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RESEARCH PAPER

# The investigation of several soliton solutions to the complex Ginzburg-Landau model with Kerr law nonlinearity

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## Abstract

This work investigates the complex Ginzburg-Landau equation (CGLE) with Kerr law in nonlinear optics, which represents soliton propagation in the presence of a detuning factor. The $\varphi^6$-model expansion approach is used to find optical solitons such as dark, bright, singular, and periodic as well as the combined soliton solutions to the model. The results presented in this study are intended to improve the CGLE's nonlinear dynamical characteristics, it might also assist in comprehending some of the physical implications of various nonlinear physics models. The hyperbolic sine, for example, appears in the calculation of the Roche limit and gravitational potential of a cylinder, while the hyperbolic cotangent appears in the Langevin function for magnetic polarization. The current research is frequently used to report a variety of fascinating physical phenomena, such as the Kerr law of non-linearity, which results from the fact that an external electric field causes non-harmonic motion of electrons bound in molecules, which causes nonlinear responses in a light wave in an optical fiber. The obtained solutions' 2-dimensional, 3-dimensional, and contour plots are shown.

**Key words**: $\varphi^6$-model expansion method; complex Ginzburg-Landau equation; traveling wave solution; Kerr law nonlinearity
**AMS 2020 Classification**: 35Qxx; 35C07; 35Q51

## 1 Introduction

Partial differential equations were first employed for the study of surfaces in geometry [1, 2, 3, 4, 5] and a vast range of mechanical issues. Renowned mathematicians from throughout the world were keenly interested in studying a wide range of issues brought on by partial differential equations in the late 19th century [6]. Since optical solitons which are the solutions of the NPDEs can be used as information carriers for transmitting digital signals over long distances in optical fiber networks, the propagation of optical solitons in nonlinear optical fibers has received a lot of attention [7, 8, 9, 10, 11]. Maintaining a moderate balance between nonlinearity and group velocity dispersion is the fundamental concept for the presence of the optical solitons. The study of exact solutions of the nonlinear partial differential equations NLPDEs, as scientific methods of the concepts, will help one to clarify these phenomena. Many successful methods for obtaining exact solutions of NLPDEs, such as the Adomian's decomposition method [12], exponential rational function method [13], the $F$-expansion method [14], the $\left(\frac{1}{G'}\right)$-expansion method [15, 16], Jacobi elliptic function technique [17, 18], the modified sub-equation method [19], the $\left(\frac{G'}{G}\right)$-expansion method [20], the auto-Bäcklund transformation method [21], extended direct algebraic method [22], the homoclinic technique [23], reduction perturbation method [24],

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the $\varphi^6$-model expansion method [25, 26, 27, 28], the nonstandard finite difference [29]. The recent developments in the field of mathematical modelling as well as its applications have been introduced in the last few decades [30, 31, 32].

Many researchers have recently solved the CGLE. Chu et al. [33] have solved this equation with the help of modified extended tanh technique and received different forms of solitons, such as, hyperbolic and trigonometric functions. The modified simple equation method is used to obtain some bright, dark and singular soliton solutions by Arnous and Ahmed [34]. Liu and Yu [35] used the modified Hirota bilinear method and obtained Kink waves and period waves. In [36, 37], first integral method and $\left(\frac{G'}{G}\right)$-expansion method is used to secure the hyperbolic, trigonometric as well as rational function solution. Several integration techniques are used to obtain multiple soliton solutions such as bright, dark and singular soliton by Mirzazadeh and Ekici [38]. The other methods include GRE method [39], ansatz functions technique [40], and so on.

The main idea about this paper is to derive new solitons such as dark, bright, singular, rational, combined periodic, combined singular and periodic solitary wave solutions to the CGLE model using Kerr law nonlinearity with the help of the newly developed $\varphi^6$-model expansion method [41] which has not been studied yet based on our knowledge. The nonlinear responses that an external electric field-induced nonharmonic motion of electrons trapped in molecules causes to a light wave in an optical fiber give rise to the Kerr law of nonlinearity. The authors achieve their aims by retrieving new solutions which are different from the previous work.

The following is the outline for this paper: In Section 2, the mathematical analysis of the model is studied. The new $\varphi^6$-model expansion approach is described in Section 3. Section 4 consists of application of the proposed method on the complex Ginzburg-Landau equation using Kerr law nonlinearity to retrieve solitons such as dark, bright, singular, periodic, combined singular and combined periodic soliton solutions. Some of the traveling wave solution's physical structures are graphically displayed in the related 3D, 2D, and contour graphs. In Section 5, the result of the derived solutions is discussed, while the whole work is concluded in Section 6.

## 2 Mathematical analysis of the model

Arnous, Ahmed H., et al. [34] gives the dimensionless shape of (GCLE) that will be investigated in this article.

$$iq_t + aq_{xx} + cF(|q|^2)q = \frac{1}{|q|^2 q^*}\left[\alpha\,|q|^2\,(|q|^2)_{xx} - \beta\left\{(|q|^2)_x\right\}^2\right] + \gamma q, \tag{1}$$

where $q = q(x,t)$ is a complex function that describes the wave profile seen in a variety of phenomena such as nonlinear optics and plasma physics, $x$ is the non-dimensional distance along the fibers, $t$ is time in dimensionless form, $q^*$ is a conjugate of $q$, $a, c, \alpha, \beta$ and $\gamma$ are valued constants. The coefficients $a$ and $c$ are determined by the group velocity dispersion (GVD) and nonlinearity, respectively. The terms with $\alpha$, $\beta$ and $\gamma$ result from perturbation effects, specifically detuning.

In Eq. (1), $F$ is a real-valued algebraic function that must be smooth. $F(|q|^2)q$ is continuously differentiable $k$ times, implying that

$$F(|q|^2)q \in \cup_{m,n=1}^{\infty} C^k\left((-n,n)\times(-m,m)\,;R^2\right). \tag{2}$$

By setting up

$$\alpha = 2\beta, \tag{3}$$

Eq. (1) turns to

$$iq_t + aq_{xx} + cF(|q|^2)q = \frac{\beta}{|q|^2 q^*}\left[2\,|q|^2\,(|q|^2)_{xx} - \left\{(|q|^2)_x\right\}^2\right] + \gamma q. \tag{4}$$

To solve Eq. (1), the standard decomposition into phase-amplitude components yields:

$$q\,(x,t) = P(\zeta)e^{i(-kx+wt+\theta)}, \tag{5}$$

and the wave variable $\zeta$ is given by

$$\zeta = \lambda\,(x - vt)\,. \tag{6}$$

The function $P$ represents the pulse shape here, $v$ is the soliton's velocity. In the phase factor, $k$ denotes the frequency of the soliton, $\omega$ the soliton wave number and the phase constant $\theta$. Substituting the phase-amplitude decomposition into Eq. (4) results in the following couple of equations after breaking into real and imaginary parts [33, 34]:

$$-\left(ak^2 + \gamma + \omega\right)P + cF\left(P^2\right)P + (a - 4\beta)\,P'' = 0, \tag{7}$$

and

$$v = -2ka. \tag{8}$$

In the following part after the description of the method, Eq. (7) will be examined using Kerr's nonlinearity law.

## 3 Description of the method

According to Zayed et al. [28] the following are the key steps of a recent $\varphi^6$-model expansion method:

**Step-1**: Consider the following nonlinear evolution equation for $q = q(x,t)$

$$F(q, q_x, q_t, q_{xx}, q_{xt}, q_{tt}, \ldots) = 0, \tag{9}$$

there $F$ is a polynomial of $q(x,t)$ and its highest order partial derivatives, including its nonlinear terms.

**Step-2**: Making use of the wave transformation

$$q(x,t) = q(\zeta), \quad \zeta = \lambda\,(x - vt)\,, \tag{10}$$

where $v$ represents wave speed, then, Eq. (9) can be converted into the nonlinear ordinary differential equation shown below

$$\Omega(q, q^{'}, qq^{'}, q^{''}, \ldots) = 0, \tag{11}$$

where the derivatives with respect to $\zeta$ are represented by prime.

**Step-3**: Suppose that the formal solution to Eq. (11) exists:

$$q\,(\zeta) = \sum_{i=0}^{2N} \alpha_i U^i(\zeta), \tag{12}$$

where $\alpha_i (i = 0, 1, 2, \ldots, N)$ are to be determined constants, $N$ can be obtained using the balancing rule and $U(\zeta)$ satisfies the auxiliary NLODE;

$$U'^2(\zeta) = h_0 + h_2 U^2(\zeta) + h_4 U^4(\zeta) + h_6 U^6(\zeta), \tag{13}$$
$$U^{''}(\zeta) = h_2 U(\zeta) + 2h_4 U^3(\zeta) + 3h_6 U^5(\zeta),$$

where $h_i (i = 0, 2, 4, 6)$ are real constants that will be discovered later.

**Step-4**: It is well known that the answer to Eq. (13) is as follows;

$$U(\zeta) = \frac{P(\zeta)}{\sqrt{fP^2(\zeta) + g}}, \tag{14}$$

provided that $0 < fP^2(\zeta) + g$ and $P(\zeta)$ is the Jacobi elliptic equation solution

$$P'^2(\zeta) = l_0 + l_2 P^2(\zeta) + l_4 P^4(\zeta), \tag{15}$$

where $l_i (i = 0, 2, 4)$ are unknown constants to be determined, $f$ and $g$ are given by

$$f = \frac{h_4(l_2 - h_2)}{(l_2 - h_2)^2 + 3l_0 l_4 - 2l_2(l_2 - h_2)}, \tag{16}$$
$$g = \frac{3l_0 h_4}{(l_2 - h_2)^2 + 3l_0 l_4 - 2l_2(l_2 - h_2)},$$

under the restriction condition

$$h_4^2(l_2 - h_2)[9l_0 l_4 - (l_2 - h_2)(2l_2 + h_2)] + 3h_6[-l_2^2 + h_2^2 + 3l_0 l_4]^2 = 0. \tag{17}$$

**Step-5:** According to [28], it is well known that the Jacobi elliptic solutions of Eq. (15) can be calculated when $0 < m < 1$. We can have the exact solutions of Eq. (9) by substituting Eqs. (14) and (15) into Eq. (12).

| Function | $m \to 1$ | $m \to 0$ | Function | $m \to 1$ | $m \to 0$ |
|---|---|---|---|---|---|
| $sn(\zeta, m)$ | $tanh(\zeta)$ | $sin(\zeta)$ | $ds(\zeta, m)$ | $csch(\zeta)$ | $csc(\zeta)$ |
| $cn(\zeta, m)$ | $sech(\zeta)$ | $cos(\zeta)$ | $sc(\zeta, m)$ | $sinh(\zeta)$ | $tan(\zeta)$ |
| $dn(\zeta, m)$ | $sech(\zeta)$ | 1 | $sd(\zeta, m)$ | $sinh(\zeta)$ | $sin(\zeta)$ |
| $ns(\zeta, m)$ | $coth(\zeta)$ | $csc(\zeta)$ | $nc(\zeta, m)$ | $cosh(\zeta)$ | $sec(\zeta)$ |
| $cs(\zeta, m)$ | $csch(\zeta)$ | $cot(\zeta)$ | $cd(\zeta, m)$ | 1 | $cos(\zeta)$ |

## 4 Application of the $\varphi^6$-model expansion method

The Kerr law of nonlinearity is derived from the fact that a light wave in an optical fiber experiences nonlinear reactions due to non-harmonic electron motion in the presence of an external electric field. Since $F(u) = u$ for Kerr law nonlinearity, Eq. (4) is reduced to [33]

$$iq_t + aq_{xx} + c(|q|^2)q = \frac{\beta}{|q|^2 q^*}\left[2|q|^2(|q|^2)_{xx} - \left\{(|q|^2)_x\right\}^2\right] + \gamma q, \tag{18}$$

and Eq. (7) is transformed

$$-\left(ak^2 + \gamma + \omega\right)P + cP^3 + \lambda^2(a - 4\beta)P'' = 0, \tag{19}$$

from Eq. (19), we get $N = 1$ by balancing $P''$ with $P^3$, we can obtain the following by substituting $N = 1$ in Eq. (12)

$$P(\zeta) = \alpha_0 + \alpha_1 U(\zeta) + \alpha_2 U^2(\zeta), \tag{20}$$

where $\alpha_0, \alpha_1$ and $\alpha_2$ are constants to be determined.
We obtain the following algebraic equations by substituting Eq. (20) along with Eq. (13) into Eq. (19) and setting the coefficients of all powers of $U^i(\zeta)$, $i = 0, 1, \ldots, 6$ to be equal to zero;

$$\begin{aligned}
&U^0(\zeta);\ -\alpha_0\left(ak^2 + \gamma + \omega - c\alpha_0^2\right) + 2a\lambda^2 h_0\alpha_2 - 8\beta\lambda^2 h_0\alpha_2 = 0,\\
&U^1(\zeta);\ -\alpha_1\left(ak^2 + \gamma + \omega\right) + a\lambda^2 h_2\alpha_1 - 4\beta\lambda^2 h_2\alpha_1 + 3c\alpha_0^2\alpha_1 = 0,\\
&U^2(\zeta):\ 3c\alpha_0\alpha_1^2 - \alpha_2\left(ak^2 + \gamma + \omega\right) + 4a\lambda^2 h_2\alpha_2 - 16\beta\lambda^2 h_2\alpha_2 + 3c\alpha_0^2\alpha_2 = 0,\\
&U^3(\zeta):\ 2a\lambda^2 h_4\alpha_1 - 8\beta\lambda^2 h_4\alpha_1 + c\alpha_1^3 + 6c\alpha_0\alpha_1\alpha_2 = 0,\\
&U^4(\zeta):\ 6a\lambda^2 h_4\alpha_2 - 24\beta\lambda^2 h_4\alpha_2 + 3c\alpha_1^2\alpha_2 + 3c\alpha_0\alpha_2^2 = 0,\\
&U^5(\zeta):\ 3a\lambda^2 h_6\alpha_1 - 12\beta\lambda^2 h_6\alpha_1 + 3c\alpha_1\alpha_2^2 = 0,\\
&U^6(\zeta):\ 8a\lambda^2 h_6\alpha_2 - 32\beta\lambda^2 h_6\alpha_2 + c\alpha_2^3 = 0,
\end{aligned}$$

we get the following result after solving the resulting system:

$$\begin{aligned}
&\alpha_0 = 0,\quad \alpha_1 = \frac{\sqrt{2h_4}\sqrt{-a + 4\beta\lambda}}{\sqrt{c}},\quad \alpha_2 = 0,\\
&h_2 = \frac{ak^2 + \gamma + \omega}{(a - 4\beta)\lambda^2},\quad h_6 = 0.
\end{aligned} \tag{21}$$

In view of Eqs. (14), (20) and (21) along with the Jacobi elliptic functions in the table above, we obtain the following exact solutions of Eq. (18).
**1.** If $l_0 = 1$, $l_2 = -(1 + m^2)$, $l_4 = m^2$, $0 < m < 1$, then $P(\zeta) = sn(\zeta, m)$ or $P(\zeta) = cd(\zeta, m)$, and we have

$$q_{1,1}(x,t) = \frac{\sqrt{2h_4}\sqrt{-a + 4\beta\lambda}}{\sqrt{c}}\left[\frac{sn(\zeta, m)}{\sqrt{f\,(sn(\zeta, m))^2 + g}}\right]e^{i(-kx + wt + \theta)}, \tag{22}$$

or

$$q_{1,2}(x,t) = \frac{\sqrt{2h_4}\sqrt{-a + 4\beta\lambda}}{\sqrt{c}}\left[\frac{cd(\zeta, m)}{\sqrt{f\,(cd(\zeta, m))^2 + g}}\right]e^{i(-kx + wt + \theta)}, \tag{23}$$

such that $0 < c$, $\zeta = \lambda(x - vt)$ and $f$ and $g$ in Eqs. (16) are given by

$$\begin{aligned}
f &= \frac{(1 + m^2 + h_2)h_4}{1 - m^2 + m^4 - h_2^2},\\
g &= \frac{-3h_4}{1 - m^2 + m^4 - h_2^2},
\end{aligned} \tag{24}$$

under the restriction condition

$$-h_4^2\left(-1 - m^2 - h_2\right)\left(-1 + 2m^2 - h_2\right)\left(-2 + m^2 + h_2\right) = 0. \tag{25}$$

If $m \to 1$, then the dark optical soliton is obtained

$$q_{1,3}(x,t) = \frac{\sqrt{2h_4}\sqrt{-a+4\beta}\,\lambda\tanh(\zeta)}{\sqrt{c}\sqrt{\frac{(a-4\beta)\lambda^2 h_4\left(-3(a-4\beta)\lambda^2+\left(ak^2+\gamma+\omega+2(a-4\beta)\lambda^2\right)\tanh^2(\zeta)\right)}{-(ak^2+\gamma+\omega)^2+(a-4\beta)^2\lambda^4}}}e^{i(-kx+wt+\theta)}, \tag{26}$$

such that

$$-h_4^2(2+h_2)\left[-1+h_2\right]^2 = 0. \tag{27}$$

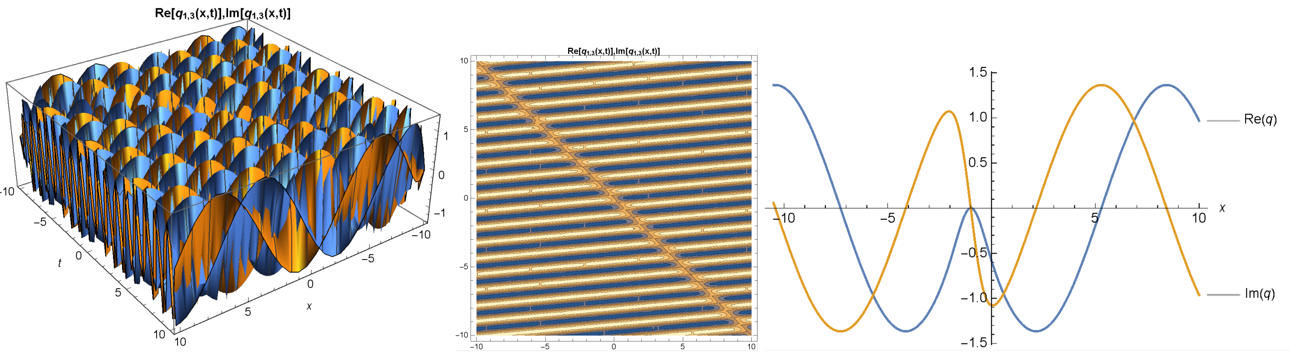


**Figure 1.** The numerical simulations corresponding to $|q_{1,3}|$ given by Eq. (26), for $m = 1$ ; (a) is the 3D graphic while (b) is the contour and (c) is the 2D graphic

If $m \to 0$, then the periodic solution is obtained

$$q_{1,4}(x,t) = \frac{\sqrt{2h_4}\sqrt{-a+4\beta}\,\lambda\sin(\zeta)}{\sqrt{c}\sqrt{\frac{(a-4\beta)\lambda^2 h_4\left(-3(a-4\beta)\lambda^2+\left(ak^2+\gamma+\omega+(a-4\beta)\lambda^2\right)\sin^2(\zeta)\right)}{-(ak^2+\gamma+\omega)^2+(a-4\beta)^2\lambda^4}}}e^{i(-kx+wt+\theta)}, \tag{28}$$

such that

$$h_4^2(-1-h_2)\left[(-2+h_2)(1+h_2)\right] = 0. \tag{29}$$

**2.** If $l_0 = 1-m^2$, $l_2 = 2m^2-1$, $l_4 = -m^2$, $0 < m < 1$, then $P(\zeta) = cn(\zeta, m)$, therefore

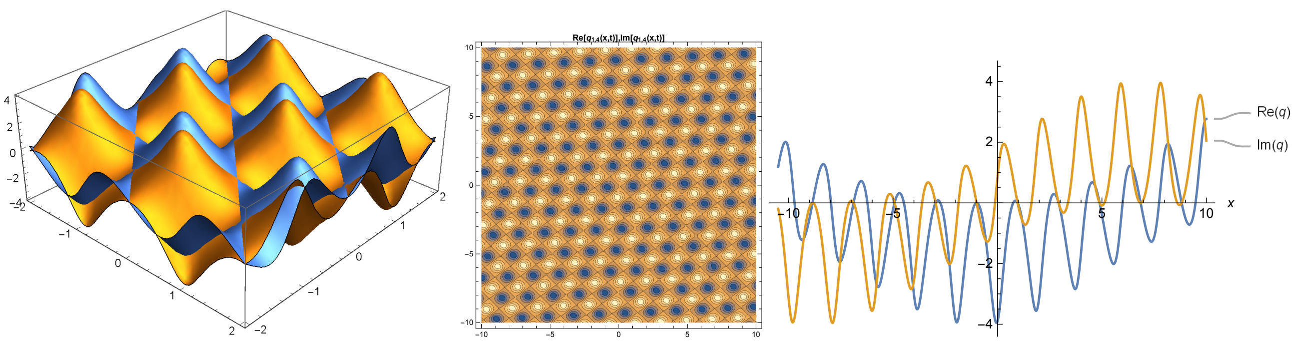


**Figure 2.** The numerical simulations corresponding to $|q_{1,4}|$ given by Eq. (28), for $m = 0$ ; (a), (b) and (c) are the 3D graphic, contour and 2D graphic, respectively

$$q_2(x,t) = \frac{\sqrt{2h_4}\sqrt{-a+4\beta}\,\lambda}{\sqrt{c}}\left[\frac{cn(\zeta,m)}{\sqrt{f\,(cn(\zeta,m))^2+g}}\right]e^{i(-kx+wt+\theta)}, \tag{30}$$

where $f$ and $g$ are determined by

$$f = -\frac{(-1+2m^2-h_2)h_4}{1-m^2+m^4-h_2^2}, \qquad (31)$$
$$g = \frac{3\left(-1+m^2\right)h_4}{1-m^2+m^4-h_2^2},$$

under the constraint condition

$$h_4^2\left(-1+2m^2-h_2\right)\left[\left(-2+m^2+h_2\right)\left(1+m^2+h_2\right)\right]=0. \qquad (32)$$

If $m\to 1$, then the bright optical soliton solution is retrieved

$$q_{2,1}(x,t)=\frac{\sqrt{2h_4}\sqrt{-a+4\beta}\lambda sech(\zeta)}{\sqrt{c}\sqrt{\frac{(a-4\beta)\lambda^2 h_4 sech^2(\zeta)}{ak^2+\gamma+\omega+(a-4\beta)\lambda^2}}}e^{i(-kx+wt+\theta)}, \qquad (33)$$

provided that

$$h_4^2(1-h_2)\left[h_2^2+h_2-2\right]=0. \qquad (34)$$

If $m\to 0$, then the periodic solution is obtained

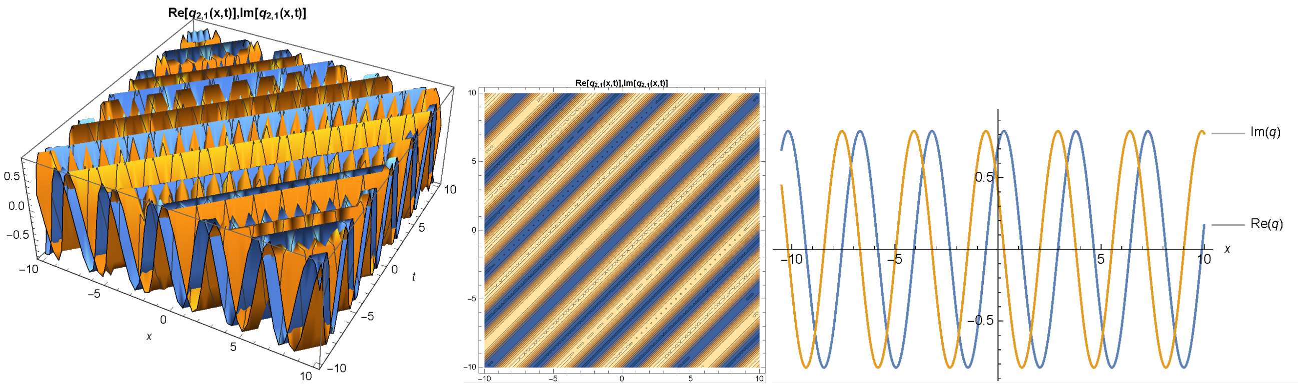


**Figure 3.** The numerical simulations corresponding to $|q_{2,1}|$ given by Eq. (33), for $m=1$; (a), (b) and (c) are the 3D graphic, contour and 2D graphic, respectively

$$q_{2,2}(x,t)=\frac{\sqrt{2h_4}\sqrt{-a+4\beta}\lambda\sin(\zeta)}{\sqrt{c}\sqrt{\frac{(a-4\beta)\lambda^2 h_4\left(-3(a-4\beta)\lambda^2+(ak^2+\gamma+\omega+(a-4\beta)\lambda^2)\sin^2(\zeta)\right)}{-(ak^2+\gamma+\omega)^2+(a-4\beta)^2\lambda^4}}}e^{i(-kx+wt+\theta)}, \qquad (35)$$

such that

$$h_4^2(-1-h_2)\left[(-2+h_2)(1+h_2)\right]=0. \qquad (36)$$

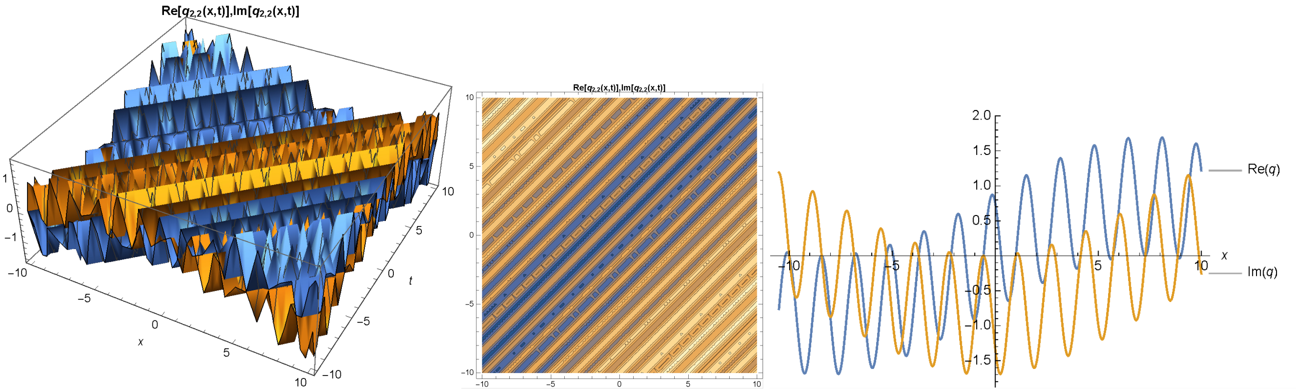


**Figure 4.** The numerical simulations corresponding to $|q_{2,2}|$ given by Eq. (35), for $m = 0$ ; (a), (b) and (c) are the 3D graphic, contour and 2D graphic, respectively

**3.** If $l_0 = m^2 - 1$, $l_2 = 2 - m^2$, $l_4 = -1$, $0 < m < 1$, then $P(\zeta) = dn(\zeta, m)$ which gives

$$q_3(x,t) = \frac{\sqrt{2h_4}\sqrt{-a+4\beta}\lambda}{\sqrt{c}}\left[\frac{dn(\zeta,m)}{\sqrt{f\,(dn(\zeta,m))^2+g}}\right]e^{i(-kx+wt+\theta)}, \tag{37}$$

where $f$ and $g$ are determined by

$$f = \frac{(-2+m^2+h_2)h_4}{1-m^2+m^4-h_2^2},$$
$$g = \frac{-3\left(-1+m^2\right)h_4}{1-m^2+m^4-h_2^2}, \tag{38}$$

under the restriction condition

$$h_4^2\left(2-m^2-h_2\right)\left[-\left(-1+2m^2+h_2\right)\left(1+m^2+h_2\right)\right] = 0. \tag{39}$$

If $m \to 1$, then the bright optical soliton solution is obtained

$$q_{3,1}(x,t) = \frac{\sqrt{2h_4}\sqrt{-a+4\beta}\lambda sech(\zeta)}{\sqrt{c}\sqrt{\frac{-(a-4\beta)\lambda^2 h_4 sech^2(\zeta)}{ak^2+\gamma+\omega+(a-4\beta)\lambda^2}}}e^{i(-kx+wt+\theta)}, \tag{40}$$

provided that

$$h_4^2(1-h_2)\left[-2+h_2+h_2^2\right] = 0. \tag{41}$$

If $m \to 0$, then the rational solution is obtained

$$q_{3,2}(x,t) = \frac{\sqrt{2h_4}\sqrt{-a+4\beta}\lambda}{\sqrt{c}\sqrt{\frac{-(a-4\beta)\lambda^2 h_4}{4\beta\lambda^2+\gamma+\omega+a(k-\lambda)(k+\lambda)}}}e^{i(-kx+wt+\theta)}, \tag{42}$$

such that

$$h_4^2(2-h_2)\left[(1+h_2)^2\right] = 0. \tag{43}$$

**4.** If $l_0 = m^2$, $l_2 = -\left(1+m^2\right)$, $l_4 = 1$, $0 < m < 1$, $P(\zeta) = ns(\zeta, m)$ or $P(\zeta) = dc(\zeta, m)$ then

$$q_{4,1}(x,t) = \frac{\sqrt{2h_4}\sqrt{-a+4\beta}\lambda}{\sqrt{c}}\left[\frac{ns(\zeta,m)}{\sqrt{f\,(ns(\zeta,m))^2+g}}\right]e^{i(-kx+wt+\theta)}, \tag{44}$$

or

$$q_{4,2}(x,t) = \frac{\sqrt{2h_4}\sqrt{-a+4\beta\lambda}}{\sqrt{c}}\left[\frac{dc(\zeta,m)}{\sqrt{f\left(dc(\zeta,m)\right)^2+g}}\right]e^{i(-kx+wt+\theta)}, \tag{45}$$

where f and g are given by

$$f = \frac{(1+m^2+h_2)h_4}{1-m^2+m^4-h_2^2}, \qquad g = \frac{-3m^2h_4}{1-m^2+m^4-h_2^2}, \tag{46}$$

under the constraint condition

$$h_4^2\left(-1-m^2-h_2\right)\left[-\left(-1+2m^2-h_2\right)\left(-2+m^2+h_2\right)\right] = 0. \tag{47}$$

If $m \to 1$, then the dark singular soliton solution is obtained

$$q_{4,3}(x,t) = \frac{\sqrt{2h_4}\sqrt{-a+4\beta\lambda}\coth(\zeta)}{\sqrt{c}\sqrt{\frac{(a-4\beta)\lambda^2\left(-3(a-4\beta)\lambda^2+(ak^2+\gamma+\omega+2(a-4\beta)\lambda^2)\coth^2(\zeta)\right)h_4}{-(ak^2+\gamma+\omega)^2+(a-4\beta)^2\lambda^4}}}e^{i(-kx+wt+\theta)}, \tag{48}$$

such that

$$h_4^2(-2-h_2)\left[(-1+h_2)^2\right] = 0. \tag{49}$$

If $m \to 0$, then the periodic solution is obtained

$$q_{4,4}(x,t) = \frac{\sqrt{2h_4}\sqrt{-a+4\beta\lambda}\csc(\zeta)}{\sqrt{c}\sqrt{\frac{-(a-4\beta)\lambda^2h_4\csc^2(\zeta)}{4\beta\lambda^2+\gamma+\omega+a(k-\lambda)(k+\lambda)}}}e^{i(-kx+wt+\theta)}, \tag{50}$$

such that

$$h_4^2(-1-h_2)\left[(-2+h_2)(1+h_2)\right] = 0. \tag{51}$$

**5.** If $l_0 = -m^2$, $l_2 = 2m^2-1$, $l_4 = 1-m^2$, $0<m<1$, then $P(\zeta) = nc(\zeta,m)$ and we have

$$q_5(x,t) = \frac{\sqrt{2h_4}\sqrt{-a+4\beta\lambda}}{\sqrt{c}}\left[\frac{nc(\zeta,m)}{\sqrt{f\left(nc(\zeta,m)\right)^2+g}}\right]e^{i(-kx+wt+\theta)}, \tag{52}$$

where f and g are given by

$$f = \frac{-(-1+2m^2-h_2)h_4}{1-m^2+m^4-h_2^2}, \qquad g = \frac{3m^2h_4}{1-m^2+m^4-h_2^2}, \tag{53}$$

under the constraint condition

$$h_4^2\left(-1+2m^2-h_2\right)\left[\left(-2+m^2+h_2\right)\left(1+m^2+h_2\right)\right] = 0. \tag{54}$$

If $m \to 1$, then the singular soliton solution is obtained

$$q_{5,1}(x,t) = \frac{\sqrt{2h_4}\sqrt{-a+4\beta}\lambda\cosh(\zeta)}{\sqrt{c}\sqrt{\frac{-(a-4\beta)\lambda^2\left(-3(a-4\beta)\lambda^2-(4\beta\lambda^2+\gamma+\omega+a(k-\lambda)(k+\lambda))\cosh^2(\zeta)\right)h_4}{-(ak^2+\gamma+\omega)^2+(a-4\beta)^2\lambda^4}}} e^{i(-kx+wt+\theta)}, \tag{55}$$

such that

$$h_4^2(1-h_2)\left[-2+h_2+h_2^2\right] = 0. \tag{56}$$

If $m \to 0$, then the periodic solution is obtained

$$q_{5,2}(x,t) = \frac{\sqrt{2h_4}\sqrt{-a+4\beta}\lambda\sec(\zeta)}{\sqrt{c}\sqrt{\frac{-(a-4\beta)\lambda^2\sec^2(\zeta)h_4}{4\beta\lambda^2+\gamma+\omega+a(k-\lambda)(k+\lambda)}}} e^{i(-kx+wt+\theta)}, \tag{57}$$

such that

$$h_4^2(-1-h_2)\left[(-2+h_2)(1+h_2)\right] = 0. \tag{58}$$

6. If $l_0 = -1$, $l_2 = 2-m^2$, $l_4 = -\left(1-m^2\right)$, $0 < m < 1$, then $P(\zeta) = nd(\zeta, m)$ and we have

$$q_6(x,t) = \frac{\sqrt{2h_4}\sqrt{-a+4\beta}\lambda}{\sqrt{c}}\left[\frac{nd(\zeta,m)}{\sqrt{f\,(nd(\zeta,m))^2+g}}\right] e^{i(-kx+wt+\theta)}, \tag{59}$$

where f and g are given by

$$f = \frac{(-2+m^2+h_2)h_4}{1-m^2+m^4-h_2^2},\qquad g = \frac{3h_4}{1-m^2+m^4-h_2^2}, \tag{60}$$

under the constraint condition

$$h_4^2\left(2-m^2-h_2\right)\left[-\left(-1+2m^2-h_2\right)\left(1+m^2+h_2\right)\right] = 0. \tag{61}$$

**7.** If $l_0 = 1$, $l_2 = 2-m^2$, $l_4 = 1-m^2$,$0 < m < 1$ $P(\zeta) = sc(\zeta, m)$ then we have

$$q_7(x,t) = \frac{\sqrt{2h_4}\sqrt{-a+4\beta}\lambda}{\sqrt{c}}\left[\frac{sc(\zeta,m)}{\sqrt{f\,(sc(\zeta,m))^2+g}}\right] e^{i(-kx+wt+\theta)}, \tag{62}$$

where f and g are given by

$$f = \frac{(-2+m^2+h_2)h_4}{1-m^2+m^4-h_2^2},\qquad g = \frac{-3h_4}{1-m^2+m^4-h_2^2}, \tag{63}$$

under the constraint condition

$$h_4^2\left(2-m^2-h_2\right)\left[-\left(-1+2m^2-h_2\right)\left(1+m^2+h_2\right)\right] = 0. \tag{64}$$

If $m \to 1$, then the singular soliton solution is obtained

$$q_{7,1}(x,t) = \frac{\sqrt{2h_4}\sqrt{-a+4\beta}\lambda\sinh(\zeta)}{\sqrt{c}\sqrt{\frac{-(a-4\beta)\lambda^2\left(3(a-4\beta)\lambda^2-(4\beta\lambda^2+\gamma+\omega+a(k-\lambda)(k+\lambda))\sinh^2(\zeta)\right)h_4}{-(ak^2+\gamma+\omega)^2+(a-4\beta)^2\lambda^4}}} e^{i(-kx+wt+\theta)}, \tag{65}$$

such that

$$h_4^2\left(1-h_2\right)\left[-2+h_2+h_2^2\right]=0. \tag{66}$$

If $m \rightarrow 0$, then the periodic solution is obtained

$$q_{7,2}\left(x,t\right)=\frac{\sqrt{2h_4}\sqrt{-a+4\beta}\lambda\tan(\zeta)}{\sqrt{c}\sqrt{\frac{(a-4\beta)\lambda^2(-3(a-4\beta)\lambda^2+(ak^2+\gamma+\omega-2(a-4\beta)\lambda^2)\tan^2(\zeta))h_4}{-(ak^2+\gamma+\omega)^2+(a-4\beta)^2\lambda^4}}}e^{i(-kx+wt+\theta)}, \tag{67}$$

such that

$$h_4^2\left(2-h_2\right)\left[\left(1+h_2\right)^2\right]=0. \tag{68}$$

8. If $l_0=1$, $l_2=2m^2-1$, $l_4=-m^2\left(1-m^2\right)$, $0<m<1$, then $P(\zeta)=sd(\zeta,m)$ and we have

$$q_8\left(x,t\right)=\frac{\sqrt{2h_4}\sqrt{-a+4\beta}\lambda}{\sqrt{c}}\left[\frac{sd(\zeta,m)}{\sqrt{f\left(sd(\zeta,m)\right)^2+g}}\right]e^{i(-kx+wt+\theta)}, \tag{69}$$

where f and g are given by

$$f=\frac{(-1+2m^2-h_2)h_4}{1-m^2+m^4-h_2^2},\quad g=\frac{-3h_4}{1-m^2+m^4-h_2^2}, \tag{70}$$

under the constraint condition

$$h_4^2\left(-1+2m^2-h_2\right)\left[\left(-2+m^2+h_2\right)\left(1+m^2+h_2\right)\right]=0. \tag{71}$$

9. If $l_0=1-m^2$, $l_2=2-m^2$, $l_4=1$, $0<m<1$, then $P(\zeta)=cs(\zeta,m)$ and we have

$$q_9\left(x,t\right)=\frac{\sqrt{2h_4}\sqrt{-a+4\beta}\lambda}{\sqrt{c}}\left[\frac{cs(\zeta,m)}{\sqrt{f\left(cs(\zeta,m)\right)^2+g}}\right]e^{i(-kx+wt+\theta)}, \tag{72}$$

where $f$ and $g$ are given by

$$f=\frac{(-2+m^2+h_2)h_4}{1-m^2+m^4-h_2^2},\quad g=\frac{3(-1+m^2)h_4}{1-m^2+m^4-h_2^2}, \tag{73}$$

under the constraint condition

$$h_4^2\left(2-m^2-h_2\right)\left[-\left(-1+2m^2-h_2\right)\left(1+m^2+h_2\right)\right]=0. \tag{74}$$

If $m \rightarrow 1$, then the singular soliton solution is obtained

$$[q_{9,1}\left(x,t\right)=\frac{\lambda\sqrt{2h_4}\sqrt{-a+4\beta}csch(\zeta)}{\sqrt{c}\sqrt{\frac{-h_4(a-4\beta)\lambda^2csch^2(\zeta)}{ak^2+\gamma+\omega+(a-4\beta)\lambda^2}}}e^{i(-kx+wt+\theta)}], \tag{75}$$

such that

$$h_4^2(1-h_2)\left[-2+h_2+h_2^2\right]=0. \tag{76}$$

If $m\to 0$, then the periodic solution is obtained

$$q_{9,2}(x,t)=\frac{\sqrt{2h_4}\sqrt{-a+4\beta}\lambda\cot(\zeta)}{\sqrt{c}\sqrt{\frac{-(a-4\beta)\lambda^2(3(a-4\beta)\lambda^2-(ak^2+\gamma+\omega-2(a-4\beta)\lambda^2)\cot^2(\zeta))h_4}{-(ak^2+\gamma+\omega)^2+(a-4\beta)^2\lambda^4}}}e^{i(-kx+wt+\theta)}, \tag{77}$$

such that

$$h_4^2(2-h_2)\left[(1+h_2)^2\right]=0. \tag{78}$$

10. If $l_0=-m^2\left(1-m^2\right)$, $l_2=2m^2-1$, $l_4=1$, $0<m<1$, then $P(\zeta)=ds(\zeta,m)$ and we have

$$q_{10}(x,t)=\frac{\sqrt{2h_4}\sqrt{-a+4\beta}\lambda}{\sqrt{c}}\left[\frac{ds(\zeta,m)}{\sqrt{f\,(ds(\zeta,m))^2+g}}\right]e^{i(-kx+wt+\theta)}, \tag{79}$$

where $f$ and $g$ are given by

$$\begin{aligned} f&=\frac{-(-1+2m^2-h_2)h_4}{1-m^2+m^4-h_2^2},\\ g&=\frac{-3m^2(-1+m^2)h_4}{1-m^2+m^4-h_2^2}, \end{aligned} \tag{80}$$

under the constraint condition

$$h_4^2\left(-1+2m^2-h_2\right)\left[\left(-2+m^2+h_2\right)\left(1+m^2+h_2\right)\right]=0. \tag{81}$$

11. If $l_0=\frac{1-m^2}{4}$, $l_2=\frac{1+m^2}{2}$, $l_4=\frac{1-m^2}{4}$, $0<m<1$, then $P(\zeta)=nc(\zeta,m)\pm sc(\zeta,m)$ or $P(\zeta)=\frac{cn(\zeta,m)}{1\pm sn(\zeta,m)}$ and we have

$$q_{11,1}(x,t)=\frac{\sqrt{2h_4}\sqrt{-a+4\beta}\lambda}{\sqrt{c}}\left[\frac{nc(\zeta,m)\pm sc(\zeta,m)}{\sqrt{f\,(nc(\zeta,m)\pm sc(\zeta,m))^2+g}}\right]e^{i(-kx+wt+\theta)}, \tag{82}$$

or

$$q_{11,2}(x,t)=\frac{\sqrt{2h_4}\sqrt{-a+4\beta}\lambda}{\sqrt{c}}\left[\frac{\frac{cn(\zeta,m)}{1\pm sn(\zeta,m)}}{\sqrt{f\left(\frac{cn(\zeta,m)}{1\pm sn(\zeta,m)}\right)^2+g}}\right]e^{i(-kx+wt+\theta)}, \tag{83}$$

where $f$ and $g$ are given by

$$\begin{aligned} f&=\frac{-8(1+m^2-2h_2)h_4}{1+14m^2+m^4-16h_2^2},\\ g&=\frac{12(-1+m^2)h_4}{1+14m^2+m^4-16h_2^2}, \end{aligned} \tag{84}$$

under the constraint condition

$$h_4^2\left(\frac{1}{2}\left(1+m^2-2h_2\right)\right)\left[\frac{1}{16}(1+(-6+m)m+4h_2)(1+m(6+m)+4h_2)\right]=0. \tag{85}$$

If $m \to 1$, then the combined singular soliton solution

$$q_{11,3}(x,t) = \frac{\sqrt{2h_4}\sqrt{-a+4\beta}\,\lambda\left(\sinh(\zeta)+\cosh(\zeta)\right)}{\sqrt{c}\sqrt{\frac{-(a-4\beta)\lambda^2(\sinh(\zeta)+\cosh(\zeta))^2 h_4}{ak^2+\gamma+\omega+(a-4\beta)\lambda^2}}}e^{i(-kx+wt+\theta)}, \tag{86}$$

or dark-bright optical soliton solition is obtained

$$q_{11,4}(x,t) = \frac{\lambda\sqrt{2h_4}\sqrt{-a+4\beta}\left(\frac{sech(\zeta)}{1+\tanh(\zeta)}\right)}{\sqrt{c}\sqrt{\frac{-h_4\lambda^2\left(\frac{sech(\zeta)}{1+\tanh(\zeta)}\right)^2(a-4\beta)}{ak^2+\gamma+\omega+(a-4\beta)\lambda^2}}}e^{i(-kx+wt+\theta)}, \tag{87}$$

such that

$$h_4^2(1-h_2)\left[-2+h_2+h_2^2\right] = 0. \tag{88}$$

If $m \to 0$, then the combined periodic solution is obtained

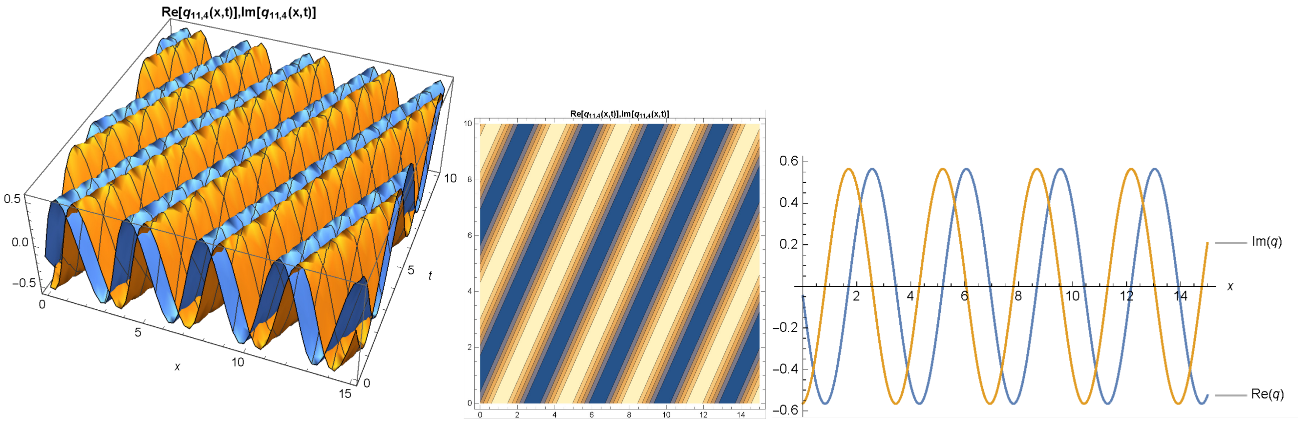


**Figure 5.** The numerical simulations corresponding to $|q_{11,4}|$ given by Eq. (87), for $m = 1$; (a), (b) and (c) are the 3D graphic, contour and 2D graphic, respectively

$$q_{11,5}(x,t) = \frac{\sqrt{h_4}\sqrt{-a+4\beta}\,\lambda\left(\sec(\zeta)+\tan(\zeta)\right)e^{i(-kx+wt+\theta)}}{\sqrt{2c}\sqrt{\frac{(a-4\beta)\lambda^2(4(ak^2+\gamma+\omega)-5(a-4\beta)\lambda^2+(4(ak^2+\gamma+\omega)+(a-4\beta)\lambda^2)\sin(\zeta))h_4}{(16(ak^2+\gamma+\omega)^2-(a-4\beta)^2\lambda^4)(-1+\sin(\zeta))}}}, \tag{89}$$

or

$$q_{11,6}(x,t) = \frac{\frac{\sqrt{h_4}\sqrt{-a+4\beta}\lambda}{\sqrt{2c}(1+\sin(\zeta))}\cos(\zeta)e^{i(-kx+wt+\theta)}}{\sqrt{\frac{(a-4\beta)\lambda^2(-4(ak^2+\gamma+\omega)+5(a-4\beta)\lambda^2+(4(ak^2+\gamma+\omega)+(a-4\beta)\lambda^2)\sin(\zeta))h_4}{(16(ak^2+\gamma+\omega)^2-(a-4\beta)^2\lambda^4)(1+\sin(\zeta))}}}, \tag{90}$$

such that

$$h_4^2\left(\frac{1}{2}-h_2\right)\left[\frac{1}{16}(1+4h_2)^2\right] = 0. \tag{91}$$

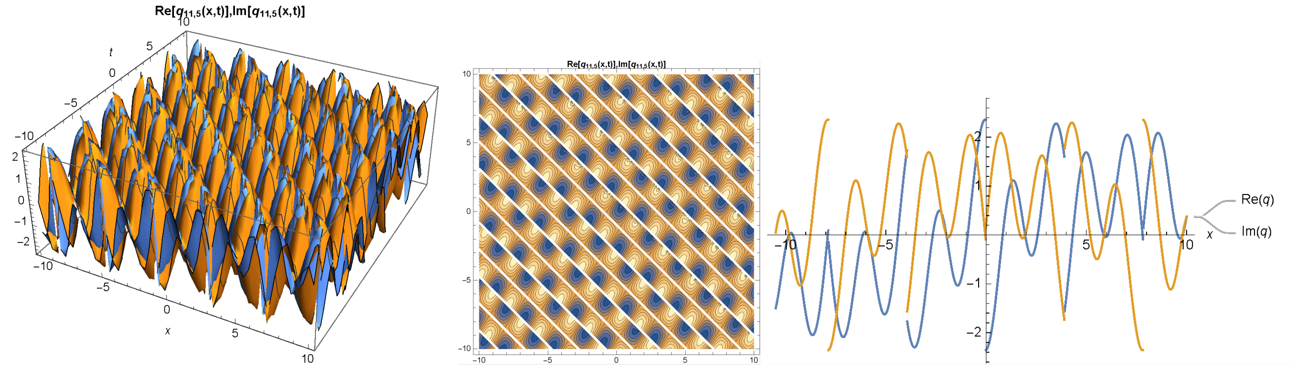


**Figure 6.** The numerical simulations corresponding to $|q_{11,5}|$ given by Eq. (89), for $m = 0$; (a), (b) and (c) are the 3D graphic, contour and 2D graphic, respectively

12. If $l_0 = \frac{-\left(1-m^2\right)^2}{4}$, $l_2 = \frac{1+m^2}{2}$, $l_4 = \frac{-1}{4}$, $0 < m < 1$, then $P(\zeta) = mcn(\zeta, m) \pm dn(\zeta, m)$ and we have

$$q_{12}(x,t) = \frac{\sqrt{2h_4}\sqrt{-a+4\beta\lambda}}{\sqrt{c}}\left[\frac{mcn(\zeta,m) \pm dn(\zeta,m)}{\sqrt{f\left(mcn(\zeta,m) \pm dn(\zeta,m)\right)^2 + g}}\right]e^{i(-kx+wt+\theta)}, \tag{92}$$

where $f$ and $g$ are given by

$$\begin{aligned} f &= \frac{-8(1+m^2-2h_2)h_4}{1+14m^2+m^4-16h_2^2}, \\ g &= \frac{12(-1+m^2)^2 h_4}{1+14m^2+m^4-16h_2^2}, \end{aligned} \tag{93}$$

under the constraint condition

$$h_4^2\left(\frac{1}{2}\left(1+m^2-2h_2\right)\right)\left[\frac{1}{16}\left(1+(-6+m)m+4h_2\right)\left(1+m(6+m)+4h_2\right)\right] = 0. \tag{94}$$

13. If $l_0 = \frac{1}{4}$, $l_2 = \frac{1-2m^2}{2}$, $l_4 = \frac{1}{4}$, $0 < m < 1$, then $P(\zeta) = \frac{sn(\zeta,m)}{1 \pm cn(\zeta,m)}$ and we have

$$q_{13}(x,t) = \frac{\sqrt{2h_4}\sqrt{-a+4\beta\lambda}}{\sqrt{c}}\left[\frac{\frac{sn(\zeta,m)}{1\pm cn(\zeta,m)}}{\sqrt{f\left(\frac{sn(\zeta,m)}{1\pm cn(\zeta,m)}\right)^2 + g}}\right]e^{i(-kx+wt+\theta)}, \tag{95}$$

where $f$ and $g$ are given by

$$\begin{aligned} f &= \frac{8(-1+2m^2+2h_2)h_4}{1-16m^2+16m^4-16h_2^2}, \\ g &= \frac{-12h_4}{1-16m^2+16m^4-16h_2^2}, \end{aligned} \tag{96}$$

under the constraint condition

$$h_4^2\left(\frac{1}{2}-m^2-h_2\right)\left[\frac{1}{16}+2m^2-2m^4+\left(\frac{1}{2}-m^2\right)h_2+h_2^2\right] = 0. \tag{97}$$

If $m \to 1$, then the combined soliton solution is obtained

$$q_{13,1}(x,t) = \frac{\frac{\sqrt{h_4}\sqrt{-a+4\beta\lambda}}{2\sqrt{c}}\tanh(\zeta)e^{i(-kx+wt+\theta)}}{\sqrt{\frac{(a-4\beta)\lambda^2\cosh^2(\frac{\zeta}{2})sech(\zeta)(-4(ak^2+\gamma+\omega)+(a-4\beta)\lambda^2+(4(ak^2+\gamma+\omega)+5(a-4\beta)\lambda^2)sech(\zeta))h_4}{(16(ak^2+\gamma+\omega)^2-(a-4\beta)^2\lambda^4)}}}, \tag{98}$$

such that

$$h_4^2\left(\frac{-1}{2}-h_2\right)\left[\frac{1}{16}\left(1-4h_2\right)^2\right]=0. \tag{99}$$

If $m \to 0$, then the combined periodic solution is obtained

$$q_{13,2}(x,t)=\frac{\frac{\sqrt{h_4}\sqrt{-a+4\beta}\lambda}{2\sqrt{c}}\sin(\zeta)e^{i(-kx+wt+\theta)}}{\sqrt{\frac{(a-4\beta)\lambda^2\cos^2(\frac{\zeta}{2})(-4(ak^2+\gamma+\omega)+5(a-4\beta)\lambda^2+(4(ak^2+\gamma+\omega)+(a-4\beta)\lambda^2)\cos(\zeta))h_4}{(16(ak^2+\gamma+\omega)^2-(a-4\beta)^2\lambda^4)}}}, \tag{100}$$

such that

$$h_4^2\left(\frac{1}{2}-h_2\right)\left[\frac{1}{16}\left(1+4h_2\right)^2\right]=0. \tag{101}$$

14. If $l_0=\frac{1}{4}$, $l_2=\frac{1+m^2}{2}$, $l_4=\frac{\left(1-m^2\right)^2}{4}$, $0<m<1$, then $P(\zeta)=\frac{sn(\zeta,m)}{cn(\zeta,m)\pm dn(\zeta,m)}$ and we have

$$q_{14}(x,t)=\frac{\sqrt{2h_4}\sqrt{-a+4\beta}\lambda}{\sqrt{c}}\left[\frac{\frac{sn(\zeta,m)}{cn(\zeta,m)\pm dn(\zeta,m)}}{\sqrt{f\left(\frac{sn(\zeta,m)}{cn(\zeta,m)\pm dn(\zeta,m)}\right)^2+g}}\right]e^{i(-kx+wt+\theta)}, \tag{102}$$

where $f$ and $g$ are given by

$$\begin{aligned} f&=\frac{-8(1+m^2-2h_2)h_4}{1+14m^2+m^4-16h_2^2},\\ g&=\frac{-12h_4}{1+14m^2+m^4-16h_2^2}, \end{aligned} \tag{103}$$

under the constraint condition

$$h_4^2\left(\frac{1}{2}\left(1+m^2-2h_2\right)\right)\left[\frac{1}{16}\left(1+(-6+m)\,m+4h_2\right)\left(1+m\left(6+m\right)+4h_2\right)\right]=0. \tag{104}$$

If $m \to 1$, then the singular soliton solution is obtained

$$q_{14,1}(x,t)=\frac{\frac{\sqrt{2h_4}\sqrt{-a+4\beta}\lambda}{\sqrt{c}}\sinh(\zeta)}{\sqrt{\frac{-(a-4\beta)\lambda^2\left(3(a-4\beta)\lambda^2-(4\beta\lambda^2+\gamma+\omega+a(k-\lambda)(k+\lambda))\sinh^2(\zeta)\right)h_4}{-(ak^2+\gamma+\omega)^2+(a-4\beta)^2\lambda^4}}}e^{i(-kx+wt+\theta)}, \tag{105}$$

such that

$$h_4^2\left(1-h_2\right)\left[-2+h_2+h_2^2\right]=0. \tag{106}$$

If $m \to 0$, then the combined periodic solution is obtained

$$q_{14,2}(x,t)=\frac{\frac{\sqrt{h_4}\sqrt{-a+4\beta}\lambda\sin(\zeta)}{2\sqrt{c}}e^{i(-kx+wt+\theta)}}{\sqrt{\frac{(a-4\beta)\lambda^2\cos^2(\frac{\zeta}{2})(-4(ak^2+\gamma+\omega)+5(a-4\beta)\lambda^2+(4(ak^2+\gamma+\omega)+(a-4\beta)\lambda^2)\cos(\zeta))h_4}{(16(ak^2+\gamma+\omega)^2-(a-4\beta)^2\lambda^4)}}}, \tag{107}$$

such that

$$h_4^2\left(\frac{1}{2}-h_2\right)\left[\frac{1}{16}\left(1+4h_2\right)^2\right]=0. \tag{108}$$

## 5 Result and discussion

This study used the newly created $\varphi^6$—model expansion method to get dark, bright, singular, periodic and combined soliton solutions to the complex Ginzburg-Landau equation (CGLE) with Kerr law in nonlinear optics. The Kerr law of nonlinearity is a result of the nonlinear reactions that an external electric field-induced nonharmonic motion of trapped electrons in molecules induces in a light wave in an optical fiber. The constraint conditions ensure the existence of these solutions.
The graphics in Figures 1, 3 and 5 show the behavior of dark, bright and dark-bright solitons together with periodic and combined periodic wave solutions at any given time, which is important in the transmission of energy from one location to another. Furthermore, to examine the physical implications of the parameters in the transformation, which is known as the classical wave transformation represented by Eqs. (1) and (2). The physical meanings of the parameters in the solution of Eqs. (26), (28), (33), (35), (87) and (89) traveling waves, which contain numerous mathematical constants. It is the internal dynamics of the traveling wave for various parameter values. We may conclude that the traveling wave behavior alters for different values of each. The simulation is performed for several values of the wave frequency in order to examine the changes in the dark and bright solitons more clearly. Similarly, a similar discussion can be made for other physical parameters as well as various traveling wave solutions.

## 6 Conclcusion

This work investigates the complex Ginzburg-Landau equation (CGLE) with Kerr law in nonlinear optics, which represents soliton propagation in the presence of a detuning factor. The scheme's benefit is that the solutions are first recovered in terms of Jacobi's elliptic function. When a result, as the limiting values of the modulus of ellipticity approach 0 or unity, solitons or singular-periodic solutions are produced. The $\varphi^6$-model expansion approach is used to find dark, bright, dark-bright or combined, singular and combined singular optical soliton solutions to the CGL model with Kerr law. The $\varphi^6$-model expansion approach is found to be efficient for constructing optical soliton solutions for most nonlinear physical phenomena. The results presented in this study are intended to improve the CGLE's nonlinear dynamical characteristics. The findings of this study might assist in comprehending some of the physical implications of various nonlinear physics models. The hyperbolic sine, for example, appears in the calculation of the Roche limit and gravitational potential of a cylinder, while the hyperbolic cotangent appears in the Langevin function for magnetic polarization. In order to take into account slow-light pulses, the model will also be examined using fractional temporal evolution.

## Declarations

### List of abbreviations

CGLE: Complex Ginzburg-Landau Equation
GVE: Group Velocity Dispersion
NPDEs: Nonlinear Partial Differential Equations

### Consent for publication

Not applicable.

### Conflicts of interest

The authors declare that they have no conflict of interests.

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No funding was used in this study.

### Author's contributions

I.M.A.: Conceptualization, Methodology, Software, Visualization, Investigation, Supervision, Software, Validation, Writing-Reviewing and Editing. A.Y.: Conceptualization, Methodology Writing-Original draft preparation. Visualization, Investigation, Supervision, Software, Validation, Writing-Reviewing and Editing. All authors discussed the results and contributed to the final manuscript.

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