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Bigeometric Laplace İntegral Dönüşümü Üzerine

Sinem KAYMAK, Numan YALÇIN*

Öne Çıkanlar:
- Newtonyen olmayan analiz
- İntegral dönüşüm metotları
- Laplace İntegral dönüşümü

Anahtar Kelimeler:
- Bigeometrik analiz
- Üstel aritmetik
- Bigeometrik türev
- Bigeometrik integral
- Bigeometrik Laplace integral dönüşümü

ÖZET:

Bu çalışmanın amacı, klasik analizin integral dönüşüm metotlarından biri olan Laplace integral dönüşümünün temel tanım ve teoremleri kullanılarak, Newtonyen olmayan analizlerden biri olan bigeometrik analizde Laplace integral dönüşümünü tanımlamaktır. Öncelikli olarak Newtonyen olmayan analizlerin temelini oluşturan üstel aritmetik kavramı verilmiştir. Klasik analizde olduğu gibi bigeometrik analizde de bigeometrik limit, bigeometrik süreklilik, bigeometrik türev ve bigeometrik integral kavramlarının tanımları verilmiştir. Ardından, bigeometrik Laplace integral dönüşümünün tanımı yapılmıştır. Sonra, bigeometrik Laplace integral dönüşümünün bazı temel kavramları ve teoremleri verilmiştir. Bunun için bigeometrik analizde yer alan bigeometrik türev, bigeometrik belirsiz integral ve bigeometrik belirli integral kavramlarının tanımları ve bu kavramların özellikleri kullanılmıştır. Ayrıca, bigeometrik Laplace integral dönüşümünün özellikleri incelenmiştir. Son olarak bigeometrik Laplace integral dönüşümü yardımıyla bigeometrik lineer diferansiyel denklemlerin çözümleri araştırılmıştır.

On Bigeometric Laplace Integral Transform

Highlights:
- Non-Newtonian analysis
- Integral transform method
- Laplce integral transform

Keywords:
- Bigeometric analysis
- Exponential arithmetic
- Bigeometric derivative
- Bigeometric integral
- Bigeometric Laplace integral transform

ABSTRACT:

The purpose of this study is to mention the Laplace integral transform in bigeometric analysis, which is one of the non-Newtonian analysis by using the fundamental definitions and theorems of the Laplace integral transform, which is one of the integral transform methods of classical analysis. First of all, the concept of exponential arithmetic, which forms the basis of non-Newtonian analysis, is given. As in classical analysis, definitions of the concepts of bigeometric limit, bigeometric continuity, bigeometric derivative and bigeometric integral are given in bigeometric analysis. Here, the definition of the bigeometric Laplace integral transform in bigeometric analysis is given. Then, some basic concepts and theorems of the bigeometric Laplace integral transform are given. For this purpose, the definitions of the concepts of bigeometric derivative and bigeometric indefinite integral and bigeometric definite integral in bigeometric analysis and the properties of these concepts are used. In addition, the properties of the bigeometric Laplace integral transform are investigated. Finally, solutions of bigeometric linear differential equations are investigated with the help of the bigeometric Laplace integral transform.

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INTRODUCTION

It is seen that integral transforms started to be used in the first half of the 19th century. However, it was G. W. Leibniz (1646-1716) who first introduced the idea of symbolic transform. Later, J. L. Lagrange (1736-1813) and P. S. Laplace (1749-1827) worked on this transform.

The importance of integral transform methods has emerged with Laplace and Fourier transforms. Sumudu and other integral transforms are based on Laplace and Fourier integral transforms. The Laplace transform is an integral transform; it has an important place in physics, mechanics, engineering, telecommunications, mathematics and other applied sciences. It was described by the French mathematician and astronomer P. S. Laplace (1749 - 1827). This transform, provides great convenience in solving differential equations and is also used in mathematical methods of physics.

With the help of different arithmetic operations, new analysis that differ from the classical analysis has been defined. American scientists Michael Grossman and Robert Katz defined Non-Newtonian analysis from 1967 to 1970. They defined classical analysis, geometric analysis, harmonic analysis and quadratic analysis in July 1967, and then bigeometric analysis, anageometric analysis, biharmonic analysis, anaharmonic analysis, biquadratic analysis and anaquadratic analysis in August 1970. They used the adjective "non-Newtonian" to distinguish these new analysis from classical analysis.

The most popular non-Newtonian analysis are geometric multiplicative analysis and bigeometric analysis. There are many applications of these two analysis in the literature; multiplicative analysis has a wide range of applications, both geometrically and bigeometrically (Boruah and Hazarika, 2021a,b: Boruah and Hazarika, 2022; Erdoğan and Duyar, 2018; Güngör, 2021). Multiplicative analysis has applications in many fields such as science, engineering and mathematics. Some of those; interest rates, elasticity in economics (Elasticity) theory, blood viscosity, computer science including image processing and artificial intelligence, biology, differential equations and probability theory (Bashirov vd. 2011; Filip vd. 2014; Córdova-Lepe, 2006). Also, Rybaczuk and Stopel explored fractal growth in materials science with the help of multiplicative analysis (Rybaczuk and Stopel, 2000). Bashirov et al. defined the basic concepts of multiplicative analysis and gave some applications (Bashirov vd. 2008). Florak and Assen used multiplicative analysis in biomedical image analysis (Florak and Assen, 2012). Çakmak and Başar studied double integrals in multiplicative analysis (Çakmak and Başar, 2014). Kadak and Özlük generalized the Runge-Kutta method for ordinary differential equations in multiplicative analysis. (Kadak and Özlük, 2014). Córdova-Lepe worked on the measurement of elasticity in the economy with the help of multiplicative analysis (Córdova-Lepe, 2015). Yalçın et al. worked on multiplicative linear differential equations (Yalçın, 2016; Yalçın and Çelik, 2018; Yalçın vd., 2016; Yalçın, 2019; Yalçın, 2021; Yalçın and Dedetürk, 2021; Yalçın and Dedetürk, 2022).

In this study, the definition of the bigeometric Laplace integral transform in the sense of bigeometric analysis is made and some properties are given by using the Laplace integral transform known from classical analysis. In addition, solutions of bigeometric linear differential equations are made with the help of the bigeometric Laplace integral transform.

MATERIALS AND METHODS

In this section, the basic definitions and theorems of bigeometric analysis will be discussed (Grossman and Katz, 1972; Grossman, 1983; Boruah and Hazarika, 2018 a,b).

Basic Definitions and Theorems of Bigeometric Analysis

Let us first give the concept of exponential arithmetic, which forms the basis of non-Newtonian analysis. (Türkmen and Başar, 2012; Çakmak and Başar, 2012; Boruah and Hazarika, 2018 a,b)

Definition 1. Exponential sets of numbers are defined as follows:

a) The set of exponential real numbers: $\mathbb{R}_{exp} = \{e^t | t \in \mathbb{R}\}$ (1)

b) The set of exponential positive real numbers: $\mathbb{R}_{exp}^+ = \{e^t | t \in \mathbb{R},\ t > 0\} = (1, \infty)$ (2)

c) The set of exponential negative real numbers: $\mathbb{R}_{exp}^- = \{e^t | t \in \mathbb{R},\ t < 0\} = (0,1)$ (3)

d) The set of exponential integers: $\mathbb{Z}_{exp} = \{e^t |\ t \in \mathbb{Z}\}$ (4)

Definition 2. The operations of exponential addition, exponential subtraction, exponential multiplication and exponential division are defined as follows (Çakmak and Basar, 2012) : $\forall u, v \in \mathbb{R}_{exp}$

a) $u \oplus v = e^{\ln(u)+\ln(v)} = u \cdot v$ (5)

b) $u \ominus v = e^{\ln(u)-\ln(v)} = \frac{u}{v}$ (6)

c) $u \odot v = e^{\ln(u)\cdot\ln(v)} = u^{\ln(v)}$ (7)

d) $u \oslash v = e^{\frac{\ln(u)}{\ln(v)}} = u^{\frac{1}{\ln(v)}},\ (v \neq 1)$ (8)

Definition 3. The function $|\cdot|_*: \mathbb{R}_{exp} \to \mathbb{R}_{exp}^{+,0}$ which is given by

$$|t|_* = \exp|\ln(t)| = e^{|\ln(t)|} \tag{9}$$

is called multiplicative (geometric) absolute value function (Türkmen and Başar, 2012).

Definition 4. Multiplicative (geometric) absolute value function satisfies the identities below.(Grossman and Katz, 1972; Grossman 1983; Türkmen and Başar, 2012; Boruah and Hazarika, 2018a,b).

$$|t|_* = \begin{cases} t, & 1 \leq t, \\ \ominus t, & 0 < t < 1 \end{cases} \Rightarrow |t|_* = \begin{cases} t, & 1 \leq t, \\ \frac{1}{t}, & 0 < t < 1. \end{cases} \tag{10}$$

Here, the inverse of $t \in \mathbb{R}_{exp}$ with respect to exponential addition (multiplication in the classical sense) is represented by "$\ominus t$".

Definition 5. Let A be a subset of $\mathbb{R}_{exp}$. A function $f_{BG}: A \to \mathbb{R}_{exp}$ is called a bigeometric function.

Definition 6. (Bigeometric Limit)

Let $f: A \subseteq \mathbb{R}_{exp} \to \mathbb{R}_{exp}$ be a bigeometric function and a be an accumulation point of the set A according to geometric (exponential) neighbourhood. The bg (bigeometric) function f has the bigeometric limit at the point $a \in \mathbb{R}_{exp}$ and it is a unique exponential number $L \in \mathbb{R}_{exp}$ if and only if for all $\varepsilon > 1$ there exists an exponential number $\delta = \delta(\varepsilon) > 1$ such that $f(t) \in (L \ominus \varepsilon, L \oplus \varepsilon) = \left(\frac{L}{\varepsilon}, L \cdot \varepsilon\right)$ whenever $t \in (a \ominus \delta, a \oplus \delta) \setminus \{a\} = \left(\frac{a}{\delta}, a \cdot \delta\right) \setminus \{a\}$ (Grossman and Katz, 1972; Grossman 1983; Boruah and Hazarika, 2018a).

The bigeometric limit of a function is denoted as below.

$\pi \lim_{t \to a} f(t) = L$ or $f(t) \overset{\pi}{\to} L$ for $t \overset{\pi}{\to} a$.

Definition 7. (Bigeometric One-Sided Limit)

Let $f: A \subseteq \mathbb{R}_{exp} \to \mathbb{R}_{exp}$ be a bigeometric function and $a \in A$. The bigeometric right and bigeometric left limits of the function f at the point a are defined for $\varepsilon > 1$, respectively, as follows.

$$f(a^{bg+}) = \pi \lim_{\varepsilon \to 1} f(\varepsilon \cdot a), \tag{11}$$

$$f(a^{bg-}) = \pi \lim_{\varepsilon \to 1} f\left(\frac{a}{\varepsilon}\right), \tag{12}$$

If the function f has the bigeometric limit at the point a, the following equation is valid.

$$\pi \lim_{t \to a} f(t) = f(a^{bg+}) = f(a^{bg-}). \tag{13}$$

Definition 8. (Bigeometric Continuity)

Let $f: A \subseteq \mathbb{R}_{exp} \to \mathbb{R}_{exp}$ be a bigeometric function and $a \in A$. The function f is said to be bigeometric continuous at the point $a \in A$ if for all $\varepsilon > 1$ there exists an exponential number $\delta = \delta(\varepsilon) > 1$ such that $|f(t) \ominus f(a)|_* < \varepsilon$ whenever $|t \ominus a|_* < \delta$.

If a bigeometric function $f(t)$ is continuous at the point $t = a$ then we have

$$\pi \lim_{t \to a} f(t) = f(a). \tag{14}$$

Definition 9. (Bigeometric Uniform Discontinuity Point)

Let $f: A \subseteq \mathbb{R}_{exp} \to \mathbb{R}_{exp}$ be a bigeometric function and $a \in A$. If the bigeometric right and bigeometric left limits of the bigeometric function f at the point a have finite values and f is not bigeometric continuous at the point a, then the point a is called a bigeometric uniform discontinuity point of f function.

Definition 10. (Bigeometric Piecewise Continuous Function)

A bigeometric function that is bigeometric continuous in an interval [a,b] except for a finite number of bigeometric uniform discontinuity points is said to be bigeometric piecewise continuous function in that interval.

Definition 11. (Bigeometric Derivative)

Let $f: \mathbb{R}_{exp} \to \mathbb{R}_{exp}$ be a bigeometric function. If the limit

$$\lim_{h \to 0} \left[\frac{f[(1+h)\cdot t]}{f(t)}\right]^{1/h} \tag{15}$$

or the bigeometric limit

$$\pi \lim_{h \to 1} \left[\frac{f(h\cdot t)}{f(t)}\right]^{1/\ln h} \tag{16}$$

exists then this limit (bigeometric limit) value is called the bigeometric derivative of $f$and it is denoted by

$$\frac{d^\pi f}{dt^\pi}(t) = f^\pi(t) = \lim_{h \to 0} \left\{\frac{f[(1+h)\cdot t]}{f(t)}\right\}^{1/h} \tag{17}$$

(Grossman and Katz, 1972; Grossman 1983; Boruah and Hazarika, 2018a,b).

Remark 1. The bigeometric derivative defined above can also be expressed by,

$$f^\pi(t) = \pi \lim_{\varphi \to t} \{[f(\varphi) \ominus f(t)] \oslash [\varphi \ominus t]\} = \lim_{\varphi \to t} \left[\frac{f(\varphi)}{f(t)}\right]^{1/\ln \varphi - \ln t} \tag{18}$$

with the help of exponential arithmetic.

If the limit above exists then it is represented by $f^\pi(t)$ and it is called the bigeometric derivative of f at the point t. Also the bigeometric derivative $f^\pi: \mathbb{R}_{exp} \to \mathbb{R}_{exp}$ can be given by the formula below

$$f^\pi(t) = \pi \lim_{h \to 1} [f(t \oplus h) \ominus f(t)] \oslash h = \pi \lim_{h \to 1} \left[\frac{f(h\cdot t)}{f(t)}\right]^{1/\ln h} \tag{19}$$

(Grossman and Katz, 1972; Grossman 1983; Boruah and Hazarika, 2018a,b).

Remark 2. From the definition of bigeometric derivative we have the following equality.

$$f^\pi(t) = \exp\left[t \cdot \frac{f'(t)}{f(t)}\right] = e^{t\cdot(\ln\circ f)'(t)}. \tag{20}$$

This gives the relation between bigeometric derivative and classical derivative.

Definition 12. As first order bigeometric derivative is f^π, taking one more time bigeometric derivative we get the second order bigeometric derivative which is denoted by $f^{\pi\pi}$. According to this the second order bigeometric derivative can be given as below (Güngör, 2020);

$$f^{\pi\pi}(t) = \frac{d^{\pi\pi}}{dt^{\pi\pi}} f(t) = \frac{d^\pi}{dt^\pi}\left(\frac{d^\pi}{dt^\pi} f(t)\right) = \pi \lim_{h \to 1} [f^\pi(t \oplus h) \ominus f^\pi(t)] \oslash h. \tag{21}$$

Similarly the n^{th} order bigeometric derivative of f is denoted by $f^{\pi(n)}$ and it is written as (Güngör, 2020);

$$f^{\pi(n)}(t)=\frac{d^{\pi(n)}}{dt^{\pi(n)}}f(t)=\frac{d^{\pi}}{dt^{\pi}}\left(\frac{d^{\pi(n-1)}}{dt^{\pi(n-1)}}f(t)\right)=\pi\lim_{h\to1}\left[f^{\pi(n-1)}(t\oplus h)\ominus f^{\pi(n-1)}(t)\right]\oslash h. \quad (22)$$

Theorem 1. Let $f,h:(a,b)\subset\mathbb{R}_{exp}\to\mathbb{R}_{exp}$ be two bg differentiable bg function. For a constant $k\in\mathbb{R}_{exp}$ the following equalities hold (Riza and Eminağa, 2014).

a) $(k\cdot f)^{\pi}(t)=f^{\pi}(t)$ **b)** $(f+h)^{\pi}=f^{\pi}(t)^{\frac{f(t)}{f(t)+h(t)}}\cdot h^{\pi}(t)^{\frac{h(t)}{f(t)+h(t)}}$ **c)** $\left({}^{f}\!/_{h}\right)^{\pi}(t)={f^{\pi}(t)}/{h^{\pi}(t)}$

d) $(f\cdot h)^{\pi}(t)=f^{\pi}(t)\cdot h^{\pi}(t)$ **e)** $\left(f^{h}\right)^{\pi}(t)=f^{\pi}(t)^{h(t)}\cdot f(t)^{t\cdot h'(t)}$ **f)** $(f\circ h)^{\pi}(t)=\{f^{\pi}[h(t)]\}^{h'(t)}$

Now, we will give the bigeometric derivatives of some bg functions (Grossman and Katz, 1972; Grossman 1983; Boruah and Hazarika, 2018a,b);

a) $\frac{d^{\pi}}{dt^{\pi}}(c)=1$ **b)** $\frac{d^{\pi}}{dt^{\pi}}(\sin t)=e^{t\cdot\cot t}$ **c)** $\frac{d^{\pi}}{dt^{\pi}}(\cot t)=e^{-t\cdot\sec t\cdot\csc t}$

d) $\frac{d^{\pi}}{dt^{\pi}}(e^{t})=e^{t}$ **e)** $\frac{d^{\pi}}{dt^{\pi}}(\cos t)=e^{-t\cdot\tan t}$ **f)** $\frac{d^{\pi}}{dt^{\pi}}(\sec t)=e^{t\cdot\tan t}$

g) $\frac{d^{\pi}}{dt^{\pi}}(t^{n})=e^{n}$ **h)** $\frac{d^{\pi}}{dt^{\pi}}(\tan t)=e^{t\cdot\sec t\cdot\csc t}$ **i)** $\frac{d^{\pi}}{dt^{\pi}}(\csc t)=e^{-t\cdot\cot t}$

Definition 13. (Bigeometric Antiderivative)

Let $f(t)$ be a bigeometric function. If there exists a bigeometric function F satisfying the relation $F^{\pi}(t)=f(t)$, then F is called an antiderivative of f.

Theorem 2. Let F be an antiderivative of f, $F^{\pi}(t)=f(t)$, on the open interval $I=(a,b)\subset\mathbb{R}_{exp}$. Each bigeometric antiderivative of f on the open interval I is in the form

$$G(t)=C\cdot F(t). \quad (23)$$

Here $C\in\mathbb{R}_{exp}$ is an arbitrary exponential constant In other words, the ratio between two bg antiderivatives is constant.

Definition 14. (Bigeometric Indefinite Integral)

Let $f(t)$ be a bigeometric function and $F(t)$ be an bigeometric antiderivative of $f(t)$. Then, the most general bigeometric antiderivative of $f(t)$ is called the bigeometric indefinite integral of $f(t)$ and it is denoted by

$$\pi\int f(t)dt^{\pi}=C\cdot F(t) \quad (24)$$

where $C\in\mathbb{R}_{exp}$ is an arbitrary exponential constant.

This integral is also called π −integral (Grossman and Katz, 1972; Grossman 1983; Boruah and Hazarika, 2018 b). The bg indefinite integral of $f(t)$ can be calculated by the following formula

$$\pi\int f(t)dt^{\pi}=\exp\left(\int\frac{(\ln\circ f)(t)}{t}dt\right). \quad (25)$$

Now, we will give the bigeometric integrals of some bg functions (Boruah and Hazarika, 2018 b);

a) $\pi\int 1dt^{\pi}=C$ **b)** $\pi\int e^{t\cdot\cot t}dt^{\pi}=C\cdot\sin t$ **c)** $\pi\int e^{-t\cdot\sec t\cdot\csc t}dt^{\pi}=C\cdot\cot t$

d) $\pi\int t^{n}dt^{\pi}=C\cdot e^{\frac{n\cdot\ln^{2}t}{2}}$ **e)** $\pi\int e^{-t\cdot\tan t}dt^{\pi}=C\cdot\cos t$ **f)** $\pi\int e^{t\cdot\tan t}dt^{\pi}=C\cdot\sec t$

g) $\pi\int e^{n}dt^{\pi}=C\cdot t^{n}$ **h)** $\pi\int e^{t\cdot\sec t\cdot\csc t}dt^{\pi}=C\cdot\tan t$ **i)** $\pi\int e^{-t\cdot\cot t}dt^{\pi}=C\cdot\csc t$

Definition 15. (Bigeometric Definite Integral)

Let f be a bg continuous function in the interval $[a,b] \subset \mathbb{R}_{exp}$. Then the bigeometric definite integral of $f(t)$ on the interval $[a,b]$ is given by

$$\pi\int_a^b f(t)dt^{\pi} = \exp\left(\int_{t=a}^{b} \frac{(\ln\circ f)(t)}{t}dt\right) = e^{\int_{t=a}^{b} \frac{(\ln\circ f)(t)}{t}dt}. \tag{26}$$

(Grossman and Katz, 1972; Grossman 1983; Boruah and Hazarika, 2018 b).

Lemma 2. The following equality holds for bg integral of a bg function f.

$$\pi\int_a^b f(t)dt^{\pi} = \exp\left(\int_{\tau=\ln(a)}^{\tau=\ln(b)} (\ln\circ f)(e^{\tau})\,d\tau\right). \tag{27}$$

Proof: From the bg definite integral definition we know

$$\pi\int_a^b f(t)dt^{\pi} = \exp\left(\int_{t=a}^{b} \frac{(\ln\circ f)(t)}{t}dt\right) = \exp\left(\int_{t=a}^{b} (\ln\circ f)(t)\,d(\ln t)\right).$$

By changing variable $\tau = \ln t$, $(t = e^{\tau})$ we get

$$\pi\int_a^b f(t)dt^{\pi} = \exp\left(\int_{\tau=\ln(a)}^{\ln(b)} (\ln\circ f)(e^{\tau})\,d\tau\right).$$

Theorem 3. Let f, h two bg functions which are integrable in the interval $[a,b] \subset \mathbb{R}_{exp}$. For a constant $k \in \mathbb{R}$, we have the following rules (Grossman and Katz, 1972; Grossman 1983; Boruah and Hazarika, 2018 b);

a) $\pi\int_a^b [f(t)]^k dt^{\pi} = \left[\pi\int_a^b f(t)\,dt^{\pi}\right]^k,$ (28)

b) $\pi\int_a^b [f(t)\cdot h(t)]\,dt^{\pi} = \pi\int_a^b f(t)\,dt^{\pi} \cdot \pi\int_a^b h(t)\,dt^{\pi},$ (29)

c) $\pi\int_a^b \left[\frac{f(t)}{h(t)}\right] dt^{\pi} = \frac{\pi\int_a^b f(t)dt^{\pi}}{\pi\int_a^b h(t)dt^{\pi}},$ (30)

d) $\pi\int_a^b f(t)\,dt^{\pi} = \pi\int_a^c f(t)\,dt^{\pi} \cdot \pi\int_c^b f(t)\,dt^{\pi},\ (a < c < b).$ (31)

Theorem 4. (First Fundamental Theorem of Bigeometric Analysis)

Let f be a bg continuous function in the interval $[a,b] \subset \mathbb{R}_{exp}$. Also let $h(t)$ be a bg function defined on $[a,b]$ by

$$h(t) = \pi\int_a^t f(s)\,ds^{\pi}. \tag{32}$$

Then for the interval $[a,b]$ the following equality is valid:

$$h^{\pi}(t) = f(t). \tag{33}$$

(Grossman and Katz, 1972; Grossman 1983).

Theorem 5. (Second Fundamental Theorem of Bigeometric Analysis)

Let f^{π} be a bg function which is bg continuos in the interval $[a,b] \subset \mathbb{R}_{exp}$. Then the following equality holds

$$\pi\int_a^b f^{\pi}(t)\,dt^{\pi} = \frac{f(b)}{f(a)}. \tag{34}$$

(Grossman and Katz, 1972; Grossman 1983).

RESULTS AND DISCUSSION

In this section we will define bigeometric Laplace integral transform (BGLIT) in bigeometric calculus based on Laplace integral transform in Newtonian calculus. Also some fundamental properties of this new transformation will be given.

Definition 16. (Improper Bigeometric Integral)

Let $f\colon \mathbb{R}_{exp}^{+} \to \mathbb{R}_{exp}$ be a bigeometric function which is bigeometric continuous on $[a,b] \subset \mathbb{R}_{exp}^{+}$ for each exponential positive real number $b \geq a$. The bg limit

$$\pi \lim_{b\to\infty} \pi \int_a^b f(t)dt^{\pi} \tag{35}$$

is called improper bg integral of type 1 of the function f on $[a,\infty)$ and is denoted by

$$\pi \int_a^{\infty} f(t)dt^{\pi}. \tag{36}$$

If the bg limit exists and equals to an exponential number $L \in \mathbb{R}_{exp}$, then it is said that the improper bg integral is bigeometric convergent. If the bg limit does not exist or equals to ∞ or 0, then it is said that the improper bg integral is divergent (Duyar and Erdoğan, 2018).

Definition 17. (The Bigeometric Laplace Integral Transform)

Let $f\colon \mathbb{R}_{exp}^{+} \to \mathbb{R}_{exp}$ be a bigeometric function. Then the bigeometric Laplace integral transform of $f(t)$ is defined by

$$\mathcal{L}_{BG}\{f(t)\} = F_{BG}(s) = \pi \int_1^{\infty} f(t) \odot e^{(\ominus s)\odot t} dt^{\pi}. \tag{37}$$

Here $F_{BG}\colon \mathbb{R}_{exp} \to \mathbb{R}_{exp}$ is a bigeometric function.

Lemma 3. Let $f\colon \mathbb{R}_{exp}^{+} \to \mathbb{R}_{exp}$ be a bigeometric function. For the bigeometric Laplace integral transform of $f(t)$ the following equality holds ($t = e^{\tau}$)

$$\mathcal{L}_{BG}\{f(t)\} = \exp\left\{\int_{\tau=0}^{\infty} (\ln\circ f)(e^{\tau}) \cdot s^{-\tau} \cdot d\tau\right\}. \tag{38}$$

Proof: First note that

$$(\ominus s) \odot t = e^{\ln(\ominus s)\cdot \ln(t)} = e^{\ln\left(\frac{1}{s}\right)\cdot \ln(t)} = e^{-\ln(s)\cdot\ln(t)} = \left[e^{\ln(s)}\right]^{-\ln(t)} = s^{-\ln(t)}$$

and

$$\begin{aligned} f(t) \odot e^{(\ominus s)\odot t} &= f(t) \odot \exp[(\ominus s) \odot t] \\ &= e^{(\ln\circ f)(t)\cdot(\ln\circ\exp)[(\ominus s)\odot t]} \\ &= e^{(\ln\circ f)(t)\cdot[(\ominus s)\odot t]} \\ &= \left[e^{(\ln\circ f)(t)}\right]^{[(\ominus s)\odot t]} \\ &= [f(t)]^{[(\ominus s)\odot t]} \end{aligned}$$

$$f(t) \odot e^{(\ominus s)\odot t} = [f(t)]^{\left(s^{-\ln(t)}\right)}.$$

Now let us use the last equality in the definition of the bigeometric Laplace integral transform.

$$\begin{aligned} \mathcal{L}_{BG}\{f(t)\} &= \pi \int_1^{\infty} f(t) \odot e^{(\ominus s)\odot t} dt^{\pi} \\ &= \pi \int_1^{\infty} f(t)^{\left(s^{-\ln(t)}\right)} dt^{\pi}. \end{aligned}$$

Now we will use the formula

$$\pi \int_a^b h(t)dt^{\pi} = \exp\left\{\int_{\tau=\ln(a)}^{\ln(b)} (\ln\circ h)(e^{\tau})\, d\tau\right\} = \exp\left\{\int_{\tau=\ln(a)}^{\ln(b)} \ln[h(e^{\tau})]\, d\tau\right\}$$

to express the bigeometric integral by using the classical integral. Here $\tau = \ln t$.

$$\mathcal{L}_{BG}\{f(t)\} = \exp\left\{\lim_{b\to\infty} \int_{\tau=\ln(1)}^{\ln(b)} \ln\left[f(e^{\tau})^{\left(s^{-\ln(e^{\tau})}\right)}\right] d\tau\right\}$$

$$= \exp\left\{\int_{\tau=0}^{\infty} \ln\left[f(e^\tau)^{(s^{-\tau})}\right] d\tau\right\}.$$

Thus we get

$$\mathcal{L}_{BG}\{f(t)\} = \exp\left\{\int_{\tau=0}^{\infty} (\ln \circ f)(e^\tau) \cdot s^{-\tau} d\tau\right\}.$$

Lemma 4. Let $f: (0,\infty) \to \mathbb{R}$, $f(\tau)$ be a function and $F(\sigma)$ be its classical Laplace transform, i.e. $\mathcal{L}\{f(\tau)\} = F(\sigma)$. Then for $t = e^\tau$ and $s = e^\sigma$ the following equality holds:

$$\mathcal{L}_{BG}\{(\exp \circ f \circ \ln)(t)\} = (\exp \circ F \circ \ln)(s). \quad (39)$$

Proof:

By using the equality (38), we have

$$\mathcal{L}_{BG}\{(\exp \circ f \circ \ln)(t)\} = \exp\left\{\int_{\tau=0}^{\infty} (\ln \circ \exp \circ f \circ \ln)(e^\tau) \cdot s^{-\tau} \cdot d\tau\right\}$$

$$= \exp\left\{\int_{\tau=0}^{\infty} f(\tau) \cdot s^{-\tau} \cdot d\tau\right\}$$

$$= \exp\left\{\int_{\tau=0}^{\infty} f(\tau) \cdot e^{-\tau \ln s} \cdot d\tau\right\}$$

$$= \exp\{F(\ln s)\}$$

$$\mathcal{L}_{BG}\{(\exp \circ f \circ \ln)(t)\} = (\exp \circ F \circ \ln)(s).$$

The equality above can also be expressed as

$$\mathcal{L}_{BG}\left\{e^{f(\ln t)}\right\} = e^{F(\ln s)}.$$

According to this the bigeometric Laplace integral transforms of some bg functions are written in the table below.

$f(\tau)$	$\mathcal{L}\{f(\tau)\} = F(\sigma)$	$e^{f(\ln t)}$	$\mathcal{L}_{BG}\left\{e^{f(\ln t)}\right\} = e^{F(\ln s)}$
0	$\mathcal{L}\{0\} = 0$	1	$\mathcal{L}_{BG}\{1\} = 1$
1	$\mathcal{L}\{1\} = 1/\sigma$	e	$\mathcal{L}_{BG}\{e\} = e^{\frac{1}{\ln s}}$
$\ln a, \quad (a > 0)$	$\mathcal{L}\{\ln a\} = \frac{\ln a}{\sigma}$	$a, \quad (a \in \mathbb{R}_{exp})$	$\mathcal{L}_{BG}\{a\} = a^{\frac{1}{\ln s}}$
e^τ	$\mathcal{L}\{e^\tau\} = \frac{1}{\sigma-1}, \quad (\sigma > 1)$	e^t	$\mathcal{L}_{BG}\{e^t\} = e^{\frac{1}{\ln(s)-1}}$
$e^{a\tau}, \quad (a \in \mathbb{R})$	$\mathcal{L}\{e^{a\tau}\} = \frac{1}{\sigma-a}, \quad (\sigma > a)$	$e^{(t^a)}, \quad (a \in \mathbb{R})$	$\mathcal{L}_{BG}\left\{e^{(t^a)}\right\} = e^{\frac{1}{\ln(s)-a}}$
$\tau^n, \quad (n \in \mathbb{N})$	$\mathcal{L}\{\tau^n\} = \frac{n!}{\sigma^{n+1}}$	$e^{[(\ln t)^n]}$	$\mathcal{L}_{BG}\left\{e^{[(\ln t)^n]}\right\} = e^{\frac{n!}{(\ln s)^{n+1}}}$
$\cos(a\tau), (a \in \mathbb{R})$	$\mathcal{L}\{\cos(a\tau)\} = \frac{\sigma}{\sigma^2+a^2}$	$e^{\cos(\ln(at))}, (a \in \mathbb{R})$	$\mathcal{L}_{BG}\left\{e^{\cos(\ln(at))}\right\} = e^{\frac{\ln s}{(\ln s)^2+a^2}}$
$\sin(a\tau), (a \in \mathbb{R})$	$\mathcal{L}\{\sin(a\tau)\} = \frac{1}{\sigma^2+a^2}$	$e^{\sin(\ln(at))}, (a \in \mathbb{R})$	$\mathcal{L}_{BG}\left\{e^{\sin(\ln(at))}\right\} = e^{\frac{1}{(\ln s)^2+a^2}}$

Definition 18. Let $f: \mathbb{R}_{exp}^+ \to \mathbb{R}_{exp}$ be a bigeometric function defined on $(1,\infty)$. If there exists $t_0 \in \mathbb{R}_{exp}^+$, $K \in \mathbb{R}_{exp}, \alpha \in \mathbb{R}_{exp}$ constants such that for $t > t_0$

$$|f(t)|_* \leq K^{(\alpha^{\ln t})}. \quad (40)$$

Then the bg function f is said to be of α −bigeometric exponential order for $t > t_0$.

Theorem 6. (Existence of Bigeometric Laplace Integral Transform)

Let a bigeometric function $f: \mathbb{R}_{exp}^{+} \to \mathbb{R}_{exp}$ be of α −bigeometric exponential order for $t > t_0$ and be piecewise bg continuous. Then $\mathcal{L}_{BG}\{f(t)\}$ exists for $s > \alpha$.

Proof. As f is of α −bigeometric exponential order there exist $K \in \mathbb{R}_{exp}^{+}$, $(K = e^k,\ k \in \mathbb{R}^+)$ and $\alpha \in \mathbb{R}_{exp}$ such that

$$|f(t)|_* \leq K^{(\alpha^{\ln t})}$$

for $t > t_0$. Without loss of generality we will take $t_0 = 1$. Using a change of variables $t = e^\tau$, $(\tau \in \mathbb{R})$ the inequality

$$|f(t)|_* \leq K^{(\alpha^{\ln t})} = e^{k \cdot \alpha^{\ln t}}$$

can be written as

$$|f(e^\tau)|_* \leq e^{k \cdot \alpha^\tau}.$$

Since $\ln|f(e^\tau)|_* = |\ln f(e^\tau)|$, taking natural logarithm of both sides we see that

$$|\ln f(e^\tau)| \leq k \cdot \alpha^\tau.$$

From the definition of bigeometric Laplace transform for $t = e^\tau$, $s \in \mathbb{R}_{exp}$ we know that

$$\mathcal{L}_{BG}\{f(t)\} = e^{\int_{\tau=0}^{\infty} \ln f(e^\tau) \cdot s^{-\tau} d\tau}.$$

Let us define the above integral as $I = \int_{\tau=0}^{\infty} \ln f(e^\tau) \cdot s^{-\tau} d\tau$. If the integral I is convergent then $\mathcal{L}_{BG}\{f(t)\} = e^I$. Now, consider the following inequality regarding the integral I.

$$\left| \int_{\tau=0}^{\infty} \ln f(e^\tau) \cdot s^{-\tau} d\tau \right| \leq \int_{\tau=0}^{\infty} |\ln f(e^\tau)| \cdot s^{-\tau} d\tau \leq \int_{\tau=0}^{\infty} k \cdot \alpha^\tau \cdot s^{-\tau} d\tau = k \int_{\tau=0}^{\infty} \left(\frac{s}{\alpha}\right)^{-\tau} d\tau < \infty.$$

We see that the integral I is convergent for $s > \alpha$. Thus $\mathcal{L}_{BG}\{f(t)\}$ exists for $s > \alpha$.

Theorem 7. (Bigeometric Linearity Property)

Bigeometric Laplace integral transform is an exponentially linear transform. In other words, let k_1, k_2 be arbitrary real constants and $f_1(t), f_2(t)$ be two bigeometric functions which have bg Laplace transforms. Then, the following equality holds

$$\mathcal{L}_{BG}\{[f_1(t)]^{k_1} \cdot [f_2(t)]^{k_2}\} = \{\mathcal{L}_{BG}[f_1(t)]\}^{k_1} \cdot \{\mathcal{L}_{BG}[f_2(t)]\}^{k_2} \tag{41}$$

Proof. From the definition of bigeometric Laplace transform we have

$$\begin{aligned}
\mathcal{L}_{BG}\{[f_1(t)]^{k_1} \cdot [f_2(t)]^{k_2}\} &= \exp\left\{\int_0^\infty \ln([f_1(e^\tau)]^{k_1} \cdot [f_2(e^\tau)]^{k_2}) \cdot s^{-\tau} d\tau\right\} \\
&= \exp\left\{\int_0^\infty [k_1 \cdot \ln f_1(e^\tau) + k_2 \cdot \ln f_2(e^\tau)] \cdot s^{-\tau} d\tau\right\} \\
&= \exp\left\{k_1 \cdot \int_0^\infty \ln f_1(e^\tau) \cdot s^{-\tau} d\tau + k_2 \cdot \int_0^\infty \ln f_2(e^t) \cdot s^{-\tau} d\tau\right\} \\
&= \left\{\exp \int_0^\infty \ln f_1(e^\tau) \cdot s^{-\tau} d\tau\right\}^{k_1} \cdot \left\{\exp \int_0^\infty \ln f_2(e^\tau) \cdot s^{-\tau} d\tau\right\}^{k_2}
\end{aligned}$$

$$\mathcal{L}_{BG}\{[f_1(t)]^{k_1} \cdot [f_2(t)]^{k_2}\} = \{\mathcal{L}_{BG}[f_1(t)]\}^{k_1} \cdot \{\mathcal{L}_{BG}[f_2(t)]\}^{k_2}.$$

Theorem 8. (Bigeometric First Shifting Property)

Let $f: \mathbb{R}_{exp}^{+} \to \mathbb{R}_{exp}$ be a bigeometric function and $F_{BG}(s) = \mathcal{L}_{BG}\{f(t)\}$ be its bigeometric Laplace transform. Then the following equality holds

$$\mathcal{L}_{BG}\left\{f(t)^{\left(a^{\ln t}\right)}\right\} = F_{BG}\left(\frac{s}{a}\right) \tag{42}$$

for $a \in \mathbb{R}_{\exp}$.

Proof. Using the definition of bigeometric Laplace transform we get

$$\mathcal{L}_{BG}\left\{f(t)^{\left(a^{\ln t}\right)}\right\} = \exp\left\{\int_0^\infty \ln\left[f(e^\tau)^{\left(a^{\ln e^\tau}\right)}\right] \cdot s^{-\tau} d\tau\right\}$$

$$= \exp\left\{\int_0^\infty \ln[f(e^\tau)]^{(a^\tau)} \cdot s^{-\tau} d\tau\right\}$$

$$= \exp\left\{\int_0^\infty \ln f(e^\tau) \cdot a^\tau \cdot s^{-\tau} d\tau\right\}$$

$$= \exp\left\{\int_0^\infty \ln f(e^\tau) \cdot \left(\frac{s}{a}\right)^{-\tau} d\tau\right\}$$

$$\mathcal{L}_{BG}\left\{f(t)^{\left(a^{\ln t}\right)}\right\} = F_{BG}\left(\frac{s}{a}\right).$$

Theorem 9. (Bigeometric Second Shifting Property)

Let $f: \mathbb{R}_{exp}^{+} \to \mathbb{R}_{exp}$ be a bigeometric function and $F_{BG}(s) = \mathcal{L}_{BG}\{f(t)\}$ be its bigeometric Laplace transform. And also let

$$g(t) = \begin{cases} 1, & 1 < t < a \\ f\left(\frac{t}{a}\right), & t > a. \end{cases}$$

Then the following equality is satisfied.

$$\mathcal{L}_{BG}\{g(t)\} = \{F_{BG}(s)\}^{(\ominus s)\odot a} = \{F_{BG}(s)\}^{s^{-\ln a}}. \tag{43}$$

Proof. From the definition of bigeometric Laplace transform we can write

$$\mathcal{L}_{BG}\{g(t)\} = \exp\left\{\int_0^\infty \ln g(e^\tau) \cdot s^{-\tau} d\tau\right\}$$

$$= \exp\left\{\int_{t=1}^\infty \ln g(t) \cdot s^{-\ln t} \cdot d(\ln t)\right\}$$

$$= \exp\left\{\underbrace{\int_1^a \ln \underbrace{g(t)}_{1} \cdot s^{-\ln t} \cdot d(\ln t)}_{0} + \int_a^\infty \ln \underbrace{g(t)}_{f\left(\frac{t}{a}\right)} \cdot s^{-\ln t} \cdot d(\ln t)\right\}$$

$$\mathcal{L}_{BG}\{g(t)\} = \exp\left\{\int_a^\infty \ln f\left(\frac{t}{a}\right) \cdot s^{-\ln t} \cdot d(\ln t)\right\}.$$

Now, using a change of variables, $u = \frac{t}{a}$, $(t = a \cdot u)$ for $a > 1$ we have

$$\mathcal{L}_{BG}\{g(t)\} = \exp\left\{\int_{u=1}^\infty \ln f(u) \cdot s^{-\ln(a\cdot u)} \cdot d(\ln(a \cdot u))\right\}.$$

Here, noting that

$d(\ln(a \cdot u)) = d(\ln a + \ln u) = \mathrm{d}(\ln a) + \mathrm{d}(\ln u) = 0 + \mathrm{d}(\ln u) = \mathrm{d}(\ln u)$

we get the following equality.

$$\mathcal{L}_{BG}\{g(t)\} = \exp\left\{\int_{u=1}^{\infty} \ln f(u) \cdot s^{-(\ln a+\ln u)} \cdot d(\ln u)\right\}$$
$$= \exp\left\{\left[\int_{u=1}^{\infty} \ln f(u) \cdot s^{-\ln u} \cdot d(\ln u)\right] \cdot s^{-\ln a}\right\}$$
$$= \left\{\exp\left[\int_{u=1}^{\infty} \ln f(u) \cdot s^{-\ln u} \cdot d(\ln u)\right]\right\}^{s^{-\ln a}}$$
$$\mathcal{L}_{BG}\{g(t)\} = \{F_{BG}(s)\}^{s^{-\ln a}}.$$

Theorem 10. (Bigeometric Change of Scale Property)

Let $f: \mathbb{R}_{exp}^{+} \to \mathbb{R}_{exp}$ be a bigeometric function and $F_{BG}(s) = \mathcal{L}_{BG}\{f(t)\}$ be its bigeometric Laplace transform. Then the following equality holds for $a \in \mathbb{R}_{exp}^{+}$

$$\mathcal{L}_{BG}\{f(a \odot t)\} = \left\{F_{BG}\left(s^{1/\ln a}\right)\right\}^{\frac{1}{\ln a}}. \tag{44}$$

Proof. We will replace $a \odot t = e^{\ln a \cdot \ln t} = a^{\ln t}$ by using the definition of exponential multiplication and write

$$\mathcal{L}_{BG}\{f(a^{\ln t})\} = \exp\left\{\int_{\tau=0}^{\infty} \ln f\left(a^{\ln e^{\tau}}\right) \cdot s^{-\tau} d\tau\right\}$$
$$= \exp\left\{\int_{\tau=0}^{\infty} \ln f(a^{\tau}) \cdot s^{-\tau} d\tau\right\}.$$

Here using a change of variables, $u = a^{\tau}, \left(\tau = \frac{\ln u}{\ln a}\right)$ for $a > 1$ we get

$$\mathcal{L}_{BG}\{f(a^{\ln t})\} = \exp\left\{\int_{u=1}^{\infty} \ln f(u) \cdot s^{-\left(\frac{\ln u}{\ln a}\right)} \cdot d\left(\frac{\ln u}{\ln a}\right)\right\}$$
$$= \exp\left\{\int_{u=1}^{\infty} \ln f(u) \cdot \left(s^{1/\ln a}\right)^{-\ln u} \cdot \frac{d(\ln u)}{\ln a}\right\}$$
$$= \exp\left\{\left[\int_{u=1}^{\infty} \ln f(u) \cdot \left(s^{1/\ln a}\right)^{-\ln u} \cdot d(\ln u)\right] \cdot \frac{1}{\ln a}\right\}$$
$$= \left\{\exp\left[\int_{u=1}^{\infty} \ln f(u) \cdot \left(s^{1/\ln a}\right)^{-\ln u} \cdot d(\ln u)\right]\right\}^{\frac{1}{\ln a}}$$
$$\mathcal{L}_{BG}\{f(a \odot t)\} = \left\{F_{BG}\left(s^{1/\ln a}\right)\right\}^{\frac{1}{\ln a}}.$$

Theorem 11. Let $f: \mathbb{R}_{exp}^{+} \to \mathbb{R}_{exp}$ be a bigeometric function and $F_{BG}(s) = \mathcal{L}_{BG}\{f(t)\}$ be its bigeometric Laplace transform. Then the following equality holds

$$\mathcal{L}_{BG}\left\{f(t)^{\ln t}\right\} = [F_{BG}^{\pi}(s)]^{-1}. \tag{45}$$

Proof.

$$F_{BG}^{\pi}(s)=\frac{d^{\pi}}{ds^{\pi}}\left[\exp\left(\int_{\tau=0}^{\infty}(\ln\circ f)(e^{\tau})\cdot s^{-\tau}\cdot d\tau\right)\right]$$

$$=\exp\left[s\cdot\left(\frac{d}{ds}\int_{\tau=0}^{\infty}(\ln\circ f)(e^{\tau})\cdot s^{-\tau}\cdot d\tau\right)\right]$$

$$=\exp\left[s\cdot\left(\int_{\tau=0}^{\infty}\ln[f(e^{\tau})]\cdot(-\tau)\cdot s^{-\tau-1}\cdot d\tau\right)\right]$$

$$=\exp\left\{(-1)\cdot\int_{\tau=0}^{\infty}\ln[f(e^{\tau})]^{\tau}\cdot s^{-\tau}\cdot d\tau\right\}$$

$$=\left\{\exp\left(\int_{\tau=0}^{\infty}\ln\left[\left(f(e^{\tau})\right)^{\ln(e^{\tau})}\right]\cdot s^{-\tau}\cdot d\tau\right)\right\}^{-1}$$

$$=\left(\mathcal{L}_{BG}\{f(t)^{\ln t}\}\right)^{-1}$$

$$\mathcal{L}_{BG}\{f(t)^{\ln t}\}=[F_{BG}^{\pi}(s)]^{-1}.$$

Theorem 12. Let $f:\mathbb{R}_{exp}^{+}\to\mathbb{R}_{exp}$ be a bigeometric function and $F_{BG}(s)=\mathcal{L}_{BG}\{f(t)\}$ be its bigeometric Laplace transform. Then the following equality holds

$$\mathcal{L}_{BG}\{f(t)^{(\ln t)^{n}}\}=\left[F_{BG}^{\pi(n)}(s)\right]^{(-1)^{n}},\ \ (n=1,2,\dots).\tag{46}$$

Proof. We will do the proof by mathematical induction.

i) For $k=1$ we know that the equality $\mathcal{L}_{BG}\{f(t)^{\ln t}\}=[F_{BG}^{\pi}(s)]^{-1}$ is satisfied from Theorem 11.

ii) For $k=n$, let us assume that the equality

$$\mathcal{L}_{BG}\{f(t)^{(\ln t)^{n}}\}=\left[F_{BG}^{\pi(n)}(s)\right]^{(-1)^{n}}$$

holds.

iii) Now for $k=n+1$ we write

$$\mathcal{L}_{BG}\{f(t)^{(\ln t)^{n+1}}\}=\left[F_{BG}^{\pi(n+1)}(s)\right]^{(-1)^{n+1}}.$$

$$\mathcal{L}_{BG}\{f(t)^{(\ln t)^{n+1}}\}=\mathcal{L}_{BG}\left\{\left[f(t)^{(\ln t)^{n}}\right]^{\ln t}\right\}$$

$$=\left\{\frac{d^{\pi}}{ds^{\pi}}\left[\mathcal{L}_{BG}\left(f(t)^{(\ln t)^{n}}\right)\right]\right\}^{-1}$$

$$=\left\{\frac{d^{\pi}}{ds^{\pi}}\left[F_{BG}^{\pi(n)}(s)\right]^{(-1)^{n}}\right\}^{-1}$$

$$=\left\{\left[F_{BG}^{\pi(n+1)}(s)\right]^{(-1)^{n}}\right\}^{-1}.$$

Thus, we get

$$\mathcal{L}_{BG}\{f(t)^{(\ln t)^{n+1}}\}=\left[F_{BG}^{\pi(n+1)}(s)\right]^{(-1)^{n+1}}$$

for $k=n+1$. And, this proves the theorem.

By the theorem above we can write

$$\mathcal{L}_{BG}\{f(t)^{(\ln t)^n}\} = \left[F_{BG}^{\pi(n)}(s)\right]^{(-1)^n} = \begin{cases} 1/F_{BG}^{\pi(n)}(s), & \text{if } n \text{ is odd,} \\ F_{BG}^{\pi(n)}(s), & \text{if } n \text{ is even.} \end{cases} \quad (47)$$

Definition 19. (Bigeometric Convolution Property)

Let $f(t)$ and $g(t)$ be two bigeometric functions. In this case, the convolution of $f(t)$ with $g(t)$ is defined as follows;

$$f(t) *^{\pi} g(t) = \pi \int_{x=1}^{t} f(x) \odot g(t \ominus x) dx^{\pi}. \quad (48)$$

According to this definition the following equation can be written;

$$f(t) *^{\pi} g(t) = \pi \int_{x=1}^{t} [f(x)]^{(\ln\circ g)\left(\frac{t}{x}\right)} \cdot dx^{\pi}. \quad (49)$$

Theorem 13. Let $f(t)$ and $g(t)$ be two bigeometric functions. In this case, the following equation is valid (Kaymak, 2023);

$$\mathcal{L}_{BG}\{f(t) *^{\pi} g(t)\} = \mathcal{L}_{BG}\{f(t)\} \odot \mathcal{L}_{BG}\{g(t)\}. \quad (50)$$

Theorem 14. Let $f(t)$ be a bigeometric function. In this case, the following equation is valid for $t = e^{\tau}$ (Kaymak, 2023);

$$\left(\ln\circ f^{\pi(n)} \circ \exp\right)(\tau) = (\ln\circ f \circ \exp)^{(n)}(\tau). \quad (51)$$

Result 1. From the theorem 13 and theorem 14 above, the following equality is satisfied;

$$f^{\pi(n)}(t) = e^{(\ln\circ f\circ \exp)^{(n)}(\tau)}, \ (t = e^{\tau}). \quad (52)$$

Theorem 15. The bigeometric Laplace integral transform of the first-order bigeometric derivative is given as follows (Kaymak, 2023);

$$\mathcal{L}_{BG}\{f^{\pi}(t)\} = [F_{BG}(s)]^{\ln s} \cdot [f(1)]^{-1}. \quad (53)$$

Theorem 16. The bigeometric Laplace integral transform of the n^{th}-order bigeometric derivative is given as follows (Kaymak, 2023);

$$\mathcal{L}_{BG}\{f^{\pi(n)}(t)\} = [F_{BG}(s)]^{(\ln s)^n} \cdot \left[\prod_{k=1}^{n} \left(f^{\pi(k-1)}(1)\right)^{(\ln s)^{n-k}}\right]^{-1}. \quad (54)$$

Definition 20. $F_{BG}: \mathbb{R}_{exp} \to \mathbb{R}_{exp}$ is a given bigeometric function and $f: \mathbb{R}_{exp}^{+} \to \mathbb{R}_{exp}$ is a bigeometric piecewise continuous function which is of α −bigeometric exponential order such as $\mathcal{L}_{BG}\{f(t)\} = F_{BG}(s)$.

Then, the bigeometric function $f(t)$ is called the bigeometric inverse Laplace transform of $F_{BG}(s)$ and it is shown as;

$$f(t) = \mathcal{L}_{BG}^{-1}\{F_{BG}(s)\}. \quad (55)$$

Theorem 17. The bigeometric inverse Laplace transform is also bigeometrically linear. In other words, if k_1, k_2 are arbitrary constant exponents and $f_1(t)$, $f_2(t)$ are two given continuous functions, which have bigeometric Laplace transforms $\mathcal{L}_{BG}\{f_1(t)\} = F_1$, $\mathcal{L}_{BG}\{f_2(t)\} = F_2$, respectively. Then (Kaymak, 2023);

$$\mathcal{L}_{BG}^{-1}\{F_1^{k_1} \cdot F_2^{k_2}\} = \mathcal{L}_{BG}^{-1}\{F_1\}^{k_1} \cdot \mathcal{L}_{BG}^{-1}\{F_2\}^{k_2}. \quad (56)$$

Applications to bigeometric linear differential equations

In this section, the aim is to show how bigeometric Laplace transform is used to solve initial-value problems for bigeometric linear differential equations (Kaymak, 2023). Bigeometric Laplace transform is especially useful for bigeometric linear differential equations with constant exponents. The solution is obtained by applying the bigeometric Laplace transform to both sides of such an equation.

Example 4.1. Consider the following bigeometric differential equation with the initial condition $y^{\pi}(t) \cdot [y(t)]^3 = e^{6\ln t+5}, y(1) = e$.

Here, $y: \mathbb{R}_{exp}^{+} \to \mathbb{R}_{exp}$ be a bigeometric function and $\mathcal{L}_{BG}\{y(t)\} = Y_{BG}(s)$ be its bigeometric Laplace integral transform. Taking the bigeometric Laplace transform of both sides of the bigeometric differential equation and using the given conditions, we have

$$\mathcal{L}_{BG}\{y^{\pi}(t)\cdot[y(t)]^3\} = \mathcal{L}_{BG}\{e^{6\ln t+5}\}$$
$$\mathcal{L}_{BG}\{y^{\pi}(t)\}\cdot\mathcal{L}_{BG}\{[y(t)]^3\} = e^{\mathcal{L}\{6\tau+5\}}, \qquad (\tau = \ln t)$$
$$[Y_{BG}(s)]^{\ln s}\cdot[y(1)]^{-1}\cdot[Y_{BG}(s)]^3 = e^{\frac{6}{\sigma^2}+\frac{5}{\sigma}}, \qquad (\sigma = \ln s)$$
$$[Y_{BG}(s)]^{\ln s+3} = y(1)\cdot\left(e^{\frac{6}{\sigma^2}+\frac{5}{\sigma}}\right)$$
$$[Y_{BG}(s)]^{\ln(s)+3} = e\cdot e^{\frac{6}{\sigma^2}+\frac{5}{\sigma}} = e^{\frac{\sigma^2+5\sigma+6}{\sigma^2}}$$
$$[Y_{BG}(s)]^{\ln(s)+3} = e^{\frac{\ln^2(s)\,+5\ln(s)+6}{\ln^2(s)}} = e^{\frac{(\ln(s)+2)\cdot(\ln(s)+3)}{\ln^2(s)}}$$
$$Y_{BG}(s) = e^{\frac{(\ln(s)+2)\cdot(\ln(s)+3)}{\ln^2(s)}\cdot\frac{1}{(\ln(s)+3)}} = e^{\frac{(\ln(s)+2)}{\ln^2(s)}} = e^{\frac{1}{\ln(s)}}\cdot e^{\frac{2}{\ln^2(s)}}.$$

Now, taking the bigeometric inverse Laplace transform, we obtain

$$\mathcal{L}_{BG}^{-1}\{Y_{BG}(s)\} = \mathcal{L}_{BG}^{-1}\left\{e^{\frac{1}{\ln(s)}}\cdot e^{\frac{2}{\ln^2(s)}}\right\}$$
$$\mathcal{L}_{BG}^{-1}\{Y_{BG}(s)\} = \mathcal{L}_{BG}^{-1}\left\{e^{\frac{1}{\ln(s)}}\cdot\left[e^{\frac{1}{\ln^2(s)}}\right]^2\right\} = \mathcal{L}_{BG}^{-1}\left\{e^{\frac{1}{\ln(s)}}\right\}\cdot\mathcal{L}_{BG}^{-1}\left\{e^{\frac{1}{\ln^2(s)}}\right\}^2$$
$$y(t) = e^1\cdot\left(e^{\ln t}\right)^2$$
$$y(t) = e\cdot t^2.$$

CONCLUSION

In this study, the definition of the bigeometric Laplace integral transform in bigeometric analysis is made and some basic properties of this new transform are examined. It has been seen that the bigeometric Laplace integral transform has properties such as linearity, first shifting, second shifting, change of scale and bigeometric convolution. In addition, the existence of the bigeometric Laplace transform has been proven. Then, the bigeometric Laplace transform of the bigeometric derivative of the bigeometric function is given and the bigeometric inverse Laplace transform is defined. Finally, solutions of some bigeometric initial-value problems are investigated with the help of bigeometric Laplace integral transform.

Conflict of Interest

The article authors declare that there is no conflict of interest between them.

Author's Contributions

The authors declare that they have contributed equally to the article.

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