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Ricci Soliton Lightlike Submanifolds with Co-Dimension 2

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Abstract

The necessary requirements for half-lightlike and coisotropic lightlike submanifolds to be a Ricci soliton are obtained. Some examples of Ricci soliton half-lightlike and Ricci soliton coisotropic lightlike submanifolds are given. The Ricci soliton equation is investigated on totally geodesic, totally umbilical, and irrotational lightlike submanifolds.

1. Introduction

The concept of Ricci solitons has become a fascinating issue in the differential geometry and this concept has been studied on various submanifolds of Riemannian manifolds. For some applications on Ricci solitons, we touch on [1–8], etc. A (semi-) Riemannian manifold (M,g) is said to be a Ricci soliton if

$$L_{\zeta}g(X_1,X_2)+2\mathrm{Ric}(X_1,X_2)=2\kappa g(X_1,X_2) \tag{1.1}$$

is satisfied for each tangent vector fields X_1 and X_2. In (1.1), L_{ζ} indicates the Lie derivative, Ric denotes the Ricci tensor, κ is a scalar and ζ is called the potential vector field. If κ is a function then (M,g) is said to be an almost Ricci soliton. Considering the description of Ricci solitons, every Einstein manifold is a Ricci soliton. This issue is known as shrinking when $\kappa>0$, steady when $\kappa=0$ and expanding when $\kappa<0$.

In addition to this, concircular vector fields are one of the most utilized vector field to characterize a smooth manifold. Concircular vector fields were initially observed in the definition of torse-forming vector field which was introduced by K. Yano [9]. There exist remarkable applications of concircular vector fields in the literature (cf. [10–15]). A vector field ζ is said to be concircular if there is a differentiable function φ such that

$$\nabla_X\zeta=\varphi X$$

is satisfied for each tangent vector field X. If $\varphi=1$, ζ becomes concurrent.

In the degenerate geometry, Ricci soliton lightlike hypersurfaces were studied in [16]. As a continuation of this study, we examine Ricci soliton half-lightlike submanifolds and Ricci soliton coisotropic lightlike submanifolds admitting concircular and concurrent potential vector fields in this paper.

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2. Half-Lightlike Submanifolds

Let $(\widetilde{M},\widetilde{g})$ be a semi-Riemannian manifold equipped with a semi-Riemannian metric $\widetilde{g}$. Presume that (M,g) to be a lightlike submanifold $(\widetilde{M},\widetilde{g})$ with the co-dimension 2. Then the radical distribution $\mathrm{Rad}(T_pM)$ at $p\in M$ is determined by

$$\mathrm{Rad}(T_pM)=\left\{\xi\in T_pM:\widetilde{g}(\xi,X)=0,\forall X\in T_pM\right\}.$$

Denote a complementary non-degenerate vector bundle of $\mathrm{Rad}(TM)$ by $\mathrm{S}(TM)$. We note that a lightlike submanifold is usually indicated by the triple $(M,g,\mathrm{S}(TM))$ [17]. Here, the distribution $\mathrm{S}(TM)$ is said to be a screen distribution and we put

$$TM=\mathrm{Rad}(TM)\oplus\mathrm{S}(TM)$$

where $\oplus_{orth}$ indicates the orthogonal direct sum. If the dimension of $\mathrm{Rad}(TM)$ is equal to 1, then $(M,g,\mathrm{S}(TM))$ is said to be a half-lightlike submanifold [18].

For each half-lightlike submanifold, there are a $1-$dimensional non-degenerate subbundle $\mathbb{D}$ and a $1-$dimensional degenerate subbundle $\mathrm{ltr}(TM)$ such that $\mathbb{D}=Span\{U\}$, $ltr(TM)=Span\{N\}$ and

$$\begin{aligned}\widetilde{g}(\xi,U)&=\widetilde{g}(N,U)=0,\quad \widetilde{g}(U,U)\neq 0,\\ \widetilde{g}(\xi,N)&=1,\quad \widetilde{g}(N,U)=0\end{aligned}$$

are satisfied.

Let $\widetilde{\nabla}$ denotes the Levi-Civita connection on $(\widetilde{M},\widetilde{g})$ and P denotes the projection from $\Gamma(TM)$ to $\Gamma(\mathrm{S}(TM))$. The Gauss and Weingarten type formulas are formulated by

$$\widetilde{\nabla}_{X_1}X_2=\nabla_{X_1}X_2+B(X_1,X_2)N+D(X_1,X_2)U,\tag{2.1}$$
$$\widetilde{\nabla}_{X_1}N=-A_NX_1+\rho_1(X_1)N+\rho_2(X_1)U,\tag{2.2}$$
$$\widetilde{\nabla}_{X_1}U=-A_UX_1+\mu_1(X_1)N+\mu_2(X_1)U,\tag{2.3}$$

and

$$\nabla_{X_1}PX_2=\nabla^*_{X_1}PX_2+C(X_1,X_2)\xi,\tag{2.4}$$
$$\nabla_{X_1}\xi=-A^*_\xi X_1-\rho_1(X_1)\xi,\tag{2.5}$$

where $\nabla_{X_1}X_2,A_NX_1,A_UX_1\in\Gamma(TM)$, $\nabla^*_{X_1}PX_2,A^*_\xi X_1\in\Gamma(\mathrm{S}(TM))$, A_N and A_U are the shape operators on $\Gamma(TM)$, A^*_ξ is the shape operator on $\Gamma(\mathrm{S}(TM))$, $B(X_1,X_2)$ and $D(X_1,X_2)$ are ingredients of the second fundamental form, ρ_1,ρ_2, μ_1,μ_2 are $1-$forms. From (2.1)-(2.5), it is known that the relations

$$\begin{aligned}B(X_1,X_2)&=g(A^*_\xi X_1,X_2),\\ C(X_1,X_2)&=g(A_NX_1,X_2),\\ D(X_1,X_2)&=g(A_UX_1,X_2)-\mu_1(X_1)\eta(X_2),\end{aligned}$$

are satisfied. Here $\eta(X_2)=\widetilde{g}(X_2,N)$ [19].

We note that B and D are symmetric but C is not symmetric. Since the relation

$$(\nabla_{X_3}g)(X_1,X_2)=B(X_3,X_2)\eta(X_1)+B(X_3,X_1)\eta(X_2)\tag{2.6}$$

is satisfied, ∇ is not a metric connection. The Lie derivative of g is formulated by

$$\begin{aligned}(L_{X_3}g)(X_1,X_2)&=X_3g(X_1,X_2)-\widetilde{g}([X_3,X_1],X_2)-\widetilde{g}([X_3,X_2],X_1)\\ &=X_3g(X_1,X_2)-g(\nabla_{X_3}X_1,X_2)-g(\nabla_{X_3}X_2,X_1)+g(\nabla_{X_1}X_3,X_2)+g(\nabla_{X_2}X_3,X_1).\end{aligned}\tag{2.7}$$

From (2.6) and (2.7), we also have

$$(L_{X_3}g)(X_1,X_2)=B(X_3,X_2)\eta(X_1)+B(X_3,X_1)\eta(X_2)+g(\nabla_{X_1}X_3,X_2)+g(\nabla_{X_2}X_3,X_1)\tag{2.8}$$

or we put

$$(L_{X_3}g)(X_1,X_2)=(\nabla_{X_3}g)(X_1,X_2)+g(\nabla_{X_1}X_3,X_2)+g(\nabla_{X_2}X_3,X_1).$$

Indicate the Riemannian curvatures of $(M,g,S(TM))$ and its ambient manifold by R and $\widetilde{R}$ respectively. Then the following equality holds:

$$\begin{aligned}\widetilde{g}(\widetilde{R}(X_1,X_2)PX_3,PX_4) &= g(R(X_1,X_2)PX_3,PX_4)+B(X_1,PX_3)C(X_2,PX_4)-B(X_2,PX_3)C(X_1,PX_4)\\ &\quad +D(X_1,PX_3)D(X_2,PX_4)-D(X_2,PX_3)D(X_1,PX_4).\end{aligned} \tag{2.9}$$

Let $\{Y_1,Y_2,\ldots,Y_n\}$ be a orthonormal frame field on $\Gamma(S(TM))$. Then the Ricci type tensor $R^{(0,2)}$ is determined by

$$R^{(0,2)}(X_1,X_2)=\sum_{j=1}^{n} g(R(Y_j,X_1)X_2,Y_j)+\widetilde{g}(R(\xi,X_1)X_2,N). \tag{2.10}$$

We note that $R^{(0,2)}$ is not symmetric. If $S(TM)$ is integrable, then $R^{(0,2)}$ is symmetric [17].

A half-lightlike submanifold is called totally geodesic if $B=D=0$ and it is said to be irrotational if $B(X,\xi)=0$ for any $X\in\Gamma(TM)$ [20]. A half-lightlike submanifold is said to be totally umbilical if there exists a differentiable transversal vector field H satisfying

$$B(X_1,X_2)N+D(X_1,X_2)U=g(X_1,X_2)H \tag{2.11}$$

for any $X_1,X_2\in\Gamma(TM)$ [21]. In view of (2.11), it is clear that there exist two differentiable functions H_1 and H_2 satisfying

$$B(X_1,X_2)=H_1g(X_1,X_2) \quad \text{and} \quad D(X_1,X_2)=H_2g(X_1,X_2).$$

3. Ricci Soliton Half-Lightlike Submanifolds

Let $(M,g,S(TM))$ be a half-lighlike submanifold of $(\widetilde{M},\widetilde{g})$ Assume that $\zeta\in\Gamma(T\widetilde{M})$. Hence one can put

$$\zeta=\zeta^{\top}+f_1N+f_2U$$

where $\zeta^{\top}\in\Gamma(TM)$, f_1 and f_2 are differentiable functions on $\Gamma(T\widetilde{M})$. Thus, we get

$$f_1=\widetilde{g}(\zeta,\xi) \quad \text{and} \quad f_2=\varepsilon\widetilde{g}(\zeta,U)$$

where $\varepsilon=\widetilde{g}(U,U)=\mp 1$.

Proposition 3.1. *If ζ is concircular, then the following equalities are satisfied for any $X\in\Gamma(TM)$:*

$$\nabla_X\zeta^{\top} = \varphi X+f_1A_NX+f_2A_UX, \tag{3.1}$$
$$B(X,\zeta^{\top}) = -X[f_1]-f_1\rho_1(X)-f_2\mu_1(X), \tag{3.2}$$
$$D(X,\zeta^{\top}) = -X[f_2]-f_1\rho_2(X)-f_2\mu_2(X). \tag{3.3}$$

Proof. If ζ is concircular, then we put

$$\widetilde{\nabla}_X\zeta=\varphi X=\widetilde{\nabla}_X\zeta^{\top}+\widetilde{\nabla}_X(f_1N)+\widetilde{\nabla}_X(f_2U).$$

Using (2.1), (2.2) and (2.3) in the last formula, we get

$$\begin{aligned}\varphi X &= \nabla_X\zeta+B(X,\zeta^{\top})N+D(X,\zeta)U+X[f_1]N-f_1A_NX+f_1\rho_1(X)N+f_1\rho_2(X)U+X[f_2]U\\ &\quad -f_2A_UX+f_2\mu_1(X)N+f_2\mu_2(X)U.\end{aligned}$$

Considering the tangential parts ltr(TM) and $\mathbb{D}$, the proofs of (3.1), (3.2) and (3.3) are straightforward. □

Putting $\varphi=1$ in Proposition 3.1, we find

Proposition 3.2. *For any half-lightlike submanifold admitting a concurrent vector field ζ, we have*

$$\nabla_X\zeta^{\top}=X+f_1A_NX+f_2A_UX.$$

As a special case for $\zeta=\zeta^{\top}$, we find:

Proposition 3.3. *For any half-lightlike submanifold admitting a concircular vector field $\zeta=\zeta^{\top}$,*

$$B(X,\zeta^{\top})=D(X,\zeta^{\top})=0 \tag{3.4}$$

is satisfied.

As a result of Proposition 3.3, we find the following theorem:

Theorem 3.4. *Let $(M, g, S(TM))$ be a half-lightlike submanifold. If there exists at least one concircular or concurrent vector field $\zeta = \zeta^{\top}$ on $\Gamma(TM)$, then $\nabla_{\zeta^{\top}} g = 0$.*

Definition 3.5. *A lightlike submanifold $(M, g, S(TM))$ with a integrable screen distribution is called a Ricci soliton if the relation*

$$(L_{\zeta} g)(X_1, X_2) + 2R^{(0,2)}(X_1, X_2) = 2\kappa g(X_1, X_2) \tag{3.5}$$

holds for any $X_1, X_2 \in \Gamma(TM)$. Here, κ is a constant and ζ is called the potential vector field of $(M, g, S(TM))$.

Remark 3.6. *It is known that the Ricci tensor for any lightlike submanifold is not symmetric in general. Hence (3.5) loses its geometrical meaning since the induced degenerate metric g is symmetric. Thus we investigate Ricci soliton lightlike submanifolds whose screen distribution is integrable throughout the study.*

We shall compute the Lie derivative for half-lightlike submanifolds.

From (2.8) and (3.1), we obtain

$$(L_{\zeta^{\top}} g)(X_1, X_2) = B(X_1, \zeta^{\top})\eta(X_2) + B(X_2, \zeta^{\top})\eta(X_1) + 2\varphi g(X_1, X_2) + 2f_1 g(A_N X_1, X_2) + 2f_2 g(A_U X_1, X_2). \tag{3.6}$$

If ζ is concurrent, then we easily obtain

$$(L_{\zeta^{\top}} g)(X_1, X_2) = B(X_1, \zeta^{\top})\eta(X_2) + B(X_2, \zeta^{\top})\eta(X_1) + 2g(X_1, X_2) + 2f_1 g(A_N X_1, X_2) + 2f_2 g(A_U X_1, X_2). \tag{3.7}$$

As a special case for $\zeta = \zeta^{\top}$, we obtain from (3.4) and (3.6) that

$$(L_{\zeta^{\top}} g)(X_1, X_2) = 2\varphi g(X_1, X_2).$$

If ζ is concurrent and $\zeta = \zeta^{\top}$, we obtain

$$(L_{\zeta^{\top}} g)(X_1, X_2) = 2g(X_1, X_2).$$

By (3.5) and (3.6), we have:

Theorem 3.7. *Let $(M, g, S(TM))$ be a half-lightlike submanifold admitting a concircular vector field ζ. Then, $(M, g, S(TM))$ is a Ricci soliton such that $\zeta^{\top}$ is the potential vector field if and only if the equation*

$$R^{(0,2)}(X_1, X_2) = -\frac{1}{2}B(X_1, \zeta^{\top})\eta(X_2) - \frac{1}{2}B(\zeta^{\top}, X_2)\eta(X_1) - f_1 g(A_N X_1, X_2) - f_2 g(A_U X_1, X_2) + (\kappa - \varphi) g(X_1, X_2)$$

holds for any $X_1, X_2 \in \Gamma(TM)$.

In view of (3.5) and (3.7), we have:

Theorem 3.8. *Let $(M, g, S(TM))$ be a half-lightlike submanifold admitting a concurrent vector field ζ. Then, $(M, g, S(TM))$ is a Ricci soliton such that $\zeta^{\top}$ is the potential vector field if and only if the equation*

$$R^{(0,2)}(X_1, X_2) = -\frac{1}{2}B(X_1, \zeta^{\top})\eta(X_2) - \frac{1}{2}B(\zeta^{\top}, X_2)\eta(X_1) - f_1 g(A_N X_1, X_2) - f_2 g(A_U X_1, X_2) + (\kappa - 1) g(X_1, X_2)$$

holds for any $X_1, X_2 \in \Gamma(TM)$.

Theorem 3.9. *Let $(M, g, S(TM))$ be a half-lightlike submanifold admitting a concircular vector field $\zeta = \zeta^{\top}$. Then, the submanifold is a Ricci soliton such that $\zeta^{\top}$ is the potential vector field if and only if the equation*

$$R^{(0,2)}(X_1, X_2) = (\kappa - \varphi) g(X_1, X_2) \tag{3.8}$$

holds for any $X_1, X_2 \in \Gamma(TM)$. It means that the submanifold is an Einstein manifold.

Theorem 3.10. *Let $(M, g, S(TM))$ be a half-lightlike submanifold admitting a concurrent vector field $\zeta = \zeta^{\top}$. The submanifold is a Ricci soliton if and only if the equation*

$$R^{(0,2)}(X_1, X_2) = (\kappa - 1) g(X_1, X_2)$$

holds for any $X_1, X_2 \in \Gamma(TM)$. In other words, the submanifold is an Einstein manifold. Furthermore if the soliton is steady, then the Ricci curvature of $(M, g, S(TM))$ is negative defined.

In view of Theorem 3.7 we find:

Corollary 3.11. *Let $(M,g,S(TM))$ be a totally geodesic half-lightlike submanifold admitting a concircular vector field ζ. If the submanifold is a Ricci soliton, then we find*

$$R^{(0,2)}(X_1,X_2)=-f_1g(A_NX_1,X_2)-f_2g(A_UX_1,X_2)+(\kappa-\varphi)g(X_1,X_2)$$

for any $X_1,X_2\in\Gamma(S(TM))$.

In view of Theorem 3.8 we find:

Corollary 3.12. *Let $(M,g,S(TM))$ be a totally geodesic half-lightlike submanifold admitting a concurrent vector field ζ. If the submanifold is a Ricci soliton, then we find*

$$R^{(0,2)}(X_1,X_2)=-f_1g(A_NX_1,X_2)-f_2g(A_UX_1,X_2)+(\kappa-1)g(X_1,X_2)$$

for any $X_1,X_2\in\Gamma(S(TM))$.

Corollary 3.13. *Let $(M,g,S(TM))$ be an irrotational half-lightlike submanifold admitting a concircular vector field $\zeta=\xi$. If the submanifold is a Ricci soliton, then we find*

$$R^{(0,2)}(X_1,X_2)=(\kappa-\varphi)g(X_1,X_2)$$

for any $X_1,X_2\in\Gamma(S(TM))$.

Corollary 3.14. *Let $(M,g,S(TM))$ be an irrotational half-lightlike submanifold admitting a concurrent vector field $\zeta=\xi$. If the submanifold is a Ricci soliton, then we have*

$$R^{(0,2)}(X_1,X_2)=(\kappa-1)g(X_1,X_2)$$

for any $X_1,X_2\in\Gamma(S(TM))$.

Example 3.15 ([22]). *Consider a submanifold in $\mathbb{R}^5_2$ with the signature $(-,-,+,+,+)$ and local coordinate system $\{z_1,z_2,z_3,z_4,z_5\}$ given by*

$$z_4=(z_1^2+z_2^2)^{\frac{1}{2}},\quad z_3=(1-z_5^2)^{\frac{1}{2}},\quad z_5,z_1,z_2>0.$$

Thus we find

$$\begin{aligned}
Rad(TM) &= Span\{\xi=z_1\partial_1+z_2\partial_2+z_4\partial_4\},\\
S(TM) &= Span\{X_1=z_4\partial_1+z_1\partial_4, X_2=-z_5\partial_3+z_3\partial_5\},\\
ltr(TM) &= Span\left\{N=\frac{1}{2z_3^2}(z_1\partial_1-z_2\partial_2+z_4\partial_4)\right\},
\end{aligned}$$

and

$$\mathbb{D}=Span\{U=z_3\partial_3+z_5\partial_5\}.$$

Here $\{\partial_1,\partial_2,\partial_3,\partial_4,\partial_5\}$ is the natural frame field on $\mathbb{R}^5_2$. By a straightforward computation, we obtain that $(M,g,S(TM))$ is a half-lightlike submanifold and

$$\widetilde{\nabla}_{X_1}\xi=X_1,\quad \widetilde{\nabla}_{X_2}\xi=0,\quad \widetilde{\nabla}_{\xi}\xi=\xi$$

which show that

$$\begin{aligned}
B(X_1,\xi) &= B(X_2,\xi)=B(X_1,X_2)=B(X_2,X_2)=0,\quad B(X_1,X_1)=z_2^2,\\
C(X_1,\xi) &= C(X_2,\xi)=C(X_1,X_2)=C(X_2,X_2)=0,\quad C(X_1,X_1)=-\frac{1}{2},\\
D(X_1,\xi) &= D(X_2,\xi)=D(X_1,X_2)=D(X_1,X_1)=0,\quad D(X_2,X_2)=1.
\end{aligned}$$

From (2.9), we find

$$R^{(0,2)}(X_1,X_1)=R^{(0,2)}(X_2,X_2)=R^{(0,2)}(X_1,X_2)=0.$$

If we consider $\zeta=\varphi(\xi+U)$, where φ is a differentiable function on M and $\zeta^T=\varphi\xi$, then we obtain ζ is concircular and $(M,g,S(TM))$ is a shrinking Ricci soliton such that $\zeta^\top$ is the potential vector field and $\kappa=1$.

Theorem 3.16 (cf. Theorem 3.2 of [23])**.** *Let $\widetilde{M}(c)$ be an indefinite space form with constant curvature c and $(M,g,S(TM))$ be a totally umbilical half-lightlike submanifold with the co-dimension 2. Then the following equality holds:*

$$\begin{aligned} R(X_1,X_2)X_3 &= c\{g(X_2,X_3)X_1 - g(X_1,X_3)X_2\} + H_1\{g(X_2,X_3)A_N X_1 - g(X_1,X_3)A_N X_2\} \\ &\quad + H_2\{g(X_2,X_3)A_U X_1 - g(X_1,X_3)A_U X_2\}. \end{aligned} \tag{3.9}$$

Theorem 3.17. *Let $(M,g,S(TM))$ be a totally umbilical half-lightlike submanifold of $\widetilde{M}(c)$ admitting a concircular vector field $\zeta = \zeta^{\top}$. The submanifold is a Ricci soliton such that $\zeta^{\top}$ is the potential vector field if and only if the equation*

$$\kappa = c + \varphi + H_1 traceA_N + H_2 traceA_U \tag{3.10}$$

is satisfied.

Proof. Let $\{Y_1,\ldots,Y_n\}$ be an orthonormal frame field of $\Gamma(\mathrm{S}(TM))$. In view of (2.10) and (3.9), it can be obtained

$$R^{(0,2)}(Y_i,Y_i) = c + H_1 \mathrm{trace} A_N + H_2 \mathrm{trace} A_U \tag{3.11}$$

for any $i \in \{1,\ldots,n\}$. In view of (3.8) and (3.11) we obtain (3.10). The converse part is straightforward. □

As a result of Theorem 3.17, we obtain

Theorem 3.18. *Let $(M,g,S(TM))$ be a totally umbilical half-lightlike submanifold of $\widetilde{M}(c)$ admitting a concurrent vector field $\zeta = \zeta^{\top}$. The submanifold is a Ricci soliton such that $\zeta^{\top}$ is the potential vector field if and only if the equation*

$$\kappa = c + 1 + H_1 traceA_N + H_2 traceA_U$$

is satisfied.

4. Coisotropic Lightlike Submanifolds

Let $(M,g,\mathrm{S}(TM))$ be a lightlike submanifold of $(\widetilde{M},\widetilde{g})$ with co-dimension 2. If the dimension of $\mathrm{Rad}(TM)$ is 2, then the submanifold is called a coisotropic lightlike submanifold [24]. For any coisotropic lightlike submanifold, there exists the following quasi-orthonormal basis on $\Gamma(T\widetilde{M})$:

$$\{Y_1,Y_2,\ldots,Y_n,\xi_1,\xi_2,N_1,N_2\}$$

where $\mathrm{S}(TM) = Span\{Y_1,Y_2,\ldots,Y_n\}$, $\mathrm{Rad}(TM) = Span\{\xi_1,\xi_2\}$ and $ltr(TM) = Span\{N_1,N_2\}$. It is known that the relations

$$\widetilde{g}(N_i,\xi_j) = \delta_{ij}, \quad \widetilde{g}(N_i,N_j) = 0$$

are satisfied for $i,j \in \{1,2\}$. The Gauss and Weingarten formulas are formulated by

$$\widetilde{\nabla}_{X_1}X_2 = \nabla_{X_1}X_2 + \sum_{k=1}^{2} D_k(X_1,X_2)N_k, \tag{4.1}$$

$$\widetilde{\nabla}_{X_1}N_k = -A_{N_k}X_1 + \sum_{l=1}^{2} \rho_{kl}(X_1)N_l, \tag{4.2}$$

$$\nabla_{X_1}X_2 = \nabla^*_{X_1}PX_2 + \sum_{k=1}^{2} C_k(X_1,X_2)\xi_k, \tag{4.3}$$

$$\nabla_{X_1}\xi_k = -A^*_{\xi_k}X_1 - \sum_{l=1}^{2} \rho_{kl},(X_1)\xi_l, \tag{4.4}$$

where $\nabla_{X_1}X_2, A_{N_k}X_1 \in \Gamma(TM)$, $\nabla^*_{X_1}PX_2, A^*_{\xi_k}X_1 \in \Gamma(\mathrm{S}(TM))$, A_{N_k} are the shape operators on $\Gamma(TM)$, $A^*_{\xi_k}$ are the shape operators on $\Gamma(\mathrm{S}(TM))$, $D_k(X_1,X_2)$, $C_k(X_1,X_2)$ are ingredients of the second fundamental forms, ρ_{kl} are $1-$forms for $k,l \in \{1,2\}$. In view of (4.1)-(4.4), we have

$$D_k(X_1,X_2) = g(A^*_{\xi}X_1,X_2), \quad C_k(X_1,X_2) = g(A_{N_k}X_1,X_2)$$

for any $k \in \{1,2\}$. We note that D_k are symmetric but C_k are not symmetric. Using the fact that $\widetilde{\nabla}$ is a metric connection and from (4.1), we find

$$(\nabla_{X_3}g)(X_1,X_2) = D_1(X_3,X_2)\eta_1(X_1) + D_1(X_3,X_1)\eta_1(X_2) + D_2(X_3,X_2)\eta_2(X_1) + D_2(X_3,X_1)\eta_2(X_2)$$

which shows that ∇ is not a metric connection [24]. Here, $\eta_1(X_1) = g(X_1,\xi_1)$ and $\eta_2(X_1) = g(X_1,\xi_2)$.

The Lie derivative is formulated by

$$\begin{aligned}(L_{X_3}g)(X_1,X_2) &= D_1(X_3,X_2)\eta(X_1)+D_1(X_3,X_1)\eta(X_2)+D_2(X_3,X_2)\eta(X_1)+D_2(X_3,X_1)\eta(X_2)\\ &\quad +g(\nabla_{X_1}X_3,X_2)+g(\nabla_{X_2}X_3,X_1)\end{aligned} \tag{4.5}$$

or, equivalently, we have

$$(L_{X_3}g)(X_1,X_2)=(\nabla_{X_3}g)(X_1,X_2)+g(\nabla_{X_1}X_3,X_2)+g(\nabla_{X_2}X_3,X_1).$$

Furthermore, the following equality

$$\widetilde{g}(\widetilde{R}(X_1,X_2)PX_3,PX_4)=g(R(X_1,X_2)PX_3,PX_4)+\sum_{k=1}^{2}D_k(X_1,PX_3)C_k(X_2,PX_4)-\sum_{k=1}^{2}D_k(X_2,PX_3)C_k(X_1,PX_4)$$

holds for any $X_1,X_2,X_3,X_4\in\Gamma(TM)$. The induced Ricci type tensor $R^{(0,2)}$ of M is formulated by

$$R^{(0,2)}(X_1,X_2)=\sum_{j=1}^{n}g(R(Y_j,X_1)X_2,Y_j)+\sum_{k=1}^{2}\widetilde{g}(R(X_1,\xi_k)X_2,N_k). \tag{4.6}$$

We note that $R^{(0,2)}$ is not symmetric. If $S(TM)$ is integrable, then $R^{(0,2)}$ is symmetric.

A coisotropic lightlike submanifold with co-dimension 2 is said to be totally geodesic if $D_1=D_2=0$ and it is said to be irrotational [20] if $D_k=0$ on $\mathrm{Rad}(TM)$ for any $k\in\{1,2\}$. A coisotropic lightlike submanifold is known as totally umbilical if there is a differentiable transversal vector field H such that

$$D_1(X_1,X_2)N_1+D_2(X_1,X_2)N_2=g(X_1,X_2)H$$

holds for any $X_1,X_2\in\Gamma(TM)$ [22]. In this case, there exist two differentiable functions H_1 and H_2 satisfying

$$D_i(X_1,X_2)=H_i g(X_1,X_2)$$

for any $i\in\{1,2\}$.

5. Ricci Soliton Coisotropic Lightlike Submanifolds

Assume that ζ is a vector field lying on $\Gamma(T\widetilde{M})$. Hence one can put

$$\zeta=\zeta^{\top}+f_1N_1+f_2N_2$$

where $\zeta^{\top}\in\Gamma(TM)$ and f_1,f_2 are differentiable functions on $\Gamma(T\widetilde{M})$. Thus we find

$$f_1=g(\zeta,\xi_1)\quad\text{and}\quad f_2=g(\zeta,\xi_2).$$

Proposition 5.1. *Let $(M,g,S(TM))$ be a coisotropic lightlike submanifold of $(\widetilde{M},\widetilde{g})$. If ζ is concircular, then the following equalities are satisfied for any $X\in\Gamma(TM)$:*

$$\begin{aligned}\nabla_X\zeta^{\top} &= \varphi X+f_1A_{N_1}X+f_2A_{N_2}X,\\ D_1(X,\zeta^{\top}) &= -X[f_1]-f_1\rho_{11}(X)-f_2\rho_{21}(X),\\ D_2(X,\zeta^{\top}) &= -X[f_2]-f_1\rho_{12}(X)-f_2\rho_{22}(X).\end{aligned} \tag{5.1}$$

Proof. If ζ is concircular, we find

$$\widetilde{\nabla}_X\zeta=\varphi X=\widetilde{\nabla}_X\zeta^{\top}+\widetilde{\nabla}_X(f_1N_1)+\widetilde{\nabla}_X(f_2N_2).$$

In view of (4.1) and (4.2), we get

$$\begin{aligned}\varphi X &= \nabla_X\zeta^{\top}+D_1(X,\zeta^{\top})N_1+D_2(X,\zeta^{\top})N_2+X[f_1]N_1-f_1A_{N_1}X+f_1\rho_{11}(X)N_1+f_1\rho_{12}(X)N_2+X[f_2]N_2\\ &\quad -f_2A_{N_2}X+f_2\rho_{21}(X)N_1+f_2\rho_{22}(X)N_2.\end{aligned}$$

In view of the last equation, we obtain (3.1), (3.2) and (3.3) immediately. □

Putting $\varphi=1$ in Proposition 5.1, we have:

Proposition 5.2. *For any coisotropic lightlike submanifold admitting a concurrent vector field, we have*

$$\nabla_X \zeta^\top = X + f_1 A_{N_1} X + f_2 A_{N_2} X.$$

As a special case for $\zeta = \zeta^\top$, we have:

Proposition 5.3. *For any coisotropic lightlike submanifold admitting a concircular (or concurrent) vector field $\zeta = \zeta^\top$, we have*

$$D_1(X, \zeta^\top) = D_2(X, \zeta^\top) = 0.$$

Theorem 5.4. *Let $(M, g, S(TM))$ be a coisotropic lightlike submanifold. If there exists at least one concircular or concurrent vector field $\zeta = \zeta^\top$ on $\Gamma(TM)$, then $\nabla_{\zeta^\top} g = 0$.*

Using (4.5) and (5.1), we arrive at

$$\begin{aligned}(L_{\zeta^\top} g)(X_1, X_2) \;=\;& D_1(\zeta^\top, X_2)\eta_1(X_1) + D_1(\zeta^\top, X_1)\eta_1(X_2) + D_2(\zeta^\top, X_2)\eta_2(X_1) + D_2(\zeta^\top, X_1)\eta_2(X_2) \\ & + 2g(\varphi X_1, X_2) + 2f_1 g(A_{N_1} X_1, X_2) + 2f_2 g(A_{N_2} X_1, X_2)\end{aligned} \tag{5.2}$$

for any $X_1, X_2 \in \Gamma(TM)$. If ζ is concurrent, then we obtain

$$\begin{aligned}(L_{\zeta^\top} g)(X_1, X_2) \;=\;& D_1(\zeta^\top, X_2)\eta_1(X_1) + D_1(\zeta^\top, X_1)\eta_1(X_2) + D_2(\zeta^\top, X_2)\eta_2(X_1) + D_2(\zeta^\top, X_1)\eta_2(X_2) \\ & + 2g(X_1, X_2) + 2f_1 g(A_{N_1} X_1, X_2) + 2f_2 g(A_{N_2} X_1, X_2).\end{aligned} \tag{5.3}$$

As a special case for $\zeta = \zeta^\top$, we obtain

$$(L_{\zeta^\top} g)(X_1, X_2) = 2\varphi g(X_1, X_2).$$

If ζ is concurrent and $\zeta = \zeta^\top$, we find

$$(L_{\zeta^\top} g)(X_1, X_2) = 2g(X_1, X_2).$$

From (3.5) and (5.2), we have

Theorem 5.5. *Let $(M, g, S(TM))$ be a coisotropic lightlike submanifold admitting a concircular vector field ζ. Then, $(M, g, S(TM))$ is a Ricci soliton such that $\zeta^\top$ is the potential vector field if and only if the equation*

$$\begin{aligned}R^{(0,2)}(X_1, X_2) \;=\;& \frac{1}{2}\Big[-D_1(\zeta^\top, X_2)\eta_1(X_1) - D_1(\zeta^\top, X_1)\eta_1(X_2) - D_2(\zeta^\top, X_2)\eta_2(X_1) \\ & - D_2(\zeta^\top, X_1)\eta_2(X_2) - 2f_1 g(A_{N_1} X_1, X_2) - 2f_2 g(A_{N_2} X_1, X_2)\Big] + (\kappa - \varphi) g(X_1, X_2)\end{aligned}$$

holds for any $X_1, X_2 \in \Gamma(TM)$.

From (3.5) and (5.3), we have:

Theorem 5.6. *Let $(M, g, S(TM))$ be a coisotropic lightlike submanifold admitting a concurrent vector field ζ. Then, $(M, g, S(TM))$ is a Ricci soliton such that $\zeta^\top$ is the potential vector field if and only if the equation*

$$R^{(0,2)}(X_1, X_2) = \frac{1}{2}\left[\begin{array}{l} -D_1(\zeta^\top, X_2)\eta_1(X_1) - D_1(\zeta^\top, X_1)\eta_1(X_2) - D_2(\zeta^\top, X_2)\eta_2(X_1) \\ -D_2(\zeta^\top, X_1)\eta_2(X_2) - 2f_1 g(A_{N_1} X_1, X_2) - 2f_2 g(A_{N_2} X_1, X_2) \end{array}\right] + (\kappa - 1) g(X_1, X_2)$$

is satisfied for any $X_1, X_2 \in \Gamma(TM)$.

Theorem 5.7. *Let $(M, g, S(TM))$ be a coisotropic lightlike submanifold admitting a concircular vector field $\zeta = \zeta^\top$. Then, the submanifold is a Ricci soliton such that $\zeta^\top$ is the potential vector field if and only if the equation*

$$R^{(0,2)}(X_1, X_2) = (\kappa - \varphi) g(X_1, X_2) \tag{5.4}$$

holds for any $X_1, X_2 \in \Gamma(TM)$. It means that $(M, g, S(TM))$ is an Einstein manifold.

Theorem 5.8. *Let $(M, g, S(TM))$ be a coisotropic lightlike submanifold admitting a concurrent vector field $\zeta = \zeta^\top$. Then, the submanifold is a Ricci soliton with the potential vector field $\zeta^\top$ if and only if the equation*

$$R^{(0,2)}(X_1, X_2) = (\kappa - 1) g(X_1, X_2)$$

is satisfied for any $X_1, X_2 \in \Gamma(TM)$. It means that the submanifold is an Einstein manifold. Furthermore if the soliton is steady, then the Ricci curvature of $(M, g, S(TM))$ is negative defined.

In view of Theorem 5.5 we find:

Corollary 5.9. *Let $(M,g,S(TM))$ be a totally geodesic coisotropic lightlike submanifold admitting a concircular vector field ζ. If the submanifold is a Ricci soliton, then the equation*

$$R^{(0,2)}(X_1,X_2)=(\kappa-\varphi)g(X_1,X_2)-f_1g(A_{N_1}X_1,X_2)-f_2g(A_{N_2}X_1,X_2)$$

is satisfied.

As a result of Theorem 5.6 we find:

Corollary 5.10. *Let $(M,g,S(TM))$ be a totally geodesic coisotropic lightlike submanifold admitting a concurrent vector field ζ. If the submanifold is a Ricci soliton, then the equation*

$$R^{(0,2)}(X_1,X_2)=(\kappa-1)g(X_1,X_2)-f_1g(A_{N_1}X_1,X_2)-f_2g(A_{N_2}X_1,X_2)$$

is satisfied.

Example 5.11. *Consider a submanifold in $\mathbb{R}^6_3$ given by*

$$z_4=(z_1^2+z_2^2)^{\frac{1}{2}},\quad z_3=(z_5^2+z_6^2)^{\frac{1}{2}},\quad z_1,z_2,z_3,z_4>0.$$

Then we find

$$\begin{aligned}
Rad(TM) &= Span\{\xi_1=z_1\partial_1+z_2\partial_2+z_4\partial_4,\quad \xi_2=z_3\partial_3+z_5\partial_5+z_6\partial_6\},\\
S(TM) &= Span\{X_1=z_4\partial_1+z_1\partial_4, X_2=z_3\partial_5+z_5\partial_3\},\\
ltr(TM) &= Span\left\{N_1=\frac{1}{2z_2^2}(z_1\partial_1-z_2\partial_2+z_3\partial_3),\quad N_2=\frac{1}{2z_5^2}(z_3\partial_3-z_5\partial_5+z_6\partial_6)\right\},
\end{aligned}$$

where $\{\partial_1,\partial_2,\partial_3,\partial_4,\partial_5,\partial_6\}$ is the natural frame field on $\mathbb{R}^6_3$. Therefore $(M,g,S(TM))$ is a coisotropic lightlike submanifold. By a straightforward computation, we obtain

$$\begin{array}{llll}
\widetilde{\nabla}_{X_1}\xi_1=X_1, & \widetilde{\nabla}_{X_1}\xi_2=0, & \widetilde{\nabla}_{X_2}\xi_1=0, & \widetilde{\nabla}_{X_2}\xi_2=X_2,\\
\widetilde{\nabla}_{\xi_1}\xi_1=\xi_1, & \widetilde{\nabla}_{\xi_1}\xi_2=\widetilde{\nabla}_{\xi_2}\xi_1=0, & \widetilde{\nabla}_{\xi_2}\xi_2=\xi_2, & \widetilde{\nabla}_{\xi_2}N_2=N_2.\\
\widetilde{\nabla}_{\xi_1}N_1=N_1, & \widetilde{\nabla}_{\xi_1}N_2=0, & \widetilde{\nabla}_{\xi_2}N_1=0, &
\end{array}$$

In view of (4.1)-(4.4), we obtain

$$D_1(\xi_1,\xi_1)=D_2(\xi_2,\xi_2)=D_1(\xi_1,\xi_2)=0.$$

Therefore, we have $R^{(0,2)}(X_1,X_2)=0$. If we consider $\zeta=\varphi(\xi_1+\xi_2)$, where φ is a differentiable function and $\zeta=\zeta^T$, then we find $(M,g,S(TM))$ is a Ricci soliton such that $\zeta^\top$ is the potential vector field and $\kappa=1$. Thus, the submanifold is a shrinking Ricci soliton.

Example 5.12. *Consider a submanifold in $\mathbb{R}^5_2$ with the signature $(-,-,+,+,+)$ given by*

$$M=\left\{(u^2,u^2w,u^2\cos v,u^2\sin v,u^2w):\ v\in[0,2\pi),u,w\in\mathbb{R}\right\}.$$

Then we have

$$\begin{aligned}
Rad(TM) &= Span\{\xi_1=2u(\partial_1+w\partial_2+\cos v\partial_3+\sin v\partial_4+w\partial_5),\quad \xi_2=u^2(\partial_2+\partial_5)\},\\
S(TM) &= Span\{X_1=u^2(-\sin v\partial_3+\cos v\partial_4)\},\\
ltr(TM) &= Span\left\{N_1=\frac{1}{2u}(-\partial_1+w\partial_2+\cos v\partial_3+\sin v\partial_4+w\partial_5),\quad N_2=\frac{1}{2u^2}(-\partial_2+\partial_5)\right\},
\end{aligned}$$

where $\{\partial_1,\partial_2,\partial_3,\partial_4,\partial_5\}$ is the natural frame field on $\mathbb{R}^5_2$. Therefore, $(M,g,S(TM))$ is a coisotropic lightlike hypersurface of $\mathbb{R}^5_2$. By a straightforward computation, we find

$$\widetilde{\nabla}_{X_1}\xi_1=\frac{2}{u}X_1,\quad \widetilde{\nabla}_{\xi_2}\xi_1=\frac{2}{u}\xi_2,\quad \widetilde{\nabla}_{\xi_1}\xi_1=\frac{2}{u}\xi_1,$$

which show that ξ_1 is a concircular vector field on $(M,g,S(TM))$ and $(L_{\xi_1}g)(X_1,X_1)=2u^2$. By a direct computation, we get

$$R(X_1,\xi_1)X_1=X_1,\quad R(X_1,\xi_2)X_1=0$$

which shows that $R^{(0,2)}(X_1,X_1)=0$. This fact shows that $(M,g,S(TM))$ is almost Ricci soliton such that $\kappa=\frac{2}{u^2}$.

Theorem 5.13 (Theorem 3.1 in [24]). *Let $(M, g, S(TM))$ be a totally umbilical lightlike submanifold of $\widetilde{M}(c)$. Then the following equality holds*

$$R(X_1,X_2)X_3 = c\{g(X_2,X_3)X_1 - g(X_1,X_3)X_2\} + \sum_{i=1}^{2}[H_i\{g(X_2,X_3)A_{N_i}X_1 - g(X_1,X_3)A_{N_i}X_2\}], \tag{5.5}$$

where the mean curvature vector field $H = H_1N + H_2U$.

Theorem 5.14. *Let $(M, g, S(TM))$ be a totally umbilical lightlike submanifold of $\widetilde{M}(c)$ admitting a concircular vector field $\zeta = \zeta^\top$. The submanifold is a Ricci soliton such that $\zeta^\top$ is the potential vector field if and only if the equation*

$$\kappa = c + \varphi + \sum_{i=1}^{2} H_i trace A_{N_i}$$

is satisfied.

Proof. Let $\{Y_1, \ldots, Y_n\}$ be an orthonormal frame field on the screen distribution. From (4.6) and (5.5), we find

$$R^{(0,2)}(Y_j,Y_j) = c + \sum_{i=1}^{2} H_1 \text{trace} A_{N_i} \tag{5.6}$$

for any $j \in \{1, \ldots, n\}$. From (5.4) and (5.6), the proof is straightforward. □

As a result of Theorem 5.14, we obtain:

Theorem 5.15. *Let $(M, g, S(TM))$ be a totally umbilical lightlike submanifold of $\widetilde{M}(c)$ admitting a concurrent vector field $\zeta = \zeta^\top$. The submanifold is a Ricci soliton with the potential vector field $\zeta^\top$ if and only if the equation*

$$\kappa = c + 1 + \sum_{i=1}^{2} H_i trace A_{N_i}$$

is satisfied.

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