http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 73, Number 1, Pages 122–130 (2024)
DOI:10.31801/cfsuasmas.1275521
ISSN 1303-5991 E-ISSN 2618-6470

COMMUNICATIONS SERIES A1

Research Article; Received: April 2, 2023; Accepted: October 6, 2023

MAJORIZATION PROPERTY FOR CERTAIN CLASSES OF ANALYTIC FUNCTIONS ASSOCIATED WITH GENERAL OPERATOR

Pranay GOSWAMI[1], Murli Manohar GOUR[2], Som Prakash GOYAL[3] and Lokesh Kumar YADAV[4]

[1]School of Liberal Studies, Dr B.R, Ambedkar University Delhi, INDIA
[2,4]Department of Mathematics, Vivekananda Global University Jaipur, INDIA
[3]Department of Mathematics, University of Rajasthan, Jaipur, INDIA

Abstract. In this study, we introduce two new classes $S_k[E, F; \mu; \gamma]$ and $T_k(\theta, \mu, \gamma)$ of analytic functions using the general integral operator. For these two classes, we study the majorization properties. Some applications of the results are discussed in the form of corollaries.

1. Introduction and Definitions

The Majorization for two analytic functions u and v is defined as follows (see [17])

$$u(\xi) \prec\prec v(\xi); \quad (\xi \in D),$$

if there is an analytic function $\psi(\xi)$, such that

$$|\psi(\xi)| \leq 1 \text{ and } u(\xi) = \psi(\xi)\, v(\xi); \quad (\xi \in D), \tag{1}$$

where $D = \{\xi \in \mathbb{C} : |\xi| < 1\}$ is an open unit disk.
The function u is subordinate to v and defined as $u(\xi) \prec v(\xi)$, if there is a *schwarz function* w, that is analytic in D with $|w(\xi)| < 1$, $w(0) = 0$, $\xi \in D$ such that $u(\xi) = v(w(\xi))$, $\xi \in D$.

Thus, by combining subordination and majorization, we may define quasi-subordination as follows:

2020 *Mathematics Subject Classification.* 30C45.
Keywords. Univalent functions, quasi-subordination, subordination, majorization property.

[1] pranaygoswami83@gmail.com; 0000-0003-1205-1975
[2] murlimanohar.gaur@vgu.ac.in-Corresponding author; 0000-0003-2907-8226
[3] somprg@gmail.com; 0009-0007-9016-9529
[4] lokeshmanjeetyadav8@gmail.com; 0000-0003-0896-5723.

We say that the function u is quasi-subordinate relative to $\phi(z)$ to the function v and defined as (See [19])

$$u(\xi) \prec_q v(\xi); \quad (\xi \in D).$$

If there are two analytic functions $\psi(\xi)$ and $w(\xi)$ in D such that $\frac{u(\xi)}{\psi(\xi)}$ is analytic and subordinate to $v(\xi)$ in D and

$$|\psi(\xi)| \leq 1 \text{ and } w(0) = 0, \ |w(\xi)| \leq 1; \quad (\xi \in D),$$

satisfying

$$u(\xi) = \psi(\xi)\, v\big(w(\xi)\big); \quad (\xi \in D). \tag{2}$$

Remark 1. *(i) We have the conventional definition of subordination if we put* $\psi(\xi) = 1$ *in* (2).
(ii) We have the conventional definition of majorization if we put $w(\xi) = \xi$ *in* (2).

Let $\mathcal{A}$ be the class of all functions of the form

$$f(\xi) = \xi + \sum_{\mathfrak{K}=2}^{\infty} a_{\mathfrak{K}}\, \xi^{\mathfrak{K}}\,; \quad (\xi \in D), \tag{3}$$

which are analytic in open unit disk D, and consider $H_s : \mathcal{A} \to \mathcal{A}$ be an operator such that $\frac{\xi H'_{s+1}(f)(\xi)}{H_{s+1}(f)(\xi)}$ is analytic in D with

$$\left.\frac{\xi H'_{s+1}(f)(\xi)}{H_{s+1}(f)(\xi)}\right|_{\xi=0} = \beta + k + \gamma.$$

and satisfies

$$\xi H'_{s+1}(f)(\xi) = k\, H_{s+1}(f)(\xi) + m\, H_s(f)(\xi), \quad \forall f \in \mathcal{A}. \tag{4}$$

for some γ, m, $k \in \mathbb{C}$, and β is a real number with $\beta > 0$ (See [2]).

Remark 2. *(i) If we take* $k = -n$, $m = n + 1$, $\beta = 1 - \eta$, *and* $\gamma = \eta + n$ *for some integers* $n > -1$ *and* $0 \leq \eta < 1$, *then the operator* H_s *reduced into the integral operator* $\mathcal{I}_n$ *introduced by Liu and Noor in [16].*
(ii) If we take $k = -b$, $m = 1+b$, $\mu = 1-\alpha$ *and* $\gamma = \alpha+b$, *for* $b \in \mathbb{C}\setminus Z_0^-$, $0 \leq \alpha < 1$, *then the operator* H_s *reduced into the Srivastava-Attiya operator* $J_{s,b}$, *(see [12] and [20]).*

Now, using the operator H_s, we express the following classes of analytic functions.

Definition 1. *The function* $f \in \mathcal{A}$ *is stated to be in the class* $S_k[E, F; \mu; \gamma]$ *if and only if*

$$1 + \frac{1}{\mu}\left(\frac{\xi\left(H_s f(\xi)\right)'}{H_s f(\xi)} - k - \gamma\right) \prec \frac{1 + E\,\xi}{1 + F\,\xi}, \tag{5}$$

with k, $\gamma \in \mathbb{C}$, $\mu \in \mathbb{C} \setminus \{0\}$ *and* $-1 \leq F < E \leq 1$.

If we take the value of k, m, β and γ as defined in Remark (1.2)(i), then this class becomes $S_n[E, F; \mu; \eta]$ which is defined by Liu and Noor in [16].
Again if we take the value of k, m, μ and γ as defined in Remark (1.2)(ii), then this class becomes $H_{s,\, b,\, \alpha}(E,\, F)$ which is defined by Kutbi and Attiya in [12].

Definition 2. *The function $f \in \mathcal{A}$ is stated to be in the class $T_k(\theta,\, \mu,\, \gamma)$ if and only if*

$$\frac{e^{i\theta}}{\mu + k + \gamma}\left(\frac{\xi\big(H_s f(\xi)\big)^{'}}{H_s f(\xi)}\right) \prec e^{\xi} cos\theta + isin\theta; \quad (\xi \in D), \tag{6}$$

where $k,\, \gamma \in \mathbb{C}$, $\mu \in \mathbb{C} \setminus \{0\}$ and $-\frac{\Pi}{2} < \theta < \frac{\Pi}{2}$.
If we take the value of k, m, β and γ as defined in Remark (1.2)(i), then this class become as $T_n[\theta; \mu; \eta]$.
If we take the value of k, m, μ and γ as defined in Remark (1.2)(ii), then this class becomes $T_{b,\, \alpha}$.

Numerous mathematicians have recently investigated various majorization problems for univalent and multivalent functions as well as meromorphic and multivalent comprising distinct operators and different groups, $\big($see [1], [6], [7], [8], [9], [10], [21], [22]$\big)$.

The majorization problems of the classes $S_k[E, F; \mu; \gamma]$ and $T_k(\theta,\, \mu,\, \gamma)$ are explored in this study as follows:

2. Main Results

Theorem 1. *Assume the function $f \in \mathcal{A}$ and that $g \in S_k[E, F; \mu; \gamma]$. If $H_s f(\xi)$ is majorized by $H_s g(\xi)$ in D, then*

$$|H_{s-1}\, f(\xi)| \le |H_{s-1}\, g(\xi)|, \quad for \quad |\xi| \le \epsilon_0, \tag{7}$$

where the least positive root of following equation is ϵ_0.

$$\begin{aligned} &|\mu(E-F) + \gamma\, F|\epsilon^3 - (2|F| + |\gamma|)\epsilon^2 - \big[2 + |\mu(E-F) + \gamma\, F|\big]\epsilon \\ &+ |\gamma| = 0, \end{aligned} \tag{8}$$

and $-1 \le F < E \le 1$, $k,\, \gamma,\, m \in \mathbb{C}$, $\mu \in \mathbb{C} \setminus \{0\}$.

Proof. Since $g \in S_k[E, F; \mu; \gamma]$ then, from (5) and definition of majorization

$$1 + \frac{1}{\mu}\left(\frac{\xi\big(H_s' g(\xi)\big)}{H_s g(\xi)} - k - \gamma\right) = \frac{1 + E\, w(\xi)}{1 + F\, w(\xi)},$$

with $w(0) = 0$ and $|w(\xi)| \le |\xi| < 1$, $\forall \xi \in D$.
Now, from the above equality

$$\frac{\xi\big(H_s' g(\xi)\big)}{H_s g(\xi)} = \frac{(k+\gamma) + (\mu\,(E-F) + (k+\gamma)F)\, w(\xi)}{1 + F\, w(\xi)}. \tag{9}$$

Using the relation (4), that is,

$$\xi\Big(H_s' g(\xi)\Big) = kH_S\, g(\xi) + m\, H_{S-1}\, g(\xi),$$

for k, $m \in \mathbb{C}$, we have from (9) as

$$\frac{H_{S-1}\, g(\xi)}{H_s\, g(\xi)} = \frac{\gamma + \big(\mu(E-F) + \gamma\, F\big) w(\xi)}{m\,\big(1 + F\, w(\xi)\big)},$$

which implies that

$$|H_s\, g(\xi)| \leq \frac{|m|\,(1 + |F|\,|\xi|)\,|H_{S-1}\, g(\xi)|}{|\gamma| - \big|\mu(E-F) + \gamma\, F\big|\,|\xi|}. \tag{10}$$

As $H_s\, f(\xi)$ is majorized by $H_S\, g(\xi)$ in open unit disk D, then

$$H_s f(\xi) = \psi(\xi)\, H_s g(\xi). \tag{11}$$

Multiplying (11) by ξ after differentiating with respect to ξ, we get

$$\xi\big(H_s'\, f(\xi)\big) = \xi\, \psi(\xi)\,\big(H_s'\, g(\xi)\big) + \xi\, \psi^{'}(\xi)\, H_s\, g(\xi),$$

on using relation (4), we have

$$m\;\, H_{s-1}\, f(\xi) = \xi\, \psi^{'}(\xi)\, H_s\, g(\xi) + m\, \psi(\xi)\, H_{s-1}\, g(\xi)$$

that implies

$$|m|\,|H_{s-1}\, f(\xi)| \leq |\xi|\,|\psi^{'}(\xi)|\,|H_s\, g(\xi)| + |m|\,|\psi(\xi)|\,|H_{s-1}\, g(\xi)|. \tag{12}$$

As a consequence, considering that the ψ (*Schwarz function*) meets the inequality, (see [18])

$$|\psi^{'}(\xi)| \leq \frac{1 - |\psi(\xi)|^2}{1 - |\xi|^2}; \qquad (\xi \in D), \tag{13}$$

on using (10) and (13) in (12), we have

$$|H_{s-1}\, f(\xi)| \leq \left[\frac{|\xi|(1 - |\psi(\xi)|^2)(1 + |F|\,|\xi|)}{(1 - |\xi|^2)\Big(|\gamma| - |\mu(E-F) + \gamma\, F|\,|\xi|\Big)} + |\psi(\xi)|\right] |H_{s-1}\, g(\xi)|. \tag{14}$$

Setting $|\xi| = \epsilon$, $|\psi(\xi)| = \kappa$, then inequality (14) leads to

$$|H_{s-1}\, f(\xi)| \leq \frac{\zeta(\epsilon, \kappa)\,|H_{s-1}\, g(\xi)|}{(1 - \epsilon^2)\Big(|\gamma| - |\mu(E-F) + \gamma\, F|\;\, \epsilon\Big)}, \tag{15}$$

where

$$\zeta(\epsilon, \kappa) = \epsilon(1 - \kappa^2)(1 + |F|\epsilon) + \kappa(1 - \epsilon^2)\big[|\gamma| - |\mu(E-F) + \gamma\, F|\epsilon\big].$$

Then, from (15)

$$|H_{s-1}\, f(\xi)| \leq \mathfrak{T}(\epsilon, \kappa)\,|H_{s-1}\, g(\xi)|, \tag{16}$$

where

$$\mathfrak{T}(\epsilon, \kappa) = \frac{\zeta(\epsilon, \kappa)}{(1-\epsilon^2)\Big(|\gamma| - |\mu(E-F) + \gamma\, F|\, \epsilon\Big)}, \tag{17}$$

from relation (16), in an attempt to prove our result, we have to specify

$$\begin{aligned} \epsilon_0 &= max\big\{\epsilon \in [0,1);\ \mathfrak{T}(\epsilon, \kappa) \le 1; \quad \forall \kappa \in [0,1]\big\} \\ &= max\big\{\epsilon \in [0,1);\ G(\epsilon, \kappa) \ge 0; \quad \forall \kappa \in [0,1]\big\}, \end{aligned}$$

where

$$\begin{aligned} G(\epsilon, \kappa) =& (1-\epsilon^2)(1-\kappa)\Big[|\gamma| - |\mu(E-F) + \gamma\, F|\, \epsilon\Big] \\ & - \epsilon(1-\kappa^2)(1+|F|\epsilon). \end{aligned}$$

A simple calculation shows that the $G(\epsilon, \kappa) \ge 0$ inequality is equivalent to

$$\begin{aligned} u(\epsilon, \kappa) =& \Big[|\gamma| - |\mu(E-F) + \gamma\, F|\epsilon\Big]\ (1-\epsilon^2) \\ & - \epsilon(1+\kappa)(1+|F|\epsilon) \ge 0, \end{aligned}$$

while the function $u(\epsilon, \kappa)$ has a least value at $\kappa = 1$, i.e.
$min\{u(\epsilon, \kappa) : \kappa \in [0,1]\} = u(\epsilon, 1) = v(\epsilon)$,
where

$$\begin{aligned} v(\epsilon) =& |\mu(E-F) + \gamma\, F|\epsilon^3 - (2|F| + |\gamma|)\epsilon^2 \\ & - \big[2 + |\mu(E-F) + \gamma\, F|\big]\epsilon + |\gamma| = 0, \end{aligned}$$

it follows that $v(\epsilon) \ge 0; \quad \forall \epsilon \in [0, \epsilon_0]$, where $\epsilon_0 = \epsilon_0(\mu, \gamma, E, F)$ is the least positive root of equation (8), which proves the conclusion of (7). □

Theorem 2. *Assume the function $f \in \mathcal{A}$ and that $g \in T_k(\theta,\, \mu,\, \gamma)$. If $H_s f(\xi)$ is majorized by $H_s g(\xi)$ in D, therefore*

$$|H_{s-1} f(\xi)| \le |H_{s-1} g(\xi)| \qquad for \quad |\xi| \le \epsilon_1, \tag{18}$$

where the least positive root of following equation is ϵ_1.

$$\epsilon^2\big(|\mu+k+\gamma|\, e^{\epsilon} - |k| - |\mu+\gamma||tan\theta|\big) + 2\epsilon|sec\theta| - \big(|\mu+k+\gamma|\, e^{\epsilon} - |k| - |\mu+\gamma||tan\theta|\big) = 0, \tag{19}$$

and $\gamma,\, k \in \mathbb{C}$, $-\frac{\Pi}{2} < \theta < \frac{\Pi}{2}$, $\mu \in \mathbb{C} \setminus \{0\}$.

Proof. Since, $g \in T_k(\theta,\, \mu,\, \gamma)$ then, from (1) and the subordination relation

$$\frac{e^{i\theta}}{\mu + k + \gamma}\left(\frac{\xi\big(H_s'\, g(\xi)\big)}{H_s g(\xi)}\right) = e^{w(\xi)} cos\theta\, + isin\theta, \tag{20}$$

with $w(0) = 0$ and $|w(\xi)| \le 1 \quad \forall \xi \in D$.
From (20), we have

$$\frac{\xi\, H_s'\, g(\xi)}{H_s g(\xi)} = (\mu + k + \gamma)\left(\frac{e^{w(\xi)} + itan\theta}{1 + itan\theta}\right). \tag{21}$$

Now, using (4) in (21), for γ, m, $k \in \mathbb{C}$ and $\mu \in \mathbb{C} \setminus \{0\}$, we have the following.

$$\frac{H_{s-1}\, g(\xi)}{H_s g(\xi)} = \frac{(\mu + k + \gamma)\, e^{w(\xi)} - k + (\gamma + \mu) i tan\theta}{m(1 + i tan\theta)}$$

which implies that

$$|H_s g(\xi)| \leq \frac{|m|\, |sec\theta|}{\big(|\mu + k + \gamma|\, e^{|\xi|} - |k| - |\mu + \gamma||tan\theta|\big)} |H_{s-1}\, g(\xi)|. \tag{22}$$

Now, since $H_s\, f(\xi)$ is majorized by $H_s\, g(\xi)$ in D, we have

$$H_s f(\xi) = \psi(\xi) H_s g(\xi). \tag{23}$$

Multiplying (23) by ξ after differentiating with respect to ξ, we get

$$\xi\big(H_s'\, f(\xi)\big) = \xi\, \psi(\xi)\, \big(H_s'\, g(\xi)\big) + \xi\, \psi^{'}(\xi)\, H_s\, g(\xi),$$

on using relation (4), we have

$$m\ H_{s-1}\, f(\xi) = \xi\, \psi^{'}(\xi)\, H_s\, g(\xi) + m\, \psi(\xi)\, H_{s-1}\, g(\xi)$$

that implies

$$|m|\, |H_{s-1}\, f(\xi)| \leq |\xi|\, |\psi^{'}(\xi)|\, |H_s\, g(\xi)| + |m|\, |\psi(\xi)|\, |H_{s-1}\, g(\xi)|. \tag{24}$$

As a consequence, considering that the ψ (*Schwarz function*) meets the inequality, (see [18])

$$|\psi^{'}(\xi)| \leq \frac{1 - |\psi(\xi)|^2}{1 - |\xi|^2}; \qquad (\xi \in D), \tag{25}$$

using (22) and (25) in (24), we have

$$|H_{s-1}\, f(\xi)| \leq \left(\frac{|\xi|(1 - |\psi(\xi)|^2)|sec\theta|}{(1 - |\xi|^2)\big(|\mu + k + \gamma|\, e^{|\xi|} - |k| - |\mu + \gamma||tan\theta|\big)} + |\psi(\xi)| \right) |H_{s-1}\, g(\xi)|. \tag{26}$$

Setting $|\xi| = \epsilon$, $|\psi(\xi)| = \kappa \quad (0 \leq \kappa \leq 1)$, then inequality (26) leads to

$$|H_{s-1}\, f(\xi)| \leq \frac{\zeta_1(\epsilon, \kappa)}{(1 - \epsilon^2)\left(|\mu + k + \gamma|\, e^{\epsilon} - |k| - |\mu + \gamma||tan\theta|\right)} |H_{s-1}\, g(\xi)|, \tag{27}$$

where

$$\zeta_1(\epsilon, \kappa) = \epsilon(1 - \kappa^2)|sec\theta| + \kappa(1 - \epsilon^2)\big(|\mu + k + \gamma|\, e^{\epsilon} - |k| - |\mu + \gamma||tan\theta|\big).$$

Then, from (27)

$$|H_{s-1}\, f(\xi)| \leq \mathfrak{T}_1(\epsilon, \kappa)\ |H_{s-1}\, g(\xi)|, \tag{28}$$

where

$$\mathfrak{T}_1(\epsilon, \kappa) = \frac{\zeta_1(\epsilon, \kappa)}{(1 - \epsilon^2)\left(|\mu + k + \gamma|\, e^{\epsilon} - |k| - |\mu + \gamma||tan\theta|\right)}, \tag{29}$$

From relation (28), in order to prove our result, we have to specify

$$\epsilon_1 = max\Big\{\epsilon \in [0, 1);\ \mathfrak{T}_1(\epsilon, \kappa) \leq 1 \quad \forall \kappa \in [0, 1]\Big\}$$

$$= max\{\epsilon \in [0,1);\ G_1(\epsilon,\kappa) \geq 0 \quad \forall \kappa \in [0,1]\},$$

where

$$G_1(\epsilon,\kappa) = (1-\epsilon^2)(1-\kappa)\big(|\mu+k+\gamma|\, e^{\epsilon} - |k| - |\mu+\gamma||tan\theta|\big) - \epsilon(1-\kappa^2)|sec\theta|.$$

A quick calculation illustrates that the inequality $G_1(\epsilon,\kappa) \geq 0$ is equivalent to

$$u_1(\epsilon,\kappa) = (1-\epsilon^2)\big(|\mu+k+\gamma|\, e^{\epsilon} - |k| - |\mu+\gamma||tan\theta|\big) - \epsilon(1+\kappa)|sec\theta| \geq 0,$$

while the function $u_1(\epsilon,\kappa)$ takes its lowest value at $\kappa = 1$, that is,

$$min\{u_1(\epsilon,\kappa) : \kappa \in [0,1]\} = u_1(\epsilon,1) = v_1(\epsilon),$$

where

$$v_1(\epsilon) = (1-\epsilon^2)\big(|\mu+k+\gamma|\, e^{\epsilon} - |k| - |\mu+\gamma||tan\theta|\big) - 2\epsilon|sec\theta| = 0,$$

It follows that $v_2(\epsilon) \geq 0 \quad \forall \epsilon \in [0,\epsilon_1]$, where $\epsilon_1 = \epsilon_1(\theta,\, \gamma,\, \mu,\, k)$ is the least positive root of equation (19), which proves the conclusion of (18). □

3. Corollaries and Consequences

Corollary 1. *Assume the function $f \in \mathcal{A}$ and that $g \in S_n[E,F;\mu;\eta]$. If $\mathcal{I}_n\, f(\xi)$ is majorized by $\mathcal{I}_n\, g(\xi)$ in D, then*

$$|\mathcal{I}_{n-1}\, f(\xi)| \leq |\mathcal{I}_{n-1}\, g(\xi)| \quad for \quad |\xi| \leq \epsilon_2, \tag{30}$$

where the least positive root of following equation is ϵ_2.

$$\big|\mu\, E+(n+\eta-\mu)F\big|\epsilon^3 - \big(2|F|+|n+\eta|\big)\epsilon^2 - \big(2+|\mu\, E+(\eta+n-\mu)F|\big)\epsilon + |\eta+n| = 0, \tag{31}$$

and $-1 \leq F < E \leq 1$, $\mu \in \mathbb{C} \setminus \{0\}$, $n > -1$, $0 \leq \eta < 1$,.

Corollary 2. *Assume the function $f \in \mathcal{A}$ and that $g \in T_n[\theta;\mu;\eta]$. If $\mathcal{I}_n\, f(\xi)$ is majorized by $\mathcal{I}_n\, g(\xi)$ in D, then*

$$|\mathcal{I}_{n-1}\, f(\xi)| \leq |\mathcal{I}_{n-1}\, g(\xi)| \quad for \quad |\xi| \leq \epsilon_3, \tag{32}$$

where the least positive root of following equation is ϵ_3.

$$(|\mu+\eta|e^{\epsilon} - |n| - |\mu+\eta+n||tan\theta|)\epsilon^2 - 2|sec\theta|\epsilon - (-|n| - |\mu+\eta+n||tan\theta| + |\mu+\eta|e^{\epsilon}) = 0, \tag{33}$$

and $n > -1$, $0 \leq \eta < 1$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$..

Corollary 3. *Assume the function $f \in \mathcal{A}$ and that $g \in H_{s,\,b,\,\alpha}(E,\, F)$. If $J_{s,b}\, f(\xi)$ is majorized by $J_{s,b}\, g(\xi)$ in D, then*

$$|J_{s-1,b}\, f(\xi)| \leq |J_{s-1,b}\, g(\xi)| \quad for \quad |\xi| \leq \epsilon_4, \tag{34}$$

where the least positive root of following equation is ϵ_4.

$$\big|(1-\alpha)E + (2\alpha+b-1)\big|\epsilon^3 - \big(2|F| + |\alpha+b|\big)\epsilon^2 - \big(2 + |(1-\alpha)E + (2\alpha+b-1)F|\big)\epsilon + |\alpha+b| = 0, \tag{35}$$

and $-1 \leq F < E \leq 1$, $b \in \mathbb{C} \setminus Z_0^-$, $0 \leq \alpha < 1$.

Corollary 4. *Assume the function $f \in \mathcal{A}$ and that $g \in T_{b,\alpha}$. If $J_{s,b}\, f(\xi)$ is majorized by $J_{s,b}\, g(\xi)$ in D, then*

$$|J_{s-1,b}\, f(\xi)| \leq |J_{s-1,b}\, g(\xi)| \quad for \quad |\xi| \leq \epsilon_5, \tag{36}$$

where the least positive root of following equation is ϵ_5.

$$\left(e^{\epsilon} - |1+b||tan\theta| - |b|\right)\epsilon^2 + 2|sec\theta|\, \epsilon - \left(e^{\epsilon} - |b| - |1+b||tan\theta|\right) = 0, \tag{37}$$

and $b \in \mathbb{C} \setminus Z_0^-$, $0 \leq \alpha < 1$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

Author Contribution Statements Each author contributes equally to the preparation of the manuscript.

Declaration of Competing Interests The author declares that he has no competing interests.

Acknowledgements The authors thank the reviewer and the editor for their constructive comments to improve the article.

References

[1] Altintas, O., Özkan, Ö., Srivastava, H. M., Majorization by starlike functions of complex order, *Complex Var. Elliptic Equ.*, (46)(3) (2001), 207–218. https://doi.org/10.1080/17476930108815409

[2] Attiya, A. A., Bulboacă, T., A general theorem associated with the Briot-Bouquet differential subordination, *J. Comput. Anal. Appl.*, 16(4) (2014), 722–730.

[3] Bulboacă, T., Classes of first-order differential subordinations, *Demonstr. Math.*, 35 (2) (2002), 287–292. https://doi.org/10.4236/am.2011.27114.

[4] Flett, T. M., The dual of an inequality of Hardy and Littlewood and some related inequalities, *J. Math. Anal. Appl.*, 38(3) (1972), 746–765. https://doi.org/10.1016/0022-247X(72)90081-9

[5] Fournier, R., Ruscheweyh, S., On two extremal problems related to univalent functions, *Rocky Mt. J. Math.*, 24 (1994), 529–538. https://doi.org/10.1216/rmjm/1181072416

[6] Goswami, P., Sharma B., Bulboacă, T., Majorization for certain classes of analytic functions using multiplier transformation, *Appl. Math. Lett.*, 23(5) (2010), 633–637. https://doi.org/10.1016/j.aml.2010.01.029

[7] Goswami, P., Aouf, M. K., Majorization properties for certain classes of analytic functions using the Salagean operator, *Appl. Math. Lett.*, 23(11) (2010), 1351–1354. https://doi.org/10.1016/j.aml.2010.06.030

[8] Gour, M. M., Goswami, P., Frasin B A., Althobaithi S., Majorization for certain classes of analytic functions defined by Fournier-Ruscheweyh integral operator, *Journal of Mathematics*, 2022 (2022), Article ID: 6580700. https://doi.org/10.1155/2022/6580700

[9] Goyal, S. P., Goswami, P., Majorization for certain classes of meromorphic functions defined by integral operator, *Ann. Univ. Mariae Curie-Skłodowska, Sect. A*, 66(2) (2012), 57-62. https://doi.org/10.2478/v10062-012-0013-1

[10] Goyal, S. P., Goswami, P., Majorization for certain classes of analytic functions defined by fractional derivatives, *Appl. Math. Lett.*, 22(12) (2009), 1855-1858. https://doi.org/10.1016/j.aml.2009.07.009

[11] Jung, I. B., Kim, Y. C., Srivastava, H. M., The Hardy space of analytic functions associated with certain one-parameter families of integral operators, *J. Math. Anal. Appl.*, 176(1) (1993), 138-147. https://doi.org/10.1006/jmaa.1993.1204

[12] Kutbi, M. A., Attiya, A. A., Differential subordination result with the Srivastava-Attiya integral operator, *J. Inequal. Appl.*, 2010 (2010), 1-10. https://doi.org/10.1155/2010/618523

[13] Li, J. L., Srivastava, H. M., Starlikeness of functions in the range of a class of integral operators, *Integral Transforms Spec. Funct.*, 15(2) (2004), 129-136. https://doi.org/10.1080/10652460310001600708

[14] Li, J. L., Srivastava, H. M., Some questions and conjectures in the theory of univalent functions, *Rocky Mt. J. Math.*, 28 (1998), 1035-1041. http://hdl.handle.net/1828/1649

[15] Li, J. L., Notes on Jung-Kim-Srivastava integral operator, *J. Math. Anal. Appl.*, 294(1) (2004), 96-103. https://doi.org/10.1016/j.jmaa.2004.01.040

[16] Liu, J. L., Noor, K. I., Some properties of Noor integral operator, *J. Nat. Geom.*, 21(1-2) (2002), 81–90. https://doi.org/10.1016/S0022-247X(03)00270-1

[17] MacGreogor, T. H., Majorization by univalent functions, *Duke Math. J.*, 34 (1967), 95-102. https://doi.org/10.1215/S0012-7094-67-03411-4

[18] Nehari Z., Conformal Mapping, MacGraw-Hill Book Company, New York, Toronto, London, 1955.

[19] Roberston M.S., Quasi-subordination and coefficient conjectures, *Bull. Am. Math. Soc.*, 76 (1970), 1-9.

[20] Srivastava, H. M., Attiya, A. A., An integral operator associated with the Hurwitz-Lerch Zeta function and differential subordination, *Integral Transforms Spec. Funct.*, 18(3) (2007), 207-216. https://doi.org/10.1080/10652460701208577

[21] Tang, H., Aouf, M. K., Deng, G. T., Majorization problems for certain subclasses of meromorphic multivalent functions associated with the Liu-Srivastava operator, *Filomat*, 29(4) (2015), 763-772. https://doi.org/10.2298/FIL1504763T

[22] Tang, H., Deng, G., Majorization problems for two subclasses of analytic functions connected with the Liu-Owa integral operator and exponential function, *J. Inequal. Appl.*, 277 (2018), 1-11. https://doi.org/10.1186/s13660-018-1865-x