http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 73, Number 2, Pages 349–364 (2024)
DOI:10.31801/cfsuasmas.1281348
ISSN 1303-5991 E-ISSN 2618-6470

Research Article; Received:April 19, 2023; Accepted: December 17, 2023

SOME PROPERTIES OF A CLASS OF GENERALIZED JANOWSKI-TYPE q-STARLIKE FUNCTIONS ASSOCIATED WITH OPOOLA q-DIFFERENTIAL OPERATOR AND q-DIFFERENTIAL SUBORDINATION

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Abstract. Without qualms, studies show that quantum calculus has received great attention in recent times. This can be attributed to its wide range of applications in many science areas. In this exploration, we study a new q-differential operator that generalized many known differential operators. The new q-operator and the concept of subordination were afterwards, used to define a new subclass of analytic-univalent functions that invariably consists of several known and new generalizations of starlike functions. Consequently, some geometric properties of the new class were investigated. The properties include coefficient inequality, growth, distortion and covering properties. In fact, we solved some radii problems for the class and also established its subordinating factor sequence property. Indeed, varying some of the involving parameters in our results led to some existing results.

1. Introduction

Define the set

$$\aleph_\jmath = \{\jmath, \jmath+1, \jmath+2, \dots\},\ \jmath = 0,1,2,\dots\ .$$

Let $\Xi = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ be the *unit disk* and let

$$\mathfrak{A} = \left\{ f : f(z) = z + \sum_{k=2}^{\infty} a_k z^k,\ f(0)=0,\ f'(0)=1, \text{ and } z \in \Xi \right\} \tag{1}$$

2020 *Mathematics Subject Classification.* 30C45, 30C55.

Keywords. Analytic function, univalent function, Schwarz function, Janowski q-starlike function, subordination, radius problem, distortion theorem, growth theorem, covering theorem, coefficient inequality, subordinating factor sequence, Opoola q-differential operator.

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be the class of *normalized analytic functions*. Also, let $\mathcal{S}$ which is a subclass of $\mathfrak{A}$ represent the class of functions that are analytic and univalent in $\varXi$. For $\varkappa \in [0,1)$, let $\mathcal{S}^\star(\varkappa)$, $\mathcal{C}(\varkappa)$ and $\mathcal{K}(\varkappa)$ represent the classes of starlike functions of order $\varkappa$, convex functions of order $\varkappa$, and close-to-convex functions of order $\varkappa$, respectively. A function f in (1) belongs to the classes $\mathcal{S}^\star(\varkappa)$, $\mathcal{C}(\varkappa)$ and $\mathcal{K}(\varkappa)$ if for $z \in \varXi$, $\mathcal{R}e(zf'/f) > \varkappa$, $\mathcal{R}e(z(f''/f') + 1) > \varkappa$ and $\mathcal{R}e(f'/h') > \varkappa$ $(h \in \mathcal{C})$, respectively. We shall let $\mathcal{S}^\star(0) = \mathcal{S}^\star$, $\mathcal{C}(0) = \mathcal{C}$ and $\mathcal{K}(0) = \mathcal{K}$ simply denote the classes of starlike functions, convex functions and close-to-convex functions, respectively.
Historically, class $\mathcal{S}^\star$ of starlike functions was introduced by Alexander [1] and it has been numerously studied in various forms, such as starlike functions of order $\varkappa$, strongly starlike functions, uniformly starlike functions, close-to-starlike functions, bi-starlike functions, Janowski-type starlike functions, Mocanu-type starlike functions, starlike functions of complex order, λ-pseudo-starlike functions, and many more. In deed, an impressive application of starlike functions was demonstrated by Rensaa [32] where the author used starlike functions to solve frequency analysis problem. A frequency analysis problem is the problem of determining unknown frequency f_k $(k \in \aleph_1)$, with its corresponding amplitude a_k $(k \in \aleph_1)$, and of a trigonometric signal $z_k(m)$ where the signal values from k observations are known. We refer readers to [15,25,39] for more information on starlike functions and to [9,41] for some details on its applications.

Suppose $f_1, f_2 \in \mathfrak{A}$, f_1 is said to be subordinate to f_2, notationally expressed as $f_1(z) \prec f_2(z)$ $(z \in \varXi)$, if there exists a Schwarz function: $\omega(z) = \omega_1 z + \omega_2 z^2 + \cdots$ $(|\omega(z)| < 1,\ z \in \varXi)$ such that

$$f_1(z) = f_2(z) \circ \omega(z) = f_2(\omega(z)). \tag{2}$$

In case $f_2(z)$ is univalent in $\varXi$, then $f_1(z) \prec f_2(z) \iff f_1(0) = f_2(0)$ and $f_1(\varXi) \subset f_2(\varXi)$.
Let $\mathcal{P}(\varkappa)$ represent the class of Carathéodory functions of order $\varkappa$ and of the form

$$p_\varkappa(z) = 1 + \sum_{k=1}^{\infty}(1-\varkappa)p_k z^k \quad \big(\mathcal{R}e\, p_\varkappa(z) > \varkappa \in [0,1),\ p_\varkappa(0) = 1,\ z \in \varXi\big). \tag{3}$$

Clearly, $\mathcal{P}(\varkappa) \subseteq \mathcal{P}(0) = \mathcal{P}$, where $\mathcal{P}$ is simply called the class of Carathéodory functions. In 2006, Polatoğlu et al. [30] generalized function is the class $\mathcal{P}$ by introducing the class

$$\mathcal{P}(\lambda; \mathcal{A}, \mathcal{B}) := \Big\{ p(z) \in \mathcal{P} : p(z) \prec (1-\lambda)\frac{1+\mathcal{A}z}{1+\mathcal{B}z} + \lambda \iff p(z) = (1-\lambda)\frac{1+\mathcal{A}\omega(z)}{1+\mathcal{B}\omega(z)} + \lambda \Big\} \tag{4}$$

where all parameters are as declared in (8). It is easily seen that $\mathcal{P}(0,1,-1) = \mathcal{P}(1,-1)$ in (3) and $\mathcal{P}(0,\mathcal{A},\mathcal{B}) = \mathcal{P}(\mathcal{A},\mathcal{B})$, the class of Janowski functions introduced in [16], see also [8,39] for more details.

Quantum calculus (simply known as q-calculus) has received a surge in research in recent years, owing to its wide range of applications in mathematics, physics and other sciences. Specifically, its application areas include, for example, quantum physics, operator theory, ordinary fractional calculus, and optimal control problems; see [5,6,17,31,40]. The application of q-calculus (that is, q-differentiation, q-integration and q-analysis,) in the development of Geometric Function Theory (GFT) is particularly noteworthy. Current development in GFT shows that the concept of q-calculus has enticed many geometric function theorists. Since the introduction of the q-derivative and the q-integral by Jackson [13,14], many researchers (see [4,18,21–24,27,28,35,42]) have in diverse ways considered them in the establishment of many properties of the subclasses of $\mathfrak{A}$. In particular, authors in [5,6,17,36] extensively discussed some areas of applications of q-operators, q-functions, q-series and q-analysis in various fields of Pure and Applied Mathematics.

For function $f \in \mathfrak{A}$ of the form (1) and for $q \in (0,1)$, the q-differential operator $\mathcal{D}_q : \mathfrak{A} \longrightarrow \mathfrak{A}$ is define by

$$\left.\begin{array}{l}
\mathcal{D}_q f(0) = f'(0) = 1 \quad (z=0) \quad \textit{if it exists}, \\
\mathcal{D}_q f(z) = \begin{cases} \frac{f(z)-f(qz)}{z(1-q)} = 1 + \sum\limits_{k=2}^{\infty} [k]_q a_k z^{k-1} \quad (z \neq 0), \\ f'(z) \text{ as } q \longrightarrow 1, \end{cases} \\
\mathcal{D}_q^2 f(z) = \mathcal{D}_q(\mathcal{D}_q f(z)) = \sum\limits_{k=2}^{\infty} [k-1]_q [k]_q a_k z^{k-2}, \\
\textit{and} \;\; [k]_q = \frac{1-q^k}{1-q} \;\; \textit{so that by L'H\check{o}pital's rule,} \;\; \lim\limits_{q \uparrow 1} [k]_q = k.
\end{array}\right\} \tag{5}$$

Using (5), then the Opoola q-differential operator $D_{q,t}^{n,b,u}$ is defined as follows.

Definition 1. *Let $f \in \mathfrak{A}$, then the Opoola q-differential operator $D_{q,t}^{n,b,u} : \mathfrak{A} \longrightarrow \mathfrak{A}$ $(q \in (0,1),\ n \in \aleph_0)$ is defined by*

$$\left.\begin{array}{l}
D_{q,t}^{0,b,u} f(z) = f(z) \\
D_{q,t}^{1,b,u} f(z) = (1 + (b-u-1)t) f(z) - zt(b-u) + zt\mathcal{D}_q f(z) = d_{q,t}(f) \\
D_{q,t}^{2,b,u} f(z) = d_{q,t}(D_{q,t}^{1,b,u} f(z)) \\
\quad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
D_{q,t}^{n,b,u} f(z) = d_{q,t}(D_{q,t}^{n-1,b,u} f(z))
\end{array}\right\} \tag{6}$$

which implies that

$$D_{q,t}^{n,b,u}f(z) = z + \sum_{k=2}^{\infty}(1 + ([k]_q + b - u - 1)t)^n a_k z^k \quad (z \in \Xi) \tag{7}$$

where

$$\left.\begin{array}{c} n \in \aleph_0,\ t \geqq 0,\ b \geqq 0,\ u \in [0,b],\ \lambda \in [0,1),\ -1 \leqq \mathcal{B} < \mathcal{A} \leqq 1, \\ q \in (0,1),\ [k]_q = \frac{1-q^k}{1-q},\ \textit{and}\ \lim_{q\uparrow 1}[k]_q = k. \end{array}\right\} \tag{8}$$

The q-operator in (6) is the q-analogue of the well-known Opoola differential operator introduced in [26]. The following properties hold for the functions in (7).

(1) $\lim_{q\uparrow 1} D_{q,t}^{0,b,u}f(z) = \lim_{q\uparrow 1} D_{q,0}^{n,b,u}f(z) = \lim_{q\uparrow 1} D_{q,0}^{0,b,u}f(z) = f(z) \in \mathfrak{A}$ in (1).

(2) $\lim_{q\uparrow 1} D_{q,1}^{n,b,b}f(z) = \lim_{q\uparrow 1} D_{q,1}^{n,u,u}f(z) = D^n f(z)$, the Sălăgean differential operator introduced in [33].

(3) $\lim_{q\uparrow 1} D_{q,t}^{n,b,b}f(z) = \lim_{q\uparrow 1} D_{q,t}^{n,u,u}f(z) = D_t^n f(z)$, the Al-Oboudi differential operator introduced in [3].

(4) $\lim_{q\uparrow 1} D_{q,t}^{n,b,u}f(z) = D_t^{n,b,u}f(z)$, the Opoola differential operator introduced in [26].

(5) $D_{q,1}^{n,b,b}f(z) = D_{q,1}^{n,u,u}f(z) = D_q^n f(z)$, the Sălăgean q-differential operator introduced by Govindaraj and Sivasubramanian [11].

(6) $D_{q,t}^{n,b,b}f(z) = D_{q,t}^{n,u,u}f(z) = D_{q,t}^n f(z)$ is herein referred to as the Al-Oboudi q-differential operator.

Instances of some recently studied q-operators in GFT can be found in [2,18,20, 29].

2. A New Class of q-Starlike Functions

In view of the geometric expression of starlike functions, the Polatoḡlu's function in (4) and the Opoola q-differential operator in Definition 1 , we therefore, present the class $\mathcal{S}_q^\star(n,b,t,u;\lambda,\mathcal{A},\mathcal{B})$ as follows.

Definition 2. *A function $f \in \mathfrak{A}$ is said to be a member of the class $\mathcal{S}_q^\star(n,b,t,u;\lambda,\mathcal{A},\mathcal{B})$ if it satisfies the q-differential subordination condition*

$$\frac{D_{q,t}^{n+1,b,u}f(z)}{D_{q,t}^{n,b,u}f(z)} \prec (1-\lambda)\frac{1+\mathcal{A}z}{1+\mathcal{B}z} + \lambda \quad (z \in \Xi) \tag{9}$$

where all parameters are as declared in (8)*.*

It can easily be seen that class $\mathcal{S}_q^\star(n,b,t,u;\lambda,\mathcal{A},\mathcal{B})$ consists of numerous subclasses of starlike functions when its involving parameters are varied. Some studies on Janowski's q-starlike functions with various definitions can be found in [7,12,19,38].

3. Applicable Lemma

Definition 3 ([43]). (Subordinating Factor Sequence). *The sequence* $\{h_k\}_{k=1}^{\infty}$ *of complex numbers is called a* subordinating factor sequence *if whenever*

$$c(z) = \sum_{k=1}^{\infty} c_k z^k \quad (c_1 = 1,\ z \in \Xi)$$

is analytic-univalently convex in Ξ, $\sum_{k=1}^{\infty} c_k h_k \prec c(z)$.

Lemma 1 ([43]). (Subordinating Factor Sequence). *From Definition 3, the sequence* $\{h_k\}_{k=1}^{\infty}$ *is called a subordinating factor sequence if and only if*

$$\mathcal{R}e\left(1 + 2\sum_{k=1}^{\infty} c_k z^k\right) > 0 \quad (z \in \Xi).$$

4. The Main Results

For brevity and in what follows from (7), let

$$\varDelta_{q,k} = (1 + ([k]_q + b - u - 1)t) \geqq 1, \tag{10}$$

so that

$$D_{q,t}^{n,b,u} f(z) = z + \sum_{k=2}^{\infty} \varDelta_{q,k}^{n} a_k z^k \quad (z \in \Xi), \tag{11}$$

$$\varLambda_q^n(k,\lambda,\mathcal{A},\mathcal{B}) = \varDelta_{q,k}^n \Big\{ \big(\varDelta_{q,k} - 1\big) + \Big| \varDelta_{q,k}\mathcal{B} - [\mathcal{A} - \lambda(\mathcal{A} - \mathcal{B})] \Big| \Big\}, \tag{12}$$

and henceforth, all parameters shall be as declared in (8).

4.1. Basic Properties.

Theorem 1 (Coefficient Inequality). *Let* $f \in \mathfrak{A}$, *then* $f \in \mathcal{S}_q^{\star}(n,b,t,u;\lambda,\mathcal{A},\mathcal{B})$ *if and only if*

$$\sum_{k=2}^{\infty} \frac{\varLambda_q^n(k,\lambda,\mathcal{A},\mathcal{B})}{(\mathcal{A} - \mathcal{B})(1-\lambda)} |a_k| \leqq 1. \tag{13}$$

All parameters are as declared in (8).

Proof. Suppose inequality (13) holds, then in view of the principle of subordination, we can express (9) as

$$\frac{D_{q,t}^{n+1,b,u} f(z)}{D_{q,t}^{n,b,u} f(z)} = \frac{1 + [\mathcal{A} - \lambda(\mathcal{A} - \mathcal{B})]\omega(z)}{1 + \mathcal{B}\omega(z)} \tag{14}$$

which simplifies to

$$\frac{D_{q,t}^{n+1,b,u} f(z) - D_{q,t}^{n,b,u} f(z)}{[\mathcal{A} - \lambda(\mathcal{A} - \mathcal{B})] D_{q,t}^{n,b,u} f(z) - \mathcal{B} D_{q,t}^{n+1,b,u} f(z)} = \omega(z). \tag{15}$$

Using (11) in (15) leads to

$$\frac{(z+\sum_{k=2}^{\infty}\Delta_{q,k}^{n+1}a_kz^k)-(z+\sum_{k=2}^{\infty}\Delta_{q,k}^{n}a_kz^k)}{[\mathcal{A}-\lambda(\mathcal{A}-\mathcal{B})](z+\sum_{k=2}^{\infty}\Delta_{q,k}^{n}a_kz^k)-\mathcal{B}(z+\sum_{k=2}^{\infty}\Delta_{q,k}^{n+1}a_kz^k)}$$
$$=\frac{\sum_{k=2}^{\infty}\Delta_{q,k}^{n}(\Delta_{q,k}-1)a_kz^{k-1}}{(\mathcal{A}-\mathcal{B})(1-\lambda)-\sum_{k=2}^{\infty}\Delta_{q,k}^{n}\{\Delta_{q,k}\mathcal{B}-[\mathcal{A}-\lambda(\mathcal{A}-\mathcal{B})]\}a_kz^{k-1}}=\omega(z). \quad (16)$$

For $|\omega(z)|<1$ and $z\in\Xi$, we have

$$\left|\frac{\sum_{k=2}^{\infty}\Delta_{q,k}^{n}(\Delta_{q,k}-1)a_kz^{k-1}}{(\mathcal{A}-\mathcal{B})(1-\lambda)-\sum_{k=2}^{\infty}\Delta_{q,k}^{n}\{\Delta_{q,k}\mathcal{B}-[\mathcal{A}-\lambda(\mathcal{A}-\mathcal{B})]\}a_kz^{k-1}}\right|$$
$$\leqq\frac{\sum_{k=2}^{\infty}\Delta_{q,k}^{n}(\Delta_{q,k}-1)|a_k|}{(\mathcal{A}-\mathcal{B})(1-\lambda)-\sum_{k=2}^{\infty}\Delta_{q,k}^{n}\Big|\Delta_{q,k}\mathcal{B}-[\mathcal{A}-\lambda(\mathcal{A}-\mathcal{B})]\Big||a_k|}\leqq 1.$$

This latter expression on the LHS is bounded above by 1 if

$$\sum_{k=2}^{\infty}\Delta_{q,k}^{n}(\Delta_{q,k}-1)|a_k|\leqq(\mathcal{A}-\mathcal{B})(1-\lambda)-\sum_{k=2}^{\infty}\Delta_{q,k}^{n}\Big|\Delta_{q,k}\mathcal{B}-[\mathcal{A}-\lambda(\mathcal{A}-\mathcal{B})]\Big||a_k|$$

so that further simplification leads to

$$\sum_{k=2}^{\infty}\Delta_{q,k}^{n}\Big\{\big(\Delta_{q,k}-1\big)+\Big|\Delta_{q,k}\mathcal{B}-[\mathcal{A}-\lambda(\mathcal{A}-\mathcal{B})]\Big|\Big\}|a_k|\leqq(\mathcal{A}-\mathcal{B})(1-\lambda) \quad (17)$$

and using (12) gives

$$\sum_{k=2}^{\infty}\Lambda_q^n(k,\lambda,\mathcal{A},\mathcal{B})|a_k|\leqq(\mathcal{A}-\mathcal{B})(1-\lambda). \quad (18)$$

Conversely, suppose $f\in\mathcal{S}_q^{\star}(n,b,t,u;\lambda,\mathcal{A},\mathcal{B})$, then from (16) we have

$$\left|\frac{\sum_{k=2}^{\infty}\Delta_{q,k}^{n}(\Delta_{q,k}-1)a_kz^{k-1}}{(\mathcal{A}-\mathcal{B})(1-\lambda)-\sum_{k=2}^{\infty}\Delta_{q,k}^{n}\{\Delta_{q,k}\mathcal{B}-[\mathcal{A}-\lambda(\mathcal{A}-\mathcal{B})]\}a_kz^{k-1}}\right|=|\omega(z)|<1$$

and since $\mathcal{R}e\, z\leqq|z|<1$, then it implies that

$$\mathcal{R}e\left\{\frac{\sum_{k=2}^{\infty}\Delta_{q,k}^{n}(\Delta_{q,k}-1)a_kz^{k-1}}{(\mathcal{A}-\mathcal{B})(1-\lambda)-\sum_{k=2}^{\infty}\Delta_{q,k}^{n}\{\Delta_{q,k}\mathcal{B}-[\mathcal{A}-\lambda(\mathcal{A}-\mathcal{B})]\}a_kz^{k-1}}\right\}<1.$$

Choosing values of z on the real axis of the complex plane and allowing $z\longrightarrow 1$, implies that

$$\frac{\sum_{k=2}^{\infty}\Delta_{q,k}^{n}(\Delta_{q,k}-1)|a_k|}{(\mathcal{A}-\mathcal{B})(1-\lambda)-\sum_{k=2}^{\infty}\Delta_{q,k}^{n}\{\Delta_{q,k}\mathcal{B}-[\mathcal{A}-\lambda(\mathcal{A}-\mathcal{B})]\}|a_k|}\leqq 1$$

so that further simplification and using (12) leads to

$$\sum_{k=2}^{\infty} \Lambda_q^n(k,\lambda,\mathcal{A},\mathcal{B})|a_k| \leqq (\mathcal{A}-\mathcal{B})(1-\lambda) \tag{19}$$

as asserted. □

Corollary 1. *Observe that from* (13), *equality occurs for function*

$$f_k(z) = z + \frac{(\mathcal{A}-\mathcal{B})(1-\lambda)}{\Lambda_q^n(k,\lambda,\mathcal{A},\mathcal{B})} z^k \quad (k \in \aleph_2,\ z \in \Xi). \tag{20}$$

Corollary 2. *Let* $f \in \mathcal{S}_q^\star(n,b,t,u;\lambda,\mathcal{A},\mathcal{B})$, *then*

$$|a_k| \leqq \frac{(\mathcal{A}-\mathcal{B})(1-\lambda)}{\Lambda_q^n(k,\lambda,\mathcal{A},\mathcal{B})} \quad (k \in \aleph_2). \tag{21}$$

with extremal function in (20).

Remark 1. *Let* $f \in \lim_{q\uparrow 1} \mathcal{S}_q^\star(0,b,1,b;0;1,-1) = \lim_{q\uparrow 1} \mathcal{S}_q^\star(0,u,1,u;0;1,-1) = \mathcal{S}^\star$, *then*

$$\sum_{k=2}^{\infty} k|a_k| \leqq 1 \quad (k \in \aleph_2).$$

This is the result of Goodman [10] and Silverman [34].

Theorem 2 (Growth Property)**.** *Let* $f \in \mathcal{S}_q^\star(n,b,t,u;\lambda,\mathcal{A},\mathcal{B})$, *then for* $r = |z| < 1$,

$$r - \frac{r^2 \Delta_{q,2}^n(\mathcal{A}-\mathcal{B})(1-\lambda)}{\Lambda_q^n(2,\lambda,\mathcal{A},\mathcal{B})} \leqq |D_{q,t}^{n,b,u} f(z)| \leqq r + \frac{r^2 \Delta_{q,2}^n(\mathcal{A}-\mathcal{B})(1-\lambda)}{\Lambda_q^n(2,\lambda,\mathcal{A},\mathcal{B})}. \tag{22}$$

Equality occurs for function

$$f_2(z) = z + \frac{(\mathcal{A}-\mathcal{B})(1-\lambda)}{\Lambda_q^n(2,\lambda,\mathcal{A},\mathcal{B})} z^2. \tag{23}$$

Proof. From (13) and for the fact that $\Delta_{q,k}$ is an increasing function of k ($\forall\ k \in \aleph_2$), then

$$\Lambda_q^n(2,\lambda,\mathcal{A},\mathcal{B}) \sum_{k=2}^{\infty} |a_k| \leqq \sum_{k=2}^{\infty} \Lambda_q^n(k,\lambda,\mathcal{A},\mathcal{B})|a_k| \leqq (\mathcal{A}-\mathcal{B})(1-\lambda)$$

which implies that

$$\sum_{k=2}^{\infty} |a_k| \leqq \frac{(\mathcal{A}-\mathcal{B})(1-\lambda)}{\Lambda_q^n(2,\lambda,\mathcal{A},\mathcal{B})}. \tag{24}$$

Recall also that for $f \in \mathfrak{A}$ and since $r^k < r = |z| < 1$, then from (11),

$$|D_{q,t}^{n,b,u} f(z)| = \left| z + \sum_{k=2}^{\infty} \Delta_{q,k}^n a_k z^k \right| \leqq r + \sum_{k=2}^{\infty} \Delta_{q,k}^n |a_k| r^k \leqq r + r^2 \Delta_{q,2}^n \sum_{k=2}^{\infty} |a_k| \tag{25}$$

so that putting (24) into (25) leads to

$$|D_{q,t}^{n,b,u}f(z)| \leqq r + \frac{r^2\Delta_{q,2}^n(\mathcal{A}-\mathcal{B})(1-\lambda)}{\Lambda_q^n(2,\lambda,\mathcal{A},\mathcal{B})}. \tag{26}$$

In the same manner, we can show that

$$|D_{q,t}^{n,b,u}f(z)| \geqq r - \frac{r^2\Delta_{q,2}^n(\mathcal{A}-\mathcal{B})(1-\lambda)}{\Lambda_q^n(2,\lambda,\mathcal{A},\mathcal{B})}. \tag{27}$$

Putting (26) and (27) together gives (22) as asserted. □

Corollary 3. *Let $f \in \mathcal{S}_q^\star(0,b,t,u;\lambda;\mathcal{A},\mathcal{B})$, then for $r=|z|<1$,*

$$r - \frac{r^2(\mathcal{A}-\mathcal{B})(1-\lambda)}{\big(\Delta_{q,2}-1\big)+\Big|\Delta_{q,2}\mathcal{B}-[\mathcal{A}-\lambda(\mathcal{A}-\mathcal{B})]\Big|} \leqq |f(z)| \leqq r + \frac{r^2(\mathcal{A}-\mathcal{B})(1-\lambda)}{\big(\Delta_{q,2}-1\big)+\Big|\Delta_{q,2}\mathcal{B}-[\mathcal{A}-\lambda(\mathcal{A}-\mathcal{B})]\Big|}.$$

Equality occurs for function

$$f_2(z) = z + \frac{(\mathcal{A}-\mathcal{B})(1-\lambda)}{\big(\Delta_{q,2}-1\big)+\Big|\Delta_{q,2}\mathcal{B}-[\mathcal{A}-\lambda(\mathcal{A}-\mathcal{B})]\Big|}z^2.$$

Theorem 3 (Distortion Property)**.** *Let $f \in \mathcal{S}_q^\star(n,b,t,u;\lambda,\mathcal{A},\mathcal{B})$, then for $r=|z|<1$,*

$$1 - \frac{[2]_q r(\mathcal{A}-\mathcal{B})(1-\lambda)}{\Lambda_q^n(2,\lambda,\mathcal{A},\mathcal{B})} \leqq |\mathcal{D}_q(D_{q,t}^{n,b,u}f(z))| \leqq 1 + \frac{[2]_q r(\mathcal{A}-\mathcal{B})(1-\lambda)}{\Lambda_q^n(2,\lambda,\mathcal{A},\mathcal{B})}. \tag{28}$$

Equality occurs for the extremal function in (23).

Proof. Recall that for $f \in \mathfrak{A}$, $r^k < r = |z| < 1$, and by using (5) in (11); we have

$$\Big|\mathcal{D}_q(D_{q,t}^{n,b,u}f(z))\Big| = \Big|1+\sum_{k=2}^{\infty}\Delta_{q,k}^n[k]_q a_k z^{k-1}\Big| \leqq 1+\sum_{k=2}^{\infty}\Delta_{q,k}^n[k]_q|a_k|r^{k-1} \leqq 1 + r[2]_q\Delta_{q,2}^n\sum_{k=2}^{\infty}|a_k| \tag{29}$$

so that using (24) in (29) leads to

$$\Big|\mathcal{D}_q(D_{q,t}^{n,b,u}f(z))\Big| \leqq 1 + \frac{r[2]_q\Delta_{q,2}^n(\mathcal{A}-\mathcal{B})(1-\lambda)}{\Lambda_q^n(2,\lambda,\mathcal{A},\mathcal{B})}. \tag{30}$$

In the same manner, we can show that

$$\Big|\mathcal{D}_q(D_{q,t}^{n,b,u}f(z))\Big| \geqq 1 - \frac{r[2]_q\Delta_{q,2}^n(\mathcal{A}-\mathcal{B})(1-\lambda)}{\Lambda_q^n(2,\lambda,\mathcal{A},\mathcal{B})}. \tag{31}$$

Putting (30) and (31) together gives (28) as asserted. □

Corollary 4. *Let $f \in \lim_{q\uparrow 1} \mathcal{S}_q^\star(0, b, t, u; \lambda; \mathcal{A}, \mathcal{B})$, then for $r = |z| < 1$,*

$$1 - \frac{2r(\mathcal{A}-\mathcal{B})(1-\lambda)}{\big(\nabla - 1\big) + \Big|\nabla\mathcal{B} - [\mathcal{A} - \lambda(\mathcal{A}-\mathcal{B})]\Big|} \leqq |f'(z)| \leqq 1 + \frac{2r(\mathcal{A}-\mathcal{B})(1-\lambda)}{\big(\nabla - 1\big) + \Big|\nabla\mathcal{B} - [\mathcal{A} - \lambda(\mathcal{A}-\mathcal{B})]\Big|}.$$

where $\nabla = \lim_{q\uparrow 1} \Delta_{q,2}$. Equality occurs for function

$$f_2(z) = z + \frac{(\mathcal{A}-\mathcal{B})(1-\lambda)}{\big(\nabla - 1\big) + \Big|\nabla\mathcal{B} - [\mathcal{A} - \lambda(\mathcal{A}-\mathcal{B})]\Big|} z^2.$$

Theorem 4 (Covering Property)**.** *Let $f \in \mathcal{S}_q^\star(n, b, t, u; \lambda, \mathcal{A}, \mathcal{B})$, then the function $D_{q,t}^{n,b,u} f(z)$ in* (11) *maps the unit disk Ξ onto a domain that covers the disk*

$$|\varpi| < \frac{\Lambda_q^n(2, \lambda, \mathcal{A}, \mathcal{B}) - \Delta_{q,2}^n(\mathcal{A}-\mathcal{B})(1-\lambda)}{\Lambda_q^n(2, \lambda, \mathcal{A}, \mathcal{B})}$$

The result is sharp for the extremal function in (23)*.*

Proof. From (22),

$$|\varpi| = |D_{q,t}^{n,b,u} f(z)| < r - \frac{r^2 \Delta_{q,2}^n(\mathcal{A}-\mathcal{B})(1-\lambda)}{\Lambda_q^n(2, \lambda, \mathcal{A}, \mathcal{B})}$$

and observe that as $|z| = r \longrightarrow 1$,

$$|\varpi| < 1 - \frac{\Delta_{q,2}^n(\mathcal{A}-\mathcal{B})(1-\lambda)}{\Lambda_q^n(2, \lambda, \mathcal{A}, \mathcal{B})}$$

where some simplifications give the assertion. □

Remark 2. *Let $f \in \lim_{q\uparrow 1} \mathcal{S}^\star(0, b, t, u; 0; 1, -1) = \mathcal{S}^\star$, then*

$$|\varpi| < \frac{1}{2}.$$

This result agrees with that of Koebe's one-quarter theorem, see [39].

4.2. **Radii Problems.**

Theorem 5 (Radius of Starlikeness)**.** *Let $f \in \mathcal{S}_q^\star(n, b, t, u; \lambda, \mathcal{A}, \mathcal{B})$, then $f \in \mathcal{S}^\star(\varkappa)$ ($\varkappa \in [0, 1)$) in the disk*

$$|z| < \mathcal{R}_{\mathcal{S}^\star} := \inf_{k \in \aleph_2} \left\{ \frac{\Lambda_q^n(k, \lambda, \mathcal{A}, \mathcal{B})(1-\varkappa)}{(\mathcal{A}-\mathcal{B})(1-\lambda)(k-\varkappa)} \right\}^{\frac{1}{k-1}}.$$

The inequality is sharp for the function in (20)*.*

Proof. From the definition of starlikeness, it is sufficient to show that

$$\frac{\frac{zf'(z)}{f(z)} - \varkappa}{1 - \varkappa} \prec \frac{1+z}{1-z} \quad (\varkappa \in [0,1)). \tag{32}$$

Using (2) in (32) leads to

$$\frac{zf'(z) - \varkappa f(z)}{(1-\varkappa)f(z)} = \frac{1+\omega(z)}{1-\omega(z)}$$

so that

$$\left| \frac{zf'(z) - f(z)}{zf'(z) + (1-2\varkappa)f(z)} \right| = |\omega(z)| < 1$$

and using (1) leads to

$$\sum_{k=2}^{\infty} \frac{k-\varkappa}{1-\varkappa} |a_k| |z|^{k-1} < 1. \tag{33}$$

Note that inequalities (13) and (33) can only be valid if

$$\frac{k-\varkappa}{1-\varkappa} |z|^{k-1} < \frac{\Lambda_q^n(k,\lambda,\mathcal{A},\mathcal{B})}{(\mathcal{A}-\mathcal{B})(1-\lambda)}$$

where some simplifications affirm the result. □

Theorem 6 (Radius of Convexity). *Let* $f \in \mathcal{S}_q^{\star}(n,b,t,u;\lambda,\mathcal{A},\mathcal{B})$, *then* $f \in \mathcal{C}(\varkappa)$ $(\varkappa \in [0,1))$ *in the disk*

$$|z| < \mathcal{R}_{\mathcal{C}} := \inf_{k \in \aleph_2} \left\{ \frac{\Lambda_q^n(k,\lambda,\mathcal{A},\mathcal{B})(1-\varkappa)}{(\mathcal{A}-\mathcal{B})(1-\lambda)k(k-\varkappa)} \right\}^{\frac{1}{k-1}}.$$

This inequality is sharp for the function in (20).

Proof. From the definition of convexity, it is sufficient to show that

$$\frac{\frac{zf''(z)}{f'(z)} + 1 - \varkappa}{1 - \varkappa} \prec \frac{1+z}{1-z} \quad (\varkappa \in [0,1)). \tag{34}$$

Using (2) in (34) leads to

$$\frac{zf''(z) + (1-\varkappa)f'(z)}{(1-\varkappa)f'(z)} = \frac{1+\omega(z)}{1-\omega(z)}$$

so that

$$\left| \frac{zf''(z)}{zf''(z) + 2(1-\varkappa)f'(z)} \right| = |\omega(z)| < 1$$

and using (1) leads to

$$\sum_{k=2}^{\infty} \frac{k(k-\varkappa)}{1-\varkappa} |a_k| |z|^{k-1} < 1. \tag{35}$$

Note that the inequalities (13) and (35) can only be valid if

$$\frac{k(k-\varkappa)}{1-\varkappa}|z|^{k-1} < \frac{\Lambda_q^n(k,\lambda,\mathcal{A},\mathcal{B})}{(\mathcal{A}-\mathcal{B})(1-\lambda)}$$

where some simplifications affirm the result. □

Theorem 7 (Radius of Close-to-convexity). *Let $f \in \mathcal{S}_q^{\star}(n,b,t,u;\lambda,\mathcal{A},\mathcal{B})$, then $f \in \mathcal{K}(\varkappa)$ $(\varkappa \in [0,1))$ in the disk*

$$|z| < \mathcal{R}_{\mathcal{K}} := \inf_{k\in\aleph_2}\left\{\frac{\Lambda_q^n(k,\lambda,\mathcal{A},\mathcal{B})(1-\varkappa)}{(\mathcal{A}-\mathcal{B})(1-\lambda)k}\right\}^{\frac{1}{k-1}}.$$

The inequality is sharp for the function in (20).

Proof. From the definition of close-to-convexity, it is sufficient to show that

$$|f'(z)-1| < 1-\varkappa \quad (\varkappa \in [0,1)).$$

Using (1) leads to

$$|f'(z)-1| = \left|\left(1+\sum_{k=2}^{\infty} k a_k z^{k-1}\right)-1\right| \leqq \sum_{k=2}^{\infty} k|a_k||z|^{k-1} < 1-\varkappa,$$

that is,

$$\sum_{k=2}^{\infty}\frac{k}{1-\varkappa}|a_k||z|^{k-1} < 1. \tag{36}$$

Note that inequalities (13) and (36) can only be valid if

$$\frac{k}{1-\varkappa}|z|^{k-1} < \frac{\Lambda_q^n(k,\lambda,\mathcal{A},\mathcal{B})}{(\mathcal{A}-\mathcal{B})(1-\lambda)}$$

where some simplifications affirm the result. □

4.3. **Subordination Property.**

Theorem 8. *Let $f \in \mathcal{S}_q^{\star}(n,b,t,u;\lambda,\mathcal{A},\mathcal{B})$ and $c \in \mathcal{C}$, then*

$$\frac{\Lambda_q^n(2,\lambda,\mathcal{A},\mathcal{B})}{2\{(\mathcal{A}-\mathcal{B})(1-\lambda)+\Lambda_q^n(2,\lambda,\mathcal{A},\mathcal{B})\}}(f\star c)(z) \prec c(z) \tag{37}$$

and

$$\mathcal{R}e\, f > -\frac{(\mathcal{A}-\mathcal{B})(1-\lambda)+\Lambda_q^n(2,\lambda,\mathcal{A},\mathcal{B})}{\Lambda_q^n(2,\lambda,\mathcal{A},\mathcal{B})}. \tag{38}$$

The constant factor

$$\frac{\Lambda_q^n(2,\lambda,\mathcal{A},\mathcal{B})}{2\{(\mathcal{A}-\mathcal{B})(1-\lambda)+\Lambda_q^n(2,\lambda,\mathcal{A},\mathcal{B})\}} \tag{39}$$

in (37) *cannot be replaced by a larger value. The symbol $\star$ is called Hadamard product or convolution.*

The following proof adopts the technique of Srivastava and Attiya [37].

Proof. Let $f \in \mathcal{S}_q^\star(n, b, t, u; \lambda, \mathcal{A}, \mathcal{B})$ and suppose $c(z) = z + \sum_{k=2}^{\infty} c_k z^k \in \mathcal{C}$, then from (37),

$$\frac{\Lambda_q^n(2, \lambda, \mathcal{A}, \mathcal{B})}{2\{(\mathcal{A}-\mathcal{B})(1-\lambda) + \Lambda_q^n(2, \lambda, \mathcal{A}, \mathcal{B})\}}(f \star c)(z)$$
$$= \frac{\Lambda_q^n(2, \lambda, \mathcal{A}, \mathcal{B})}{2\{(\mathcal{A}-\mathcal{B})(1-\lambda) + \Lambda_q^n(2, \lambda, \mathcal{A}, \mathcal{B})\}}\left(z + \sum_{k=2}^{\infty} a_k c_k z^k\right)$$
$$= \sum_{k=1}^{\infty} \frac{\Lambda_q^n(2, \lambda, \mathcal{A}, \mathcal{B})}{2\{(\mathcal{A}-\mathcal{B})(1-\lambda) + \Lambda_q^n(2, \lambda, \mathcal{A}, \mathcal{B})\}} a_k c_k z^k$$

and clearly by Definition 3, the subordination result in (37) holds if

$$\left\{\frac{\Lambda_q^n(2, \lambda, \mathcal{A}, \mathcal{B})}{2\{(\mathcal{A}-\mathcal{B})(1-\lambda) + \Lambda_q^n(2, \lambda, \mathcal{A}, \mathcal{B})\}} a_k\right\}_{k=1}^{\infty}$$

is a *subordinating factor sequence* where $a_1 = 1$. Now applying Lemma 1 gives an equivalence inequality

$$\mathcal{R}e\left(1 + \sum_{k=1}^{\infty} \frac{\Lambda_q^n(2, \lambda, \mathcal{A}, \mathcal{B})}{(\mathcal{A}-\mathcal{B})(1-\lambda) + \Lambda_q^n(2, \lambda, \mathcal{A}, \mathcal{B})} a_k z^k\right) > 0. \tag{40}$$

Observe that $\Lambda_q^n(k, \lambda, \mathcal{A}, \mathcal{B})$ is an increasing function $\forall k \in \aleph_2$, so

$$\Lambda_q^n(2, \lambda, \mathcal{A}, \mathcal{B}) \leqq \Lambda_q^n(k, \lambda, \mathcal{A}, \mathcal{B}), \quad \forall k \in \aleph_2.$$

Hence, it follows by using $|z| = r < 1$, triangle inequality and inequality (13) that

$$\mathcal{R}e\left(1 + \sum_{k=1}^{\infty} \frac{\Lambda_q^n(2, \lambda, \mathcal{A}, \mathcal{B})}{(\mathcal{A}-\mathcal{B})(1-\lambda) + \Lambda_q^n(2, \lambda, \mathcal{A}, \mathcal{B})} a_k z^k\right)$$
$$= \mathcal{R}e\left(1 + \frac{\Lambda_q^n(2, \lambda, \mathcal{A}, \mathcal{B})}{(\mathcal{A}-\mathcal{B})(1-\lambda) + \Lambda_q^n(2, \lambda, \mathcal{A}, \mathcal{B})} \sum_{k=1}^{\infty} a_k z^k\right)$$
$$= \mathcal{R}e\left(1 + \frac{\Lambda_q^n(2, \lambda, \mathcal{A}, \mathcal{B})}{(\mathcal{A}-\mathcal{B})(1-\lambda) + \Lambda_q^n(2, \lambda, \mathcal{A}, \mathcal{B})} z + \frac{\sum_{k=2}^{\infty} \Lambda_q^n(2, \lambda, \mathcal{A}, \mathcal{B}) a_k z^k}{(\mathcal{A}-\mathcal{B})(1-\lambda) + \Lambda_q^n(2, \lambda, \mathcal{A}, \mathcal{B})}\right)$$
$$\geqq 1 - \frac{\Lambda_q^n(2, \lambda, \mathcal{A}, \mathcal{B})}{(\mathcal{A}-\mathcal{B})(1-\lambda) + \Lambda_q^n(2, \lambda, \mathcal{A}, \mathcal{B})} r - \frac{\sum_{k=2}^{\infty} \Lambda_q^n(k, \lambda, \mathcal{A}, \mathcal{B}) |a_k|}{(\mathcal{A}-\mathcal{B})(1-\lambda) + \Lambda_q^n(2, \lambda, \mathcal{A}, \mathcal{B})} r^k$$
$$> 1 - \frac{\Lambda_q^n(2, \lambda, \mathcal{A}, \mathcal{B})}{(\mathcal{A}-\mathcal{B})(1-\lambda) + \Lambda_q^n(2, \lambda, \mathcal{A}, \mathcal{B})} r - \frac{(\mathcal{A}-\mathcal{B})(1-\lambda)}{(\mathcal{A}-\mathcal{B})(1-\lambda) + \Lambda_q^n(2, \lambda, \mathcal{A}, \mathcal{B})} r$$
$$= 1 - r > 0.$$

This evidently proves inequality (40) and as well as the subordination result (37). Also, the inequality (38) follows from (37) by taking the convex function

$$c_0(z) = \frac{z}{1-z} = z + \sum_{k=2}^{\infty} z^k \in \mathcal{C}.$$

To prove the sharpness of the constant (39), consider (see (20)) the function

$$f_2(z) = z + \frac{(\mathcal{A}-\mathcal{B})(1-\lambda)}{\Lambda_q^n(2,\lambda,\mathcal{A},\mathcal{B})} z^2 \in \mathcal{S}_q^\star(n,b,t,u;\lambda,\mathcal{A},\mathcal{B})$$

so that using (37) leads to

$$\frac{\Lambda_q^n(2,\lambda,\mathcal{A},\mathcal{B})}{2\{(\mathcal{A}-\mathcal{B})(1-\lambda)+\Lambda_q^n(2,\lambda,\mathcal{A},\mathcal{B})\}} f_2(z) \prec c_0(z) = \frac{z}{1-z}. \tag{41}$$

It can *easily* be verified that for $f_2(z)$,

$$\min_{|z|\leqq r} \left\{ \mathcal{R}e \left(\frac{\Lambda_q^n(2,\lambda,\mathcal{A},\mathcal{B})}{2\{(\mathcal{A}-\mathcal{B})(1-\lambda)+\Lambda_q^n(2,\lambda,\mathcal{A},\mathcal{B})\}} f_2(z) \right) \right\} = -\frac{1}{2} \quad (z \in \Xi)$$

which shows that the constant $\frac{\Lambda_q^n(2,\lambda,\mathcal{A},\mathcal{B})}{2\{(\mathcal{A}-\mathcal{B})(1-\lambda)+\Lambda_q^n(2,\lambda,\mathcal{A},\mathcal{B})\}}$ cannot be replaced by any larger value. □

5. Conclusions

The attention geared towards the study of q-operators by scientists and in particular, by geometric function theorists in recent years is overwhelming. In this study, a new q-differential operator that generalized the famous Sălăgean [33], Al-Oboudi [3] and Opoola differential [26] operators was studied. Subsequently, the q-differential operator and the principle of subordination were used to define a subclass of analytic-univalent functions. This new class was represented by $\mathcal{S}_q^\star(n,b,t,u;\lambda,\mathcal{A},\mathcal{B})$. Further, the geometric properties such as the coefficient inequality, growth, distortion and covering theorems were established for the class $\mathcal{S}_q^\star(n,b,t,u;\lambda,\mathcal{A},\mathcal{B})$. Also, the radii of starlikeness, convexity and close-to-convexity; as well as the subordinating factor sequence problems were solved for the new class. Intermittently, some key corollaries and remarks were given to demonstrate the relationship between this new class (and the new results); and some exiting classes (and their results).

Author Contribution Statements The authors contributed equally to this work. Also, the authors read and approved the final copy of this paper.

Declaration of Competing Interests The authors declare that they have no known competing financial interest or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements We unreservedly appreciate the combined efforts of the reviewers and the editors which added quality values to this work.

References

[1] Alexander, J. W., Functions which map the interior of the unit circle upon simple regions, *Ann. Math. Sec. Ser.,* 17(1) (1915), 12–22. https://doi.org/10.2307/2007212

[2] Ali-Shah, S. G., Khan, S. Hussain, S., Darus, M., q-Noor integral operator associated with starlike functions and q-conic domains, *AIMS Math.*, 7(6) (2022), 10842-10859. https://doi.org/10.3934/math.2022606

[3] Al-Oboudi, F. M., On univalent functions defined by a generalised Sălăgean operator, *Internat. J. Math. Math. Sci.,* 2004(27) (2004), 1429–1436. https://doi.org/10.1155/S0161171204108090

[4] Al-Shbeil, I., Shaba, T. G., Cătaş, A., Second Hankel determinant for the subclass of bi-univalent functions using q-Chebyshev polynomial and Hohlov operator, *Fractal Fract.,* 6(186) (2022), 19 pages. https://doi.org/10.3390/ fractalfract6040186

[5] Annaby, M. H., Mansour, Z. S., q-Fractional Calculus and Equations, Springer Science+Business Media, New York, 2012. https://doi.org/10.1007/978-3-642-30898-7

[6] Aral, A., Gupta, V., Agarwal, R. P., Applications of q-Calculus in Operator Theory, Springer Science+Business Media, New York, 2013. https://doi.org/10.1007/978-1-4614-6946-9_1

[7] Arif, M., Barukab, O. M., Khan, A. S., Abbas, M., The sharp bounds of Hankel determinants for the families of three-leaf-type analytic functions, *Fractal Fract.* 6(291) (2022), 35 pages. https://doi.org/10.3390/fractalfract6060291

[8] Cho, N. E., Kumar, V., Ravichandran, V., A survey on coefficient estimates for Carathéodory functions, *Appl. Math. E-Notes,* 19 (2019), 370–396. https://www.emis.de/journals/AMEN/2019/AMEN-180518.pdf

[9] Friedland, S., Schiffer, M., Global results in control theory with applications to univalent functions, *Bull. Amer. Math. Soc.,* 82(6) (1976), 913–915. https://doi.org/10.1090/S0002-9904-1976-14211-5

[10] Goodman, A. W., Univalent functions and nonanalytic curves, *Proc. Amer. Math. Soc.,* 8 (1957), 598–601. https://doi.org/10.1090/S0002-9939-1957-0086879-9

[11] Govindaraj, M., Sivasubramanian, S., On a class of analytic functions related to conic domains involving q-calculus, *Anal. Math.,* 43(3) (2017), 475–487. https://doi.org/10.1007/s10476-017-0206-5

[12] Hu, Q., Srivastava, H. M., Ahmad, B., Khan, N., Khan, M. G., Mashwani, W. K., Khan, B., A subclass of multivalent Janowski type q-starlike functions and its consequences, *Symmetry,* 13(1275) (2021), 14 pages. https://doi.org/10.3390/ sym13071275

[13] Jackson, F. H., On q-functions and a certain difference operator, *Trans. Roy. Soc. Edinb.,* 46(2) (1908), 253–281. https://doi.org/10.1017/S0080456800002751

[14] Jackson, F. H., On q-difference equation, *Amer. J. Math.,* 32(4) (1910), 305–314. https://doi.org/10.2307/2370183

[15] James, A. A., Lasode, A. O., Moses, B. O., Geometric conditions for starlikeness and convexity of univalent functions, *IOSR J. Math.,* 3 (2012), 15–23. https://doi.org/10.9790/5728-0361523

[16] Janowski, W., Some extremal problems for certain families of analytic functions I, *Ann. Polon. Math.,* 28(3) (1973), 297–326. https://doi.org/10.4064/ap-28-3-297-326

[17] Kac, V., Cheung, P., Quantum Calculus, Springer Science+Business Media, New York, 2002. https://doi.org/10.1007/978-1-4613-0071-7

[18] Khan, B., Liu, Z. -G., Shaba, T. G., Araci, S., Khan, N., Khan, M. G., Applications of q-derivative operator to the subclass of bi-univalent functions involving q-Chebyshev polynomials, *J. Math.,* 2022 (2022), Artice ID 8162182, 7 pages. https://doi.org/10.1155/2022/8162182

[19] Khan, B., Liu, Z. -G., Srivastava, H. M., Araci, S., Khan, N., Ahmad, Q. Z., Higher-order q-derivatives and their applications to subclasses of multivalent Janowski type q-starlike functions, *Adv. Diff. Equ.,* 2021(440) (2021), 15 pages. https://doi.org/10.1186/s13662-021-03611-6

[20] Khan, B., Srivastava, H. M., Arjika, S., Khan, S., Khan, N., Ahmad, Q. Z., A certain q-Ruscheweyh type derivative operator and its applications involving multivalent functions, *Adv. Differ. Equ.,* 2021(279) (2021), https://doi.org/10.1186/s13662-021-03441-6

[21] Lasode, A. O., Opoola, T. O., On a generalized class of bi-univalent functions defined by subordination and q-derivative operator, *Open J. Math. Anal.,* 5(2) (2021), 46–52. https://doi.org/10.30538/psrp-oma2021.0092

[22] Lasode, A. O., Opoola, T. O., Fekete-Szegö estimates and second Hankel determinant for a generalized subfamily of analytic functions defined by q-differential operator, *Gulf J. Math.,* 11(2) (2021), 36–43. https://doi.org/10.56947/gjom.v11i2.583

[23] Lasode, A. O., Opoola, T. O., Some investigations on a class of analytic and univalent functions involving q-differentiation, *Eur. J. Math. Anal.,* 2(12) (2022), 1–9. https://doi.org/10.28924/ada/ma.2.12

[24] Lasode, A. O., Opoola, T. O., Coefficient problems of a class of q-starlike functions associated with q-analogue of Al-Oboudi-Al-Qahtani integral operator and nephroid domain, *J. Class. Anal.,* 20(1) (2022), 35–47. https://doi.org/10.7153/jca-2022-20-04

[25] Lasode, A. O., Opoola, T. O., Al-Shbeil, I., Shaba, T. G., Alsaud, H., Concerning a novel integral operator and a specific category of starlike functions, *Mathematics* 11(4519) (2023), 17 pages. https://doi.org/10.3390/math11214519

[26] Opoola, T. O., On a subclass of univalent functions defined by a generalised differential operator, *Internat. J. Math. Anal.,* 11(18) (2017), 869–876. https://doi.org/10.12988/ijma.2017.7232

[27] Oyekan, E. A., Lasode, A. O., Estimates for some classes of analytic functions associated with Pascal distribution series, error function, Bell numbers and q-differential operator, *Nigerian J. Math. Appl.,* 32 (2022), 163–173. http://www.njmaman.com/articles/2022/PAPER14.pdf

[28] Oyekan, E. A., Olatunji, T. A., Lasode, A. O., Applications of (p, q)-Gegenbauer polynomials on a family of bi-univalent functions, *Earthline J. Math. Sci.*, 12(2) (2023), 271–284. https://doi.org/10.34198/ejms.12223.271284

[29] Oyekan, E. A., Swamy, S. R., Adepoju, P. O. , Olatunji, T. A., Quasi-convolution properties of a new family of close-to-convex functions involving q-p-Opoola differential operator, *Intern. J. Math. Trends Technol.,* 69(5) (2023), 70-77. https://doi.org/10.14445/22315373/IJMTT-V69I5P506

[30] Polatoğlu, Y., Bolcal, S. M., Şen, A., Yavuz, E., A study on the generalization of Janowski functions in the unit disc, *Acta Math. Acad. Paedagog. Nyházi.,* 22 (2006), 27–31.

[31] Raza, M., Srivastava, H. M., Arif, M., Ahmad, K., Coefficient estimates for a certain family of analytic functions involving a q-derivative operator, *Ramanujan J.*, 55 (2021), 53–71. https://doi.org/10.1007/s11139-020-00338-y

[32] Rensaa, R. J., Univalent functions and frequency analysis, *Rocky J. Math.,* 33(2) (2003), 743–758. https://doi.org/10.1216/rmjm/1181069976

[33] Sălăgean, G. S., Subclasses of univalent functions, *Lect. Notes Math.,* 1013 (1983), 362–372. https://doi.org/10.1007/BFb0066543

[34] Silverman, H., Univalent functions with negative coefficients, *Proc. Amer. Math. Soc.,* 51(1) (1975), 109–116. https://doi.org/10.1090/S0002-9939-1975-0369678-0

[35] Shaba, T. G., Wanas, A. K., Coefficient bounds for a certain families of m-fold symmetric bi-univalent functions associated with q-analogue of Wanas operator, *Acta Univ. Apulensis Math. Inform.*, 68 (2021), 25–37. https://doi.org/10.17114/j.aua.2021.68.03

[36] Srivastava, H. M., Operators of basic (or q-)calculus and fractional q-calculus and their applications in geometric function theory of complex analysis, *Iran. J. Sci. Technol. Trans. Sci.,* Review Paper, (2020), 18 pages. https://doi.org/10.1007/s40995-019-00815-0

[37] Srivastava, H. M., Attiya, A. A., Some subordination results associated with certain subclasses of analytic functions, *J. Inequal. Pure Appl. Math.,* 5(4) (2004), 1–14. https://www.emis.de/journals/JIPAM/images/113_04_JIPAM/113_04.pdf

[38] Srivastava, H. M., Khan, B., Khan, N., Hussain, A., Khan, N., Tahir, M., Applications of certian basic (or q-) derivatives to subclasses of multlivalemt Janowski type q-starlike functions involving conic domain, *J. Nonlinear Var. Anal.,* 5(4) (2021), 531–547. https://doi.org/10.23952/jnva.5.2021.4.03

[39] Thomas, D. K., Tuneski, N., Vasudevarao, A., Univalent Functions: A Primer, Walter de Gruyter Inc., Berlin, 2018. https://doi.org/10.1515/9783110560961-001

[40] Ul-Haq, M., Raza, M., Arif, M., Khan, Q., Tang, H., q-Analogue of differential subordinations, *Math.* 7(8) (2019), pp. 16. https://doi.org/10.3390/math7080724

[41] Vasil'ev, A., Univalent functions in the dynamics of viscous flow, *Comput. Method Funct. Theory,* 1(2) (2001), 311–337. https://doi.org/10.1007/BF03320993

[42] Wanas, A. K., Mahdi, A. M., Applications of the q-Wanas operator for a certain family of bi-univalent functions defined by subordination, *Asian-Eur. J. Math.*, 16(6) (2023), 1–16. https://doi.org/ 10.1142/S179355712350095X

[43] Wilf, H. S., Subordinating factor sequences for convex maps of the unit circle, *Proc. Amer. Math. Soc.,* 12 (1961), 689–693. https://doi.org/10.1090/S0002-9939-1961-0125214-5