\tau f_3+\kappa^2\frac{\partial}{\partial s}(f_3)+\kappa\kappa^{'}f_3)+\langle\overrightarrow{T},\frac{\partial \overrightarrow{N}}{\partial t}\rangle,\\ 0 = \frac{\partial}{\partial t}\langle\overrightarrow{T},\overrightarrow{B}\rangle &= \langle\frac{\partial \overrightarrow{T}}{\partial t},\overrightarrow{B}\rangle+\langle\overrightarrow{T},\frac{\partial \overrightarrow{B}}{\partial t}\rangle\\ &= (\kappa^2 f_1+\kappa^2\frac{\partial}{\partial s}(f_2)+\kappa\kappa^{'}f_2+\kappa^2\tau f_2)+\langle\overrightarrow{T},\frac{\partial \overrightarrow{B}}{\partial t}\rangle,\\ 0 = \frac{\partial}{\partial t}\langle\overrightarrow{N},\overrightarrow{B}\rangle &= \langle\frac{\partial \overrightarrow{N}}{\partial t},\overrightarrow{B}\rangle+\langle\overrightarrow{N},\frac{\partial \overrightarrow{B}}{\partial t}\rangle = \lambda+\langle\overrightarrow{N},\frac{\partial \overrightarrow{B}}{\partial t}\rangle.\end{aligned}$$

From previous equation we get

$$\begin{aligned}\frac{\partial \overrightarrow{N}}{\partial t} &= -\kappa^2(\frac{\partial}{\partial s}(f_3)+\frac{\kappa^{'}}{\kappa}f_3-\tau f_3)\overrightarrow{T},\\ \frac{\partial \overrightarrow{B}}{\partial t} &= -\kappa^2(f_1+\frac{\partial}{\partial s}(f_2)+\frac{\kappa^{'}}{\kappa}f_2+\tau f_2)\overrightarrow{T}.\end{aligned}$$

respectively, where $\langle\frac{\partial \overrightarrow{N}}{\partial t},\overrightarrow{B}\rangle=\lambda$ □

The next theorem states the circumstances on PDE involving the curvatures and torsion for the curve flow $F(s,t)$ to be inextensible.

Theorem 3.12. *Assume the flow curve* $\frac{\partial F}{\partial t}=f_1\overrightarrow{T}+f_2\overrightarrow{N}+f_3\overrightarrow{B}$ *on the MOFC in 3-dimensional Minkowski space is inextensible.*
Then the PDE in the following system hold:

$$\begin{aligned}\frac{\partial \kappa^2}{\partial t} =& \frac{\partial}{\partial s}(\kappa^2 f_1)+\frac{\partial}{\partial s}(\kappa^2\frac{\partial}{\partial s}f_2)+\frac{\partial}{\partial s}(\kappa\kappa^{'}f_2)+\frac{\partial}{\partial s}(\kappa^2\tau f_2)-\kappa\kappa^{'}f_1-\kappa\kappa^{'}\frac{\partial}{\partial s}f_2-\kappa^{'^2}f_2+\kappa^2\tau f_1+\kappa^2\tau\frac{\partial}{\partial s}f_2+\kappa^2\tau^2 f_2,\\ \frac{\partial \tau}{\partial t} =& -\kappa^2\frac{\partial}{\partial s}(f_3)-\kappa\kappa^{'}f_3+\kappa^2\tau f_3+\frac{\partial}{\partial t}(\frac{\kappa^{'}}{\kappa}).\end{aligned}$$

Proof. Noting that $\frac{\partial}{\partial s}\frac{\partial \overrightarrow{T}}{\partial t}=\frac{\partial}{\partial t}\frac{\partial \overrightarrow{T}}{\partial s}$,

$$\begin{aligned}\frac{\partial}{\partial s}\frac{\partial \overrightarrow{T}}{\partial t} =&\frac{\partial}{\partial s}[(f_1+\frac{\partial}{\partial s}(f_2)+\frac{\kappa^{'}}{\kappa}f_2+\tau f_2)\overrightarrow{N}+(-\tau f_3+\frac{\partial}{\partial s}(f_3)+\frac{\kappa^{'}}{\kappa}f_3)\overrightarrow{B}]\\ =&(\frac{\partial}{\partial s}(f_1)+\frac{\partial^2}{\partial s^2}(f_2)+\frac{\partial}{\partial s}(\frac{\kappa^{'}}{\kappa}f_2)+\frac{\partial}{\partial s}(\tau f_3))\overrightarrow{N}\\ &+(f_1+\frac{\partial}{\partial s}(f_2)+\frac{\kappa^{'}}{\kappa}f_2+\tau f_2)(\frac{\kappa^{'}}{\kappa}\overrightarrow{N}+\tau\overrightarrow{N})\\ &-(\frac{\partial}{\partial s}(\tau f_3)+\frac{\partial^2}{\partial s^2}(f_3)+\frac{\partial}{\partial s}(\frac{\kappa^{'}}{\kappa}f_3))\overrightarrow{B}\\ &+(-\tau f_3+\frac{\partial}{\partial s}(f_3)+\frac{\kappa^{'}}{\kappa}f_3)(-\kappa^2\overrightarrow{T}+\frac{\kappa^{'}}{\kappa}\overrightarrow{B}-\tau\overrightarrow{B}),\end{aligned}$$

while

$$\frac{\partial}{\partial t}\frac{\partial \overrightarrow{T}}{\partial s}=\frac{\partial}{\partial t}\overrightarrow{N}=-\kappa^2(\frac{\partial}{\partial s}(f_3)+\frac{\kappa^{'}}{\kappa}f_3-\tau f_3)\overrightarrow{T}.$$

Since $\frac{\partial}{\partial s}\frac{\partial \overrightarrow{B}}{\partial t}=\frac{\partial}{\partial t}\frac{\partial \overrightarrow{B}}{\partial s}$, we have

$$\begin{aligned}\frac{\partial}{\partial s}\frac{\partial \overrightarrow{B}}{\partial t} =&\frac{\partial}{\partial s}[(-\kappa^2 f_1-\kappa^2\frac{\partial}{\partial s}(f_2)-\kappa\kappa^{'}f_2-\kappa^2\tau f_2)\overrightarrow{T}]\\ =&(-\frac{\partial}{\partial s}(\kappa^2 f_1)-\frac{\partial}{\partial s}(\kappa^2\frac{\partial}{\partial s}f_2)-\frac{\partial}{\partial s}(\kappa\kappa^{'}f_2)-\frac{\partial}{\partial s}(\kappa^2\tau f_2))\overrightarrow{T}+(-\kappa^2 f_1-\kappa^2\frac{\partial}{\partial s}(f_2)-\kappa\kappa^{'}f_2-\kappa^2\tau f_2)\overrightarrow{N},\end{aligned}$$

while

$$\begin{aligned}\frac{\partial}{\partial t}\frac{\partial \overrightarrow{B}}{\partial s} =&\frac{\partial}{\partial t}(-\kappa^2\overrightarrow{T}+\frac{\kappa^{'}}{\kappa}\overrightarrow{B}-\tau\overrightarrow{B})\\ =&-\frac{\partial}{\partial t}(\kappa^2)\overrightarrow{T}-\kappa^2(f_1+\frac{\partial}{\partial s}(f_2)+\frac{\kappa^{'}}{\kappa}f_2+\tau f_2)\overrightarrow{N}\\ &-\kappa^2(-\tau f_3+\frac{\partial}{\partial s}(f_3)+\frac{\kappa^{'}}{\kappa}f_3)\overrightarrow{B}+\frac{\partial}{\partial t}(\frac{\kappa^{'}}{\kappa})\overrightarrow{B}\\ &-\frac{\kappa^{'}}{\kappa}(\kappa^2 f_1+\kappa^2\frac{\partial}{\partial s}(f_2)+\kappa\kappa^{'}f_2+\kappa^2\tau f_2)\overrightarrow{T}-\frac{\partial}{\partial t}(\tau)\overrightarrow{B}\\ &+\tau(\kappa^2 f_1+\kappa^2\frac{\partial}{\partial s}(f_2)+\kappa\kappa^{'}f_2+\kappa^2\tau f_2)\overrightarrow{T}.\end{aligned}$$

Thus we get

$$\begin{aligned}\frac{\partial}{\partial t}\frac{\partial \overrightarrow{B}}{\partial s} =&(-\frac{\partial}{\partial t}(\kappa^2)-\frac{\kappa^{'}}{\kappa}(\kappa^2 f_1+\kappa^2\frac{\partial}{\partial s}(f_2)+\kappa\kappa^{'}f_2+\kappa^2\tau f_2)+\tau(\kappa^2 f_1+\kappa^2\frac{\partial}{\partial s}(f_2)+\kappa\kappa^{'}f_2+\kappa^2\tau f_2))\overrightarrow{T}\\ &+(-\kappa^2(f_1+\frac{\partial}{\partial s}(f_2)+\frac{\kappa^{'}}{\kappa}f_2+\tau f_2))\overrightarrow{N}+(-\kappa^2(-\tau f_3+\frac{\partial}{\partial s}(f_3)+\frac{\kappa^{'}}{\kappa}f_3)+\frac{\partial}{\partial t}(\frac{\kappa^{'}}{\kappa})-\frac{\partial}{\partial t}(\tau))\overrightarrow{B}\end{aligned}$$

Hence we see that

$$\frac{\partial\kappa^2}{\partial t}=\frac{\partial}{\partial s}(\kappa^2 f_1)+\frac{\partial}{\partial s}(\kappa^2\frac{\partial}{\partial s}f_2)+\frac{\partial}{\partial s}(\kappa\kappa^{'}f_2)+\frac{\partial}{\partial s}(\kappa^2\tau f_2)-\kappa\kappa^{'}f_1-\kappa\kappa^{'}\frac{\partial}{\partial s}f_2-\kappa^{'^2}f_2+\kappa^2\tau f_1+\kappa^2\tau\frac{\partial}{\partial s}f_2+\kappa^2\tau^2 f_2,$$

and

$$\frac{\partial\tau}{\partial t}=-\kappa^2\frac{\partial}{\partial s}(f_3)-\kappa\kappa^{'}f_3+\kappa^2\tau f_3+\frac{\partial}{\partial t}(\frac{\kappa^{'}}{\kappa}).$$

□

The proof is completed.

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