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Research Article

Existence results for positive periodic solutions to first order neutral differential equations with distributed deviating arguments

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Abstract

We take into account the first order nonlinear neutral differential equation with distributed deviating arguments. Using Krasnoselskii's fixed point theorem, we give some new criteria for the existence of positive periodic solutions to this equation. The theorems we have established are illustrated by an example.

Mathematics Subject Classification (2020). 34C25, 34K13

Keywords. fixed point, distributed delay, neutral equations, positive periodic solution, first-order

1. Introduction

In the current study, we investigate the existence of positive ω-periodic solutions to the first-order neutral differential equation with distributed deviating arguments of the form

$$x'(t) = -p(t)x(t) + \int_a^b \Big[c(t,\mu)x'(t-g(t,\mu)) + q(t,x(t-g(t,\mu)))\Big]d\mu, \tag{1.1}$$

where $a > b \geqslant 0$, $p \in C(\mathbb{R},(0,\infty))$ is a ω-periodic function, first partial derivative of c with respect to t is continuous on $(\mathbb{R}\times[a,b],\mathbb{R})$ and ω-periodic in t, second partial derivative of g with respect to t is continuous on $(\mathbb{R}\times[a,b],\mathbb{R})$, ω-periodic in t, $g(t,\mu) > 0$ and $g_t(t,\mu) \neq 1$ for all $t \in [0,\omega]$ and $\mu \in [a,b]$, $q \in C(\mathbb{R}\times\mathbb{R},\mathbb{R})$ is a ω-periodic function in t, where ω is a positive constant.

Neutral differential equations, which are present in models of population, control, and blood cell production, have recently attracted a lot of attention, see [5,7,8]. In particular, investigation of the positive periodic solution to the following equation

$$x'(t) = -a(t)x(t) + c(t)x'(t-g(t)) + q(t,x(t-g(t))),$$

with $0 \leqslant \frac{c(t)}{1-g'(t)} < 1$, $-1 \leqslant \frac{c(t)}{1-g'(t)} \leqslant 0$ in [13] served as our inspiration for this article.

In this study, we generalize the findings from [13] for the equation (1.1), and we present new criteria for the existence of positive periodic solutions to the equation (1.1). Furthermore, the papers [2,3,6,9–12,14] explore the existence of positive periodic solutions in various types of first-order neutral differential equations. Additionally, the paper [4] delves into the study of positive periodic solutions to second-order neutral differential equations.

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Lastly, the paper [1] investigates the oscillation behavior of solutions in even-order neutral functional differential equations with distributed deviating arguments.

2. Main results

Let Φ_ω be a ω-periodic continuous function space with the supremum norm $\|x\| = \sup_{t\in[0,\omega]} |x(t)|$. Then, it is clear that $(\Phi_\omega, \|\cdot\|)$ is a Banach space.

Theorem 2.1. *Suppose that* $0 \leqslant c_0 \leqslant \frac{(b-a)c(t,\mu)}{1-g_t(t,\mu)} \leqslant c_1 < 1$. *Moreover, assume that there exist positive constants* m_0 *and* m_1 *with* $m_0 < m_1$ *such that*

$$\frac{(1-c_0)m_0}{b-a} \leqslant \frac{q(t,x)-r(t,\mu)x}{p(t)} \leqslant \frac{(1-c_1)m_1}{b-a}, \tag{2.1}$$

where

$$r(t,\mu) = \frac{\Big(c_t(t,\mu)+c(t,\mu)p(t)\Big)\Big(1-g_t(t,\mu)\Big)+g_{tt}(t,\mu)c(t,\mu)}{\Big(1-g_t(t,\mu)\Big)^2}, \quad t \in [0,\omega] \quad \text{and} \quad \mu \in [a,b].$$

Then, (1.1) has at least one positive ω*-periodic solution* $x(t) \in [m_0, m_1]$.

Proof. It is important to note that obtaining an ω-periodic solution of (1.1) is equivalent to doing the same for the following integral equation

$$\begin{aligned} x(t) &= \int_a^b \frac{c(t,\mu)}{1-g_t(t,\mu)} x(t-g(t,\mu))d\mu \\ &+ \int_t^{t+\omega}\int_a^b G(t,s)\Big[-r(s,\mu)x(s-g(s,\mu))+q(s,x(s-g(s,\mu)))\Big]d\mu ds, \end{aligned}$$

where

$$G(t,s) = \frac{e^{\int_t^s p(u)du}}{e^{\int_0^\omega p(u)du}-1}.$$

Let $\Phi = \{x \in \Phi_\omega : m_0 \leqslant x(t) \leqslant m_1,\ t \in [0,\omega]\}$, which is a bounded closed and convex subset of Φ_ω. Define the operators $\Gamma_1, \Gamma_2 : \Phi \to \Phi_\omega$ as follows:

$$(\Gamma_1 x)(t) = \int_a^b \frac{c(t,\mu)}{1-g_t(t,\mu)} x(t-g(t,\mu))d\mu \tag{2.2}$$

and

$$(\Gamma_2 x)(t) = \int_t^{t+\omega}\int_a^b G(t,s)\Big[-r(s,\mu)x(s-g(s,\mu))+q(s,x(s-g(s,\mu)))\Big]d\mu ds. \tag{2.3}$$

For any $x \in \Phi$ and $t \in \mathbb{R}$, from (2.2) and (2.3) it can be deduced that

$$\begin{aligned} (\Gamma_1 x)(t+\omega) &= \int_a^b \frac{c(t+\omega,\mu)}{1-g_t(t+\omega,\mu)} x(t+\omega-g(t+\omega,\mu))d\mu \\ &= \int_a^b \frac{c(t,\mu)}{1-g_t(t,\mu)} x(t-g(t,\mu))d\mu = (\Gamma_1 x)(t) \end{aligned}$$

and

$$\begin{aligned}
(\Gamma_2 x)(t+\omega) &= \int_{t+\omega}^{t+2\omega}\int_a^b G(t+\omega,s)\Big[-r(s,\mu)x(s-g(s,\mu))+q(s,x(s-g(s,\mu)))\Big]d\mu ds\\
&= \int_t^{t+\omega}\int_a^b G(t+\omega,v+\omega)\Big[-r(v+\omega,\mu)x(v+\omega-g(v+\omega,\mu))\\
&+ q(v+\omega,x(v+\omega-g(v+\omega,\mu)))\Big]d\mu dv\\
&= \int_t^{t+\omega}\int_a^b G(t,v)\Big[-r(v,\mu)x(v-g(v,\mu))+q(v,x(v-g(v,\mu)))\Big]d\mu dv\\
&= (\Gamma_2 x)(t),
\end{aligned}$$

which implies that $\Gamma_1(\Phi)\subset\Phi_\omega$ and $\Gamma_2(\Phi)\subset\Phi_\omega$. Now, we show that $\Gamma_1 y+\Gamma_2 x\in\Phi$ for all $x,y\in\Phi$ and $t\in\mathbb{R}$. By using (2.1), (2.2) and (2.3), we have

$$\begin{aligned}
(\Gamma_1 y)(t)+(\Gamma_2 x)(t) &= \int_a^b \frac{c(t,\mu)}{1-g_t(t,\mu)}y(t-g(t,\mu))d\mu\\
&+ \int_t^{t+\omega}\int_a^b G(t,s)\Big[-r(s,\mu)x(s-g(s,\mu))+q(s,x(s-g(s,\mu)))\Big]d\mu ds\\
&\leqslant c_1 m_1\\
&+ \int_t^{t+\omega} G(t,s)p(s)\int_a^b\Big[\frac{q(s,x(s-g(s,\mu)))-r(s,\mu)x(s-g(s,\mu))}{p(s)}\Big]d\mu ds\\
&\leqslant c_1 m_1+(1-c_1)m_1\int_t^{t+\omega}G(t,s)p(s)ds\\
&= m_1
\end{aligned}$$

and

$$\begin{aligned}
(\Gamma_1 y)(t)+(\Gamma_2 x)(t) &= \int_a^b \frac{c(t,\mu)}{1-g_t(t,\mu)}y(t-g(t,\mu))d\mu\\
&+ \int_t^{t+\omega}\int_a^b G(t,s)\Big[-r(s,\mu)x(s-g(s,\mu))+q(s,x(s-g(s,\mu)))\Big]d\mu ds\\
&\geqslant c_0 m_0\\
&+ \int_t^{t+\omega} G(t,s)p(s)\int_a^b\Big[\frac{q(s,x(s-g(s,\mu)))-r(s,\mu)x(s-g(s,\mu))}{p(s)}\Big]d\mu ds\\
&\geqslant c_0 m_0+(1-c_0)m_0\int_t^{t+\omega}G(t,s)p(s)ds\\
&= m_0.
\end{aligned}$$

That implies that $\Gamma_1 x+\Gamma_2 y\in\Phi$, for all $x,y\in\Phi$. We shall show that Γ_1 is a contraction mapping on Φ. For $x,y\in\Phi$, we have

$$\begin{aligned}
|(\Gamma_1 x)(t)-(\Gamma_1 y)(t))| &= \Big|\int_a^b \frac{c(t,\mu)}{1-g_t(t,\mu)}x(t-g(t,\mu))d\mu-\int_a^b \frac{c(t,\mu)}{1-g_t(t,\mu)}y(t-g(t,\mu))d\mu\Big|\\
&\leqslant \int_a^b \frac{c(t,\mu)}{1-g_t(t,\mu)}\Big|x(t-g(t,\mu))-y(t-g(t,\mu))\Big|d\mu.
\end{aligned}$$

By taking the sup norm on both sides it follows that

$$\|\Gamma_1 x-\Gamma_1 y\|\leqslant c_1\|x-y\|$$

and therefore Γ_1 is a contraction mapping.

We shall demonstrate that Γ_2 is continuous. Consider $\{x_n\}\in\Phi$ be a convergent sequence of elements such that $x_n(t)\to x(t)$ as $n\to\infty$. Since Φ is closed, $x\in\Phi$.

For $t \in [0,\omega]$, we have

$$\begin{aligned} |(\Gamma_2 x_n)(t) - (\Gamma_2 x)(t)| &= \Big| \int_t^{t+\omega} \int_a^b G(t,s)\Big[-r(s,\mu)x_n(s-g(s,\mu)) + q(s, x_n(s-g(s,\mu)))\Big] d\mu ds \\ &- \int_t^{t+\omega} \int_a^b G(t,s)\Big[-r(s,\mu)x(s-g(s,\mu)) + q(s, x(s-g(s,\mu)))\Big] d\mu ds \Big| \\ &\leqslant \int_t^{t+\omega} \int_a^b G(t,s)|r(s,\mu)|\Big|x_n(s-g(s,\mu)) - x(s-g(s,\mu))\Big| d\mu ds \\ &+ \int_t^{t+\omega} \int_a^b G(t,s)\Big|q(s,x_n(s-g(s,\mu))) - q(s,x(s-g(s,\mu)))\Big| d\mu ds. \end{aligned}$$

The Lebesgue dominated convergence theorem yields that

$$\lim_{k\to\infty} \|(\Gamma_2 x_n) - (\Gamma_2 x)\| = 0$$

because $|x_n(t) - x(t)| \to 0$ and $|q(t,x_n(t)) - q(t,x(t))| \to 0$ as $n \to \infty$. That implies that Γ_2 is continuous. Now, we show that $\Gamma_2(\Phi)$ is relatively compact, it is sufficient to demonstrate that the family of functions $\{\Gamma_2 x : x \in \Phi\}$ is uniformly bounded and equicontinuous on $[0,\omega]$. We see from (2.1) that

$$\begin{aligned} |(\Gamma_2 x)(t)| &= \Big| \int_t^{t+\omega} \int_a^b G(t,s)\Big[-r(s,\mu)x(s-g(s,\mu)) + q(s,x(s-g(s,\mu)))\Big] d\mu ds \Big| \\ &\leqslant \Big| \int_t^{t+\omega} G(t,s)p(s) \int_a^b \Big[\frac{q(s,x(s-g(s,\mu))) - r(s,\mu)x(s-g(s,\mu))}{p(s)}\Big] d\mu ds \Big| \\ &\leqslant (1-c_1)m_1 \int_t^{t+\omega} G(t,s)p(s) ds = (1-c_1)m_1 \end{aligned}$$

and it follows that

$$\|\Gamma_2 x\| \leqslant (1-c_1)m_1.$$

For uniformly boundedness, using (2.1), we obtain

$$\begin{aligned} |(\Gamma_2 x)'(t)| &= \Big| \frac{d}{dt}\Big[\int_t^{t+\omega} \int_a^b G(t,s)\Big[-r(s,\mu)x(s-g(s,\mu)) + q(s,x(s-g(s,\mu)))\Big] d\mu ds \Big| \\ &\leqslant \Big| G(t,t+\omega) \int_a^b \Big[-r(t,\mu)x(t-g(t,\mu)) + q(t,x(t-g(t,\mu)))\Big] d\mu \\ &- G(t,t) \int_a^b \Big[-r(t,\mu)x(t-g(t,\mu)) + q(t,x(t-g(t,\mu)))\Big] d\mu - p(t)(\Gamma_2 x)(t) \Big| \\ &\leqslant \Big| p(t) \int_a^b \Big[\frac{q(t,x(t-g(t,\mu))) - r(t,\mu)x(t-g(t,\mu))}{p(t)}\Big] d\mu - p(t)(\Gamma_2 x)(t) \Big| \\ &\leqslant 2(1-c_1)m_1\|p\| \end{aligned}$$

which show that $\{\Gamma_2 x : x \in \Phi\}$ is uniformly bounded and equicontinuous on $[0,\omega]$. As a result, $\Gamma_2(\Phi)$ is relatively compact. The aforementioned findings lead us to the conclusion that Γ_2 is completely continuous. According to fixed point theorem of Krasnoselskii, there is an $x \in \Phi$ such that $\Gamma_1 x + \Gamma_2 x = x$. This implies that $x(t)$ is a positive ω-periodic solution of (1.1). $\square$

Theorem 2.2. *Suppose that* $-1 < c_0 \leqslant \frac{(b-a)c(t,\mu)}{1-g_t(t,\mu)} \leqslant c_1 \leqslant 0$. *Moreover, assume that there exist positive constants* m_0 *and* m_1 *with* $m_0 < m_1$ *such that*

$$\frac{m_0 - c_0 m_1}{b-a} \leqslant \frac{q(t,x) - r(t,\mu)x}{p(t)} \leqslant \frac{m_1 - c_1 m_0}{b-a}, \tag{2.4}$$

where

$$r(t,\mu) = \frac{\Big(c_t(t,\mu) + c(t,\mu)p(t)\Big)\Big(1 - g_t(t,\mu)\Big) + g_{tt}(t,\mu)c(t,\mu)}{\Big(1 - g_t(t,\mu)\Big)^2}, \quad t \in [0,\omega] \quad \text{and} \quad \mu \in [a,b].$$

Then, (1.1) has at least one positive ω*-periodic solution* $x(t) \in [m_0, m_1]$.

Proof. We define Φ, $G(t,s)$, Γ_1 and Γ_2 as in the proof of Theorem 2.1. It can be seen from the proof of Theorem 2.1 that $\Gamma_1(\Phi) \subset \Phi_\omega$ and $\Gamma_2(\Phi) \subset \Phi_\omega$. Now, we show that $\Gamma_1 y + \Gamma_2 x \in \Phi$ for all $x, y \in \Phi$ and $t \in \mathbb{R}$. By using (2.2), (2.3) and (2.4), we have

$$\begin{aligned}
(\Gamma_1 y)(t) + (\Gamma_2 x)(t) &= \int_a^b \frac{c(t,\mu)}{1-g_t(t,\mu)} y(t - g(t,\mu))d\mu \\
&+ \int_t^{t+\omega} \int_a^b G(t,s)\Big[- r(s,\mu)x(s - g(s,\mu)) + q(s, x(s - g(s,\mu)))\Big] d\mu ds \\
&\leqslant c_1 m_0 \\
&+ \int_t^{t+\omega} G(t,s)p(s) \int_a^b \Big[\frac{q(s,x(s-g(s,\mu))) - r(s,\mu)x(s-g(s,\mu))}{p(s)}\Big] d\mu ds \\
&\leqslant c_1 m_0 + (m_1 - c_1 m_0) \int_t^{t+\omega} G(t,s)p(s)ds \\
&= m_1
\end{aligned}$$

and

$$\begin{aligned}
(\Gamma_1 y)(t) + (\Gamma_2 x)(t) &= \int_a^b \frac{c(t,\mu)}{1-g_t(t,\mu)} y(t - g(t,\mu))d\mu \\
&+ \int_t^{t+\omega} \int_a^b G(t,s)\Big[- r(s,\mu)x(s - g(s,\mu)) + q(s, x(s - g(s,\mu)))\Big] d\mu ds \\
&\geqslant c_0 m_1 \\
&+ \int_t^{t+\omega} G(t,s)p(s) \int_a^b \Big[\frac{q(s,x(s-g(s,\mu))) - r(s,\mu)x(s-g(s,\mu))}{p(s)}\Big] d\mu ds \\
&\geqslant c_0 m_1 + (m_0 - c_0 m_1) \int_t^{t+\omega} G(t,s)p(s)ds \\
&= m_0.
\end{aligned}$$

That implies that $\Gamma_1 x + \Gamma_2 y \in \Phi$, for all $x, y \in \Phi$. Now, we shall show that Γ_1 is a contraction mapping on Φ. For $x, y \in \Phi$, we have

$$\begin{aligned}
|(\Gamma_1 x)(t) - (\Gamma_1 y)(t))| &= \Big| \int_a^b \frac{c(t,\mu)}{1-g_t(t,\mu)} x(t-g(t,\mu))d\mu - \int_a^b \frac{c(t,\mu)}{1-g_t(t,\mu)} y(t-g(t,\mu))d\mu \Big| \\
&\leqslant \int_a^b \frac{-c(t,\mu)}{1-g_t(t,\mu)} \Big| x(t-g(t,\mu)) - y(t-g(t,\mu)) \Big| d\mu.
\end{aligned}$$

By taking the sup norm on both sides it follows that

$$\|\Gamma_1 x - \Gamma_1 y\| \leqslant -c_0 \|x - y\|$$

and therefore Γ_1 is a contraction mapping. Since the rest of the proof is similar to that of Theorem 2.1, it will not be given to avoid repetition. □

Example 2.3. Consider the following first-order neutral differential equation

$$\begin{aligned} x'(t) &= -5e^{\sin(t)}x(t) \qquad (2.5)\\ &+ \int_{\pi/3}^{\pi/2}\Big[e^{-\frac{\sin(t+\mu)}{5}}x'(t-e^{\frac{\sin(t+\mu)}{5}}) + e^{1-0.16\cos(t)+0.9\sin(t)}\Big(10+2x(t-e^{\frac{\sin(t+\mu)}{5}})\Big)\Big]d\mu. \end{aligned}$$

It can be seen that (2.5) is of the form (1.1) with $p(t) = 5e^{\sin(t)}$, $c(t,\mu) = e^{-\frac{\sin(t+\mu)}{5}}$, $g(t,\mu) = e^{\frac{\sin(t+\mu)}{5}}$, $\omega = 2\pi$, $q(t,x) = e^{1-0.16\cos(t)+0.9\sin(t)}(10+2x)$. After straightforward calculation, we can obtain

$$c_t(t,\mu) = -\frac{\cos(t+\mu)}{5}e^{-\frac{\sin(t+\mu)}{5}}, \qquad g_t(t,\mu) = \frac{\cos(t+\mu)}{5}e^{\frac{\sin(t+\mu)}{5}},$$

$$g_{tt}(t,\mu) = e^{\frac{\sin(t+\mu)}{5}}\Big[\frac{-\sin(t+\mu)}{5} + \frac{\cos^2(t+\mu)}{25}\Big],$$

$$\begin{aligned} r(t,\mu) &= \frac{\Big(c_t(t,\mu)+c(t,\mu)p(t)\Big)\Big(1-g_t(t,\mu)\Big) + g_{tt}(t,\mu)c(t,\mu)}{\Big(1-g_t(t,\mu)\Big)^2}\\ &= \frac{\Big(-\frac{\cos(t+\mu)}{5}+5e^{\sin(t)}\Big)\Big(e^{-\frac{\sin(t+\mu)}{5}} - \frac{\cos(t+\mu)}{5}\Big) + \Big(\frac{-\sin(t+\mu)}{5}+\frac{\cos^2(t+\mu)}{25}\Big)}{\Big(1-\frac{\cos(t+\mu)}{5}e^{\frac{\sin(t+\mu)}{5}}\Big)^2}, \end{aligned}$$

$$0 \leqslant 0.38 = c_0 \leqslant \frac{(b-a)c(t,\mu)}{1-g_t(t,\mu)} = \frac{(\pi/2-\pi/3)e^{-\frac{\sin(t+\mu)}{5}}}{1-\frac{\cos(t+\mu)}{5}e^{\frac{\sin(t+\mu)}{5}}} \leqslant c_1 = 0.69 < 1,$$

$$\frac{(1-c_0)m_0}{b-a} = \frac{(1-0.38)3}{(\pi/2-\pi/3)} = 3.5523 \quad \text{and} \quad \frac{(1-c_1)m_1}{b-a} = \frac{(1-0.69)20}{(\pi/2-\pi/3)} = 11.8211$$

and

$$3.5523 = \frac{(1-c_0)m_0}{b-a} \leqslant 4.1280 \leqslant \frac{q(t,x)-r(t,\mu)x}{p(t)} \leqslant 10.0997 \leqslant \frac{(1-c_1)m_1}{b-a} = 11.8211.$$

It shows that the conditions of Theorem 2.1 are met when $m_0 = 3$ and $m_1 = 20$. Thus (2.5) has at least one 2π-periodic positive solution $x(t)$ satisfying $3 \leqslant x(t) \leqslant 20$.

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