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Research Article

On an attraction-repulsion chemotaxis model involving logistic source

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Abstract

This paper is concerned with the attraction-repulsion chemotaxis system involving logistic source: $u_t = \Delta u - \chi\nabla\cdot(u\nabla\upsilon) + \xi\nabla\cdot(u\nabla\omega) + f(u)$, $\rho\upsilon_t = \Delta\upsilon - \alpha_1\upsilon + \beta_1 u$, $\rho\omega_t = \Delta\omega - \alpha_2\omega + \beta_2 u$ under homogeneous Neumann boundary conditions with nonnegative initial data $(u_0, \upsilon_0, \omega_0) \in \left(W^{1,\infty}(\Omega)\right)^3$, the parameters χ, ξ, α_1, α_2, β_1, $\beta_2 > 0$, $\rho \geq 0$ subject to the non-flux boundary conditions in a bounded domain $\Omega \subset \mathbb{R}^N (N \geq 3)$ with smooth boundary and $f(u) \leq au - \mu u^2$ with $f(0) \geq 0$ and $a \geq 0$, $\mu > 0$ for all $u > 0$. Based on the maximal Sobolev regularity and semigroup technique, it is proved that the system admits a globally bounded classical solution provided that $\chi + \xi < \frac{\mu}{2}$ and there exists a constant $\beta_* > 0$ is sufficiently small for all β_1, $\beta_2 < \beta_*$.

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Keywords. attraction-repulsion, chemotaxis, logistic source, global existence

1. Introduction

We deal with the attraction-repulsion chemotaxis system with logistic source:

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla\upsilon) + \xi\nabla\cdot(u\nabla\omega) + f(u), & (x,t) \in \Omega\times(0,T), \\ \rho\upsilon_t = \Delta\upsilon - \alpha_1\upsilon + \beta_1 u, & (x,t) \in \Omega\times(0,T), \\ \rho\omega_t = \Delta\omega - \alpha_2\omega + \beta_2 u, & (x,t) \in \Omega\times(0,T), \\ \frac{\partial u}{\partial\nu} = \frac{\partial\upsilon}{\partial\nu} = \frac{\partial\omega}{\partial\nu} = 0, & (x,t) \in \partial\Omega\times(0,T), \\ u(x,0) = u_0(x), \rho\upsilon(x,0) = \rho\upsilon_0(x), \rho\omega(x,0) = \rho\omega_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a bounded domain with smooth boundary, $\rho \geq 0$, $\frac{\partial}{\partial\nu}$ is the derivative with respect to the outer normal of $\partial\Omega$ and nonnegative initial data $(u_0, \upsilon_0, \omega_0)$ satisfying suitable regularity, the parameters χ, ξ, α_1, α_2, β_1, $\beta_2 > 0$, where χ and ξ are respectively measure the strength of the attraction and repulsion and $u(x,t)$, $\upsilon(x,t)$, $\omega(x,t)$ denote the cell density, the chemoattractant concentration, the chemorepellent concentration, respectively and the logistic source $f \in C^\infty([0,\infty))$ fulfills

$$f(u) \leq au - \mu u^2 \text{ with } f(0) \geq 0, a \geq 0, \mu > 0. \tag{1.2}$$

The second and the third equations in the system (1.1) state that the chemoattractant and the chemorepellent are released by cells and undergo decay. The kinetic term $f(u)$ depicts cell proliferation and cell death.

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Next, we denote $L^p(\Omega)$, $W^{1,p}(\Omega)$, $W^{2,p}(\Omega)$ as usual Lebesgue and Sobolev spaces. The norms of $L^p(\Omega)$, $W^{1,p}(\Omega)$ and $W^{2,p}(\Omega)$ are denoted by $\|u\|_{L^p(\Omega)} := \|u\|_p$, $\|u\|_{W^{1,p}(\Omega)} := \|u\|_{1,p} = \|u\|_p + \|\nabla u\|_p$, $\|u\|_{W^{2,p}(\Omega)} := \|u\|_{2,p} = \|u\|_p + \|\Delta u\|_p$ $(1 \le p \le \infty)$ respectively.

System (1.1) is a generalized version of the classical Keller-Segel model which represents a biological process in which cells interact with a combination of repulsive and attractive signal chemicals (see [12,14,15,33–35,37]).

In order to understand our paper better, we recall some papers concerning the system (1.1). Let's begin with $\xi = 0$. In this case the repulsive signal vanishes, ω is decoupled from system (1.1), and the system becomes

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla v) + f(u), (x,t) \in \Omega \times (0,T), \\ \rho v_t = \Delta v - \alpha v + \beta u, \qquad (x,t) \in \Omega \times (0,T). \end{cases} \tag{1.3}$$

In mathematical biology, system (1.3) is used to model the mechanism of chemotaxis, which is a mechanism by which cells and organisms efficiently respond to chemical stimuli in their environment, moving toward beneficial targets or environments and avoiding undesired ones. One of the best-known examples of chemotaxis is the action of the bacteria Escherichia Coli (E. Coli). The chemotaxis model also known as Keller-Segel model was introduced mathematically by Keller and Segel in [19, 20]. The research of the Keller-Segel model has attracted and continues to attract the attention of mathematicians since its establishment. The correlation study can be divided into two types: parabolic-elliptic chemotaxis model i.e. $\rho = 0$ (see [3,39,42,47]) and parabolic-parabolic chemotaxis model i.e. $\rho \neq 0$ (see [1, 2, 17, 38, 41, 43, 46]). We refer the reader to Horstmann's paper [16] to have general knowledge about the Keller-Segel model. In [16], Horstmann summarized the results in the literature for some general forms of the classical Keller-Segel model and presented possible generalizations for more comprehensive models.

In general, because of the chemical molecules are much smaller than cells in size, chemicals diffuse much faster than cells. Hence the attraction-repulsion chemotaxis model (1.1) can be approximated by setting $\rho = 0$. In particular, when $f(u) = 0$ (the logistic source vanishes), whether or not solutions of system (1.3) blow-up in finite time has been researched. In summary, in case $\rho \ge 0$, when $N = 1$ Nagai [29] and Osaki et al. [31] obtained that the solution of the system (1.3) never blows up whereas there exists finite-time or infinite-time blow-up of the solution to the system (1.3) when $N \ge 3$ (see [16,29,44]). For the case $N = 2$, if the initial data and the domain have radial symmetry, whether will or not blow-up depends on the size of the initial data (see [29,30]). System (1.3) has been studied by many authors with logistic source $f(u) \neq 0$ (see [32, 39, 43]). In particular, if the logistic source $f(u) \le a - bu^2$ with some $a \ge 0$ and $b > 0$, Tello and Winkler [39] obtained an unique global bounded classical solution of the system (1.3) posed on a bounded domain. The damping power $\tau = 2$ in the logistic term $f(u) \le au - bu^\tau$ plays an important role in many works. For example, in [53], when $\rho = 1$, the authors dealt with the system (1.3) with logistic term $f(u) = au - \mu u^2$ with the parameters $a \in \mathbb{R}$, $\mu > 0$. The authors proved that if $\mu > 0$, then the system (1.3) has a global weak solution, and if $\mu > \frac{(N-2)_+}{N}\chi C^{\frac{1}{\frac{N}{2}+1}}_{\frac{N}{2}+1}$ where $C^{\frac{1}{\frac{N}{2}+1}}_{\frac{N}{2}+1} > 0$ is a constant which is corresponding to the maximal Sobolev regularity, then there is a bounded global classical solution of the system (1.3). Moreover, the authors also showed that if $a = 0$ and $\mu > \frac{(N-2)_+}{N}\chi C^{\frac{1}{\frac{N}{2}+1}}_{\frac{N}{2}+1}$, then both $u(\cdot,t)$ and $v(\cdot,t)$ functions decay to zero with respect to the norm in $L^\infty(\Omega)$ as $t \to \infty$. When $\rho = 0$, Winkler [42] studied the the system (1.3) in a smooth bounded domain Ω under the assumption that the generalizes the logistic function $f(u) = au - bu^\tau$ with $a \ge 0$, $b > 0$. The author introduced concept of very weak solutions to system (1.3), and obtained the existence of global solutions for any nonnegative initial data $u_0 \in L^1(\Omega)$ under the assumption that $\tau > 2 - \frac{1}{N}$, $(N \ge 2)$. Moreover, the boundedness properties of the constructed solutions are studied. In addition to these, the author showed that the solution is globally bounded if b is sufficiently large and $u_0 \in L^\infty(\Omega)$ has small norm in $L^\gamma(\Omega)$ for some $\gamma > \frac{N}{2}$. More recently, in [13] Ei et al. considered the Keller-Segel system with a logistic growth term from the spatio-temporal-oscillation point of view and showed that there are two different types of spatio-temporal oscillations of the system in certain distinct parameter regimes (see also [43, 49]).

When there is a repulsive signal, i.e. $\xi \neq 0$, Luca et al. [26] used the system (1.1) to identify the aggregation of microglia observed in Alzheimer's disease and Painter and Hillen [33] used the system (1.1) to address the quorum effect in the chemotactic process. For the case $N = 1$, when $f(u) = 0$ and $\rho = 1$, Jin [18], Liu and Wang [24] studied the existence of global solutions and of non-trivial steady states to system (1.1). Then in [36], Tao and Wang showed that if repulsion prevails over attraction in the sense that $\xi\beta_2 - \chi\beta_1 > 0$ then system (1.1) with $\rho = 0$ is globally well posed in the high dimensions ($N > 2$), and if repulsion dominates over attraction in the sense that $\xi\beta_2 - \chi\beta_1 > 0$ that the system (1.1) with $\rho = 1$ is globally well-posed in two dimensions ($N = 2$). In [23], under the critical condition that $\chi\beta_1 - \xi\beta_2 = 0$ the authors proved that the system (1.1) with $\rho = 1$ possesses a unique global solution, which is uniformly bounded in the physical domain $\Omega \subset R^N$ ($N = 2, 3$) (see also [25, 40, 50]). Recently, Liu et al. [24] studied analytically and numerically the pattern formation of the system (1.1) with $\rho = 1$. There are some papers that have also been studied under suitable conditions with related to the logistics source. In [41], when $\rho = 1$, the authors studied the global boundedness of solutions to the fully parabolic (parabolic-parabolic-parabolic) attraction-repulsion chemotaxis system (1.1) with logistic source $f(u) = a - bu^\theta$ for all $u \geq 0$ with $a \geq 0$, $b > 0$ and $\theta \geq 1$. It was shown that when the attraction cancels the repulsion (i.e. $\chi\beta_1 = \xi\beta_2$), the solution is globally bounded if $N \leq 3$, or $\theta > \theta_N := \min\left\{\frac{N+2}{4}, \frac{N\sqrt{N^2+6N+17}-N^2-3N+4}{4}\right\}$ with $N \geq 2$. When $f(u) \neq 0$, we refer to [48, 51].

In addition to these, in the past decades, scholars have conducted extensive studies on the behavior of predator-prey models with different effects involving chemotaxis. Compared with the traditional diffusion progress (random movement), the ecological models are more realistic when involving chemotaxis. In [11], the author investigated the spatiotem poral inhomogeneous pattern phenomenon of a predator–prey model with chemotaxis and time delay. Interested readers may refer to [7–10, 27, 28] and the references therein for more interesting results concerning the pattern formation of the reaction–diffusion models with chemotaxis.

In this paper when $N \geq 3$, we obtain a globally bounded classical solution of the system (1.1) by using the maximal Sobolev regularity and semigroup technique with $\chi + \xi < \frac{\mu}{2}$ and $\beta_* > 0$ is sufficiently small for all β_1, $\beta_2 < \beta_*$.

The main result of our this paper for $\rho = 1$ is expressed in Theorem 1.1 as follows.

Theorem 1.1. *Assume that $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a bounded domain with smooth boundary, the parametrs χ, ξ, α_1, α_2, β_1, $\beta_2 > 0$ and $\rho = 1$. Moreover, assume that the logistic source $f \in C^\infty([0,\infty))$ satisfies $f(u) \leq au - \mu u^2$ with $f(0) \geq 0$ and $a \geq 0$, $\mu > 0$. If*

$$\chi + \xi < \frac{\mu}{2}, \tag{1.4}$$

there exists a constant $\beta_* > 0$ is sufficiently small with β_1, $\beta_2 < \beta_*$, then for any nonnegative $(u_0, \upsilon_0, \omega_0) \in \left(W^{1,\infty}(\Omega)\right)^3$, the system (1.1) has a globally bounded classical solution (u, υ, ω) which is uniformly bounded in $\Omega \times (0, \infty)$ in the sense that there is a constant $C > 0$ such that

$$\|u(\cdot,t)\|_\infty + \|\upsilon(\cdot,t)\|_{1,\infty} + \|\omega(\cdot,t)\|_{1,\infty} \leq C \text{ for all } t > 0.$$

Remark 1.2. Li and Xiang [22], studied the system (1.1) in $\Omega \subset \mathbb{R}^N (N \geq 1)$ a bounded domain with smooth boundary, when logistic source $f(u) \leq a - bu^\tau$ for all $u > 0$, with some

$a \geq 0$, $b > 0$, and ρ, χ, ξ, $\chi \geq 0$, $\alpha_i > 0$ and $\beta_i > 0$ $(i = 1, 2)$. The authors showed that when the repulsion prevails over the attraction in the sense that $\xi\beta_2 - \chi\beta_1 > 0$, there exist global bounded classical solutions for any logistic damping $\tau \geq 1$. When the attraction dominates the repulsion in the sense that $\xi\beta_2 - \chi\beta_1 < 0$, the classical solutions are still global and bounded provided that the logistic damping is strong. For the case $\rho > 0$, the authors investigated the similar problem for $N = 1$ and $N = 2$. In a recent preprint Li [21], studied the system (1.1) for $N = 2$, $\rho = 1$ when logistic source $f(u) \leq au - bu^{\tau+1}$ with $a > 0$, $b > 0$, $\tau > 0$ and χ, $\xi > 0$, $\alpha_i > 0$, $\beta_i > 0$ $(i = 1, 2)$. Li proved that the global solution is bounded by using a different method. For the general case $\tau \geq 1$ or $N \geq 3$, however, the global existence of classical solutions for the parabolic-parabolic system (1.1) is still open. When $N \geq 3$, Zheng et al. [54] solved this problem partly (i.e. for $\alpha_1 = \alpha_2$) under the condition $\frac{\chi\beta_1+\xi\beta_2}{\mu} < \theta_0$ for some $\theta_0 > 0$.

When $N \geq 3$, we generalize and develop the results obtained by Li and Xiang [22], Zheng et al. [54] by getting $\alpha_1 \neq \alpha_2$ in the system (1.1).

The rest of the paper is organized as follows. We introduce the local existence of classical solution to the system (1.1) and necessary preliminary lemmas which play very important role in the proof of Theorem 1.1 as preliminaries in Section 2. In Section 3, we deal with the global boundedness of solution to prove Theorem 1.1

2. Preliminaries

We can obtain the local existence of a classical solution to the system (1.1) for sufficiently smooth initial data by using standard parabolic regularity theory in a suitable fixed point framework. In fact, a sufficient condition can be derived for the extensibility of a given local-in-time solution (see [5, 43]).

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^N (N \geq 3)$ be a bounded domain with smooth boundary, the parameters χ, ξ, α_1, α_2, β_1, $\beta_2 > 0$. Assume that the nonnegative initial data satisfy $(u_0, \upsilon_0, \omega_0) \in \left(W^{1,\infty}(\Omega)\right)^3$ and $f(u) \leq au - \mu u^2$ with $a \geq 0$, $\mu > 0$. Then there is a maximal existence time $T_{\max} \in (0, \infty]$ and a unique triple (u, υ, ω) of nonnegative bounded functions belong to $C^0(\overline{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max}))$ with $\rho = 1$ which is a classical solution of system (1.1) in $\Omega \times (0, T_{\max}))$. Moreover, if $T_{\max} < \infty$, then*

$$\lim_{t \to T_{\max}} \left(\|u(\cdot, t)\|_\infty + \|\upsilon(\cdot, t)\|_{1,\infty} + \|\omega(\cdot, t)\|_{1,\infty} \right) = \infty.$$

We will use the following property, referred to as a variation of the Maximal Sobolev Regularity property associated with the second equation in (1.1), which will use in the proof of our main result.

Lemma 2.2. *[6, 52]. Take $r \in (1, \infty)$, $\mu, \eta > 0$ and $U \in L^r((0, T); L^r(\Omega))$, $T \in (0, \infty]$. Let Z be the unique strong solution of the following initial boundary value problem*

$$\begin{cases} Z_t = \Delta Z - \mu Z + \eta U, & (x, t) \in \Omega \times (0, T), \\ \frac{\partial Z}{\partial \nu} = 0, & (x, t) \in \partial\Omega \times (0, T), \\ Z(x, 0) = Z_0(x), & x \in \Omega, \end{cases}$$

then there exists $C_r > 0$ depends on r and Ω, such that if $s_0 \in [0, T)$, satisfies $Z(\cdot, s_0) \in W^{2,r}(\Omega)$, $r > N$ with $\frac{\partial Z(\cdot, s_0)}{\partial \nu} = 0$ on $\partial\Omega$, and

$$\begin{aligned} &\int_{s_0}^{T} \int_\Omega e^{\mu r s} |\Delta Z|^r \, dx ds \\ \leq \; & C_r \eta^r \int_{s_0}^{T} \int_\Omega e^{\mu r s} U^r dx ds + C_r e^{\mu r s_0} \left(\|Z(\cdot, s_0)\|_r^r + \|\Delta Z(\cdot, s_0)\|_r^r \right). \end{aligned} \tag{2.1}$$

Now, we give some knowledge about the Laplacian in Ω, which is equipped with homogeneous Neumann boundary conditions, which are used in the proof of L^∞-boundedness of solutions for (1.1). For the proofs, please see [17, 44, 45].

We next give some properties of the Neumann heat semigroup which will be used later. For the proof, see Lemma 2.1 in [4] and Lemma 1.3 in [44].

Lemma 2.3. *Assume that $(e^{t\Delta})t \geq 0$ is the Neumann heat semigroup in Ω, and let $\mu_1 > 0$ denote the first nonzero eigenvalue of $-\Delta$ in Ω under Neumann boundary conditions. Then there are $k_1, k_2, k_3 > 0$ which only depend on Ω and have the following properties:*

(i) If $1 \leq q \leq p \leq \infty$, then

$$\left\| e^{t\Delta} z \right\|_p \leq k_1 t^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{p})} e^{-\mu_1 t} \|z\|_q, \forall t > 0$$

holds for all $z \in L^q(\Omega)$ satisfying $\int_\Omega z = 0$.

(ii) If $1 \leq q \leq p \leq \infty$, then

$$\left\| \nabla e^{t\Delta} z \right\|_p \leq k_2 \left(1 + t^{-\frac{1}{2}-\frac{N}{2}(\frac{1}{q}-\frac{1}{p})} \right) e^{-\mu_1 t} \|z\|_q, \forall t > 0$$

is true for each $z \in L^q(\Omega)$.

(iii) If $1 < q \leq p < \infty$, then

$$\left\| e^{t\Delta} \nabla \cdot z \right\|_p \leq k_3 \left(1 + t^{-\frac{1}{2}-\frac{N}{2}(\frac{1}{q}-\frac{1}{p})} \right) e^{-\mu_1 t} \|z\|_q, \forall t > 0 \tag{2.2}$$

is valid for any $z \in (C_0^\infty(\Omega))^N$, where $e^{t\Delta}\nabla\cdot$ is the extension of the operator $e^{t\Delta}\nabla\cdot$ on $(C_0^\infty(\Omega))^N$ to $(L^q(\Omega))^N$. Consequently, for all $t > 0$ the operator $e^{t\Delta}\nabla\cdot$ possesses a uniquely determined extension to an operator from $L^q(\Omega)$ into $L^p(\Omega)$, with norm controlled according to (2.2)*.*

3. Proof of main result

In this section, we study the proof of Theorem 1.1. We will use the maximal Sobolev regularity and semigroup technique to obtain a globally bounded classical solution of the system (1.1) under suitable conditions. The regularity obtained in (2.1) requires that the initial data satisfy homogeneous Neumann boundary conditions. Therefore, we will perform a small time shift and thus use any positive time as the "initial time" to guarantee that the respective boundary condition is satisfied naturally. Specifically, given any $s_0 \in (0, T_{\max})$ such that $s_0 \leq 1$, from the regularity principle asserted by Lemma 2.1 we know that $(u(\cdot, s_0), v(\cdot, s_0), \omega(\cdot, s_0)) \in \left(C^2(\overline{\Omega})\right)^3$ with $\frac{\partial v(\cdot, s_0)}{\partial \nu} = 0$ on $\partial\Omega$ so that in particular we can pick $M > 0$ such that

$$\sup_{0 \leq s \leq s_0} \|u(\cdot, s)\|_\infty \leq M, \sup_{0 \leq s \leq s_0} \|v(\cdot, s)\|_\infty \leq M, \|\Delta v(\cdot, s_0)\|_\infty \leq M \tag{3.1}$$

as well

$$\sup_{0 \leq s \leq s_0} \|\omega(\cdot, s)\|_\infty \leq M, \|\Delta \omega(\cdot, s_0)\|_\infty \leq M.$$

Now, we give the lemmas to be used to prove Theorem 1.1.

Lemma 3.1. *Under the assumptions of Theorem 1.1, let (u, v, ω) be a solution to* (1.1) *on $t \in (0, T_{\max})$. Then we have*

$$\|u(\cdot, t)\|_k \leq C \text{ for all } t \in (s_0, T_{\max})$$

for all $k > 1$ and $C = C(k, a, \alpha_1, \alpha_2, \mu, |\Omega|) > 0$.

Proof. For any $k > 1$ multiplying the first equation in (1.1) with u^{k-1} and using condition (1.2), integrating by parts, we have

$$\begin{aligned}
&\frac{1}{k}\frac{d}{dt}\int_\Omega u^k dx \\
\leq\ & -(k-1)\int_\Omega u^{k-2}|\nabla u|^2 dx + \chi(k-1)\int_\Omega u^{k-1}\nabla u\cdot\nabla v dx \\
& -\xi(k-1)\int_\Omega u^{k-1}\nabla u\cdot\nabla\omega dx + a\int_\Omega u^k dx - \mu\int_\Omega u^{k+1}dx \\
\leq\ & -\frac{\chi(k-1)}{k}\int_\Omega u^k\Delta v dx + \frac{\xi(k-1)}{k}\int_\Omega u^k\Delta\omega dx \\
& +a\int_\Omega u^k dx - \mu\int_\Omega u^{k+1}dx \\
=\ & -\frac{\chi(k-1)}{k}\int_\Omega u^k\Delta v dx + \frac{\xi(k-1)}{k}\int_\Omega u^k\Delta\omega dx \\
& +\left(a + \frac{\alpha(k+1)}{k}\right)\int_\Omega u^k dx - \frac{\alpha(k+1)}{k}\int_\Omega u^k dx - \mu\int_\Omega u^{k+1}dx \qquad (3.2)
\end{aligned}$$

for all $t \in (0, T_{\max})$, where $\alpha = \max\{\alpha_1, \alpha_2\}$. From the following inequality

$$a_0\xi^{i_0} - b_0\xi^{j_0} \leq a_0\left(\frac{a_0}{b_0}\right)^{\frac{i_0}{j_0-i_0}}, \forall\xi > 0, \qquad (3.3)$$

where $a_0 \geq 0$, $b_0 > 0$ and $0 \leq i_0 < j_0$, we see that

$$\left(a + \frac{\alpha(k+1)}{k}\right)\int_\Omega u^k dx - \frac{\mu}{2}\int_\Omega u^{k+1}dx \leq B_0, \qquad (3.4)$$

where $B_0 = \left(\frac{ak+\alpha(k+1)}{k}\right)\left(\frac{2(ak+\alpha(k+1))}{k\mu}\right)^k|\Omega| > 0$. On the other hand, Young's inequality also implies that

$$\begin{aligned}
-\frac{\chi(k-1)}{k}\int_\Omega u^k\Delta v dx &\leq \chi\int_\Omega u^k|\Delta v| dx \\
&\leq \frac{\chi k}{k+1}\int_\Omega u^{k+1}dx + \frac{\chi}{k+1}\int_\Omega|\Delta v|^{k+1}dx, \qquad (3.5)
\end{aligned}$$

and

$$\frac{\xi(k-1)}{k}\int_\Omega u^k\Delta\omega dx \leq \frac{\xi k}{k+1}\int_\Omega u^{k+1}dx + \frac{\xi}{k+1}\int_\Omega|\Delta\omega|^{k+1}dx. \qquad (3.6)$$

Then by (3.2), (3.4), (3.5) and (3.6), we have

$$\begin{aligned}
\frac{1}{k}\frac{d}{dt}\int_\Omega u^k dx + \frac{\alpha(k+1)}{k}\int_\Omega u^k dx \leq\ & \frac{1}{k+1}\left(\chi\int_\Omega|\Delta v|^{k+1}dx + \xi\int_\Omega|\Delta\omega|^{k+1}dx\right) \\
& -\left(\frac{\mu}{2} - \frac{k(\chi+\xi)}{k+1}\right)\int_\Omega u^{k+1}dx + B_0 \qquad (3.7)
\end{aligned}$$

with $\frac{\mu}{2}-\frac{k(\chi+\xi)}{k+1}>0$ (by the condition (1.4)). By applying the variation-of-constants formula to (3.7), we have

$$\begin{aligned}
&\frac{1}{k}\int_\Omega u^k dx\\
\leq\;& \frac{1}{k+1}\int_{s_0}^t e^{-\alpha(k+1)(t-s)}\left(\chi\int_\Omega |\Delta v|^{k+1}dx+\xi\int_\Omega|\Delta\omega|^{k+1}dx\right)ds\\
&-\left(\frac{\mu}{2}-\frac{k(\chi+\xi)}{k+1}\right)\int_{s_0}^t e^{-\alpha(k+1)(t-s)}\int_\Omega u^{k+1}dxds+B_1\\
\leq\;& \frac{1}{k+1}e^{-\alpha(k+1)t}\left(\chi\int_{s_0}^t\int_\Omega e^{\alpha(k+1)s}|\Delta v|^{k+1}dxds+\xi\int_{s_0}^t\int_\Omega e^{\alpha(k+1)s}|\Delta\omega|^{k+1}dxds\right)\\
&-\left(\frac{\mu}{2}-\frac{k(\chi+\xi)}{k+1}\right)e^{-\alpha(k+1)t}\int_{s_0}^t\int_\Omega e^{\alpha(k+1)s}u^{k+1}dxds+B_1,
\end{aligned}\tag{3.8}$$

where

$$B_1=\max_{t>s_0}\left(\frac{1}{k}e^{-\alpha(k+1)(t-s_0)}\int_\Omega u^k(\cdot,s_0)\,dx+B_0\int_{s_0}^t e^{-\alpha(k+1)(t-s)}ds\right).$$

By Lemma 2.2 to first term of last inequality in (3.8), we know that there exists a $C_{k+1}>0$ such that

$$\begin{aligned}
&\frac{\chi}{k+1}e^{-\alpha(k+1)t}\int_{s_0}^t\int_\Omega e^{\alpha(k+1)s}|\Delta v|^{k+1}dxds\\
\leq\;&\frac{\chi}{k+1}C_{k+1}\beta_1^{k+1}e^{-\alpha(k+1)t}\int_{s_0}^t\int_\Omega e^{\alpha(k+1)s}u^{k+1}dxds+B_2,
\end{aligned}\tag{3.9}$$

where

$$B_2=\frac{\chi}{k+1}C_{k+1}\max_{t>s_0}e^{-\alpha(k+1)(t-s_0)}\|v(\cdot,s_0)\|_{2,k+1}^{k+1},$$

and

$$\begin{aligned}
&\frac{\xi}{k+1}e^{-\alpha(k+1)t}\int_{s_0}^t\int_\Omega e^{\alpha(k+1)s}|\Delta\omega|^{k+1}dxds\\
\leq\;&\frac{\xi}{k+1}C_{k+1}\beta_2^{k+1}e^{-\alpha(k+1)t}\int_{s_0}^t\int_\Omega e^{\alpha(k+1)s}u^{k+1}dxds+B_3,
\end{aligned}\tag{3.10}$$

where

$$B_3=\frac{\xi}{k+1}C_{k+1}\max_{t>s_0}e^{-\alpha(k+1)(t-s_0)}\|\omega(\cdot,s_0)\|_{2,k+1}^{k+1}.$$

Then, from (3.9) and (3.10), we can write

$$\begin{aligned}
&\frac{1}{k+1}e^{-\alpha(k+1)t}\int_{s_0}^t\left(\chi\int_\Omega e^{\alpha(k+1)s}|\Delta v|^{k+1}dx+\xi\int_\Omega e^{\alpha(k+1)s}|\Delta\omega|^{k+1}dx\right)ds\\
\leq\;&\frac{C_{k+1}\left(\beta_1^{k+1}+\beta_2^{k+1}\right)}{k+1}(\chi+\xi)\,e^{-\alpha(k+1)t}\int_{s_0}^t\int_\Omega e^{\alpha(k+1)s}u^{k+1}dxds+B_4,
\end{aligned}\tag{3.11}$$

where $B_4=B_2+B_3$. Let $C_{k+1}>0$ be a constant which is corresponding to the Maximal Sobolev Regularity denoted in Lemma 2.2 for $k\in(1,\infty)$. Now we can find $\beta_*>0$ small enough such that

$$C_{k+1}\left(\beta_1^{k+1}+\beta_2^{k+1}\right)\leq 1\tag{3.12}$$

for all β_1, $\beta_2 < \beta_*$. Inserting (3.11) into (3.8), we derive

$$\begin{aligned} &\frac{1}{k}\int_\Omega u^k dx \\ \leq\ & -\left(\frac{\mu}{2}-\frac{C_{k+1}\left(\beta_1^{k+1}+\beta_2^{k+1}\right)+k}{k+1}(\chi+\xi)\right)e^{-\alpha(k+1)t}\int_{s_0}^t\int_\Omega e^{\alpha(k+1)s}u^{k+1}dxds+B_5 \\ =\ & -(\chi+\xi)\left(\frac{\mu}{2(\chi+\xi)}-\frac{C_{k+1}\left(\beta_1^{k+1}+\beta_2^{k+1}\right)+k}{k+1}\right)e^{-\alpha(k+1)t}\int_{s_0}^t\int_\Omega e^{\alpha(k+1)s}u^{k+1}dxds+B_5, \end{aligned}$$

where

$$\begin{aligned} B_5 &= \max_{t>s_0}\left(\frac{1}{k}e^{-\alpha(k+1)(t-s_0)}\int_\Omega u^k(\cdot,s_0)\,dx+B_0\int_{s_0}^t e^{-\alpha(k+1)(t-s)}ds\right) \\ &\quad +\frac{C_{k+1}}{k+1}\max_{t>s_0}e^{-\alpha(k+1)(t-s_0)}\left(\chi\left\|\upsilon(\cdot,s_0)\right\|_{2,k+1}^{k+1}+\xi\left\|\omega(\cdot,s_0)\right\|_{2,k+1}^{k+1}\right) \\ &> 0 \end{aligned}$$

for all $t\in(0,T_{\max})$. By the conditions (3.12) and (1.4), we see that $\frac{\mu}{2(\chi+\xi)}-\frac{C_{k+1}\left(\beta_1^{k+1}+\beta_2^{k+1}\right)+k}{k+1}>0$ for all $k>1$. Therefore, we have

$$\int_\Omega u^k dx\leq C$$

for all $t\in(0,T_{\max})$ and $k>1$, where $C=kB_5$. The proof of Lemma 3.1 is completed. □

Next, we give the following $L^\infty(\Omega)$-estimates for $\nabla\upsilon$ and $\nabla\omega$.

Firstly, we show that there exists a constant $D>0$ such that $\left\|u(t)\right\|_1\leq D$ for all $t>0$. Integrating the first equation in (1.1) with respect to $x\in\Omega$, we have

$$\frac{d}{dt}\int_\Omega udx\leq a\int_\Omega udx-\mu\int_\Omega u^2dx. \tag{3.13}$$

From the (3.3) inequality, we have

$$au-\frac{\mu}{2}u^2\leq D_0 \tag{3.14}$$

with $D_0=a\left(\frac{2a}{\mu}\right)>0$. Then by (3.13) and (3.14), we obtain

$$\frac{d}{dt}\int_\Omega udx\leq-\frac{\mu}{2}\int_\Omega u^2dx+D_0\left|\Omega\right|. \tag{3.15}$$

Making use of Hölder's inequality, we obtain

$$\int_\Omega u^2dx\geq C_\Omega\left(\int_\Omega udx\right)^2,$$

for some $C_\Omega>0$. Combining this inequality with (3.15) gives

$$\frac{d}{dt}\int_\Omega udx+\frac{\mu C_\Omega}{2}\left(\int_\Omega udx\right)^2\leq D_0\left|\Omega\right|.$$

Then standard ODE theory implies that

$$\int_\Omega udx\leq D \text{ for all } t>0.$$

Lemma 3.2. *Let the conditions of Theorem 1.1 hold. Then there is $C>0$ such that*

$$\left\|\nabla\upsilon\right\|_\infty\leq C \text{ and } \left\|\nabla\omega\right\|_\infty\leq C$$

for all $t\in(s_0,T_{\max})$.

Proof. For some given $k > k_0 > N$, then there exist $\beta_* > 0$ and $C_0 > 0$ such that (by Lemma 3.1)

$$\|u(\cdot,t)\|_{k_0} \le C_0$$

for all $t \in (s_0, T_{\max})$. First, it follows from the variation-of-constants formula to the second equation in (1.1), we obtain

$$\upsilon(\cdot,t) \le e^{(\Delta-1)t}\upsilon(\cdot,s_0) + \int_{s_0}^{t} e^{(\Delta-1)(t-s)}\left(u(\cdot,s) - \upsilon(\cdot,s)\right) ds$$

for all $t \in (s_0, T_{\max})$. By the standard estimate for Neumann semigroup (see [17]) to the second and third equations in (1.1) and since u, υ and $\omega \ge 0$, by using Lemma 2.3 we have

$$\begin{aligned}
&\|\nabla\upsilon(\cdot,t)\|_\infty \\
\le\ & \left\|\nabla e^{(\Delta-\alpha_1)t}\upsilon(\cdot,s_0)\right\|_\infty + \beta_1\int_{s_0}^{t}\left\|\nabla e^{(\Delta-\alpha_1)(t-s)}u(\cdot,s)\right\|_\infty ds \\
\le\ & e^{-\alpha_1 t}\|\nabla\upsilon(\cdot,s_0)\|_\infty + \beta_1 C_1\int_{s_0}^{t}\left(1+(t-s)^{-\frac{1}{2}-\frac{N}{2k_0}}\right)e^{-\alpha_1(t-s)}\|u(\cdot,s)\|_{k_0} ds \\
\le\ & e^{-\alpha_1 t}\|\nabla\upsilon(\cdot,s_0)\|_\infty + \beta_1 C_1 C_3\int_{s_0}^{t}\left(1+(t-s)^{-\frac{1}{2}-\frac{N}{2k_0}}\right)e^{-\alpha_1(t-s)} ds \\
\le\ & C,
\end{aligned}$$

and similarly it follows from the variation-of-constants formula to the third equation in (1.1), we have

$$\omega(\cdot,t) \le e^{(\Delta-1)t}\omega(\cdot,s_0) + \int_{s_0}^{t} e^{(\Delta-1)(t-s)}\left(u(\cdot,s) - \omega(\cdot,s)\right) ds,$$

and

$$\begin{aligned}
&\|\nabla\omega(\cdot,t)\|_\infty \\
\le\ & \left\|\nabla e^{(\Delta-\alpha_2)t}\omega(\cdot,s_0)\right\|_\infty + \beta_2\int_{s_0}^{t}\left\|\nabla e^{(\Delta-\alpha_2)(t-s)}u(\cdot,s)\right\|_\infty ds \\
\le\ & e^{-\alpha_2 t}\|\nabla\omega(\cdot,s_0)\|_\infty + \beta_2 C_2\int_{s_0}^{t}\left(1+(t-s)^{-\frac{1}{2}-\frac{N}{2k_0}}\right)e^{-\alpha_2(t-s)}\|u(\cdot,s)\|_{k_0} ds \\
\le\ & e^{-\alpha_2 t}\|\nabla\omega(\cdot,s_0)\|_\infty + \beta_2 C_2 C_3\int_{s_0}^{t}\left(1+(t-s)^{-\frac{1}{2}-\frac{N}{2k_0}}\right)e^{-\alpha_2(t-s)} ds \\
\le\ & C
\end{aligned}$$

with $C_2 > 0$ and $C_3 > 0$, for all $t \in (s_0, T_{\max})$. The proof of Lemma 3.2 is completed. □

Lemma 3.3. *Let the conditions of Theorem 1.1 hold. Then there is $C_0 > 0$ such that*

$$\|u(\cdot,t)\|_\infty \le C_0$$

for all $t \in (0, T_{\max})$.

Proof. The variation-of-constants formula associated with the first equation in (1.1) represents u as

$$\begin{aligned} & u(\cdot,t) \\ = \; & e^{t(\triangle-1)}u(\cdot,s_0) - \chi\int_{s_0}^{t} e^{(t-s)(\triangle-1)}\nabla\cdot(u(\cdot,s)\nabla\upsilon(\cdot,s))\,ds \\ & +\xi\int_{s_0}^{t} e^{(t-s)(\triangle-1)}\nabla\cdot(u(\cdot,s)\nabla\omega(\cdot,s))\,ds \\ & +\int_{s_0}^{t} e^{(t-s)(\triangle-1)}\Big(au(\cdot,s)-\mu u^2(\cdot,s)\Big)\,ds \\ = \; & I_1(\cdot,t)+I_2(\cdot,t)+I_3(\cdot,t)+I_4(\cdot,t). \end{aligned}$$

Since u is nonnegative, we derive

$$\|u(\cdot,t)\|_\infty \le \|I_1(\cdot,t)\|_\infty + \|I_2(\cdot,t)\|_\infty + \|I_3(\cdot,t)\|_\infty + \|I_4(\cdot,t)\|_\infty \tag{3.16}$$

for all $t\in(s_0,T_{\max})$. Thanks to the boundedness of u, υ and ω in $\Omega\times(1,\infty)$, we employ the smoothing Neumann heat semigroup estimates and the fact that $e^{\delta(\triangle-1)}$ $(\delta>0)$ is order preserving, we have

$$\|I_1(\cdot,t)\|_\infty = \left\|e^{t(\triangle-1)}u(\cdot,s_0)\right\|_k \le \|u(\cdot,s_0)\|_k \le C_0 \tag{3.17}$$

for all $t\in(s_0,T_{\max})$ with $C_0>0$.

We set

$$M := \sup_{t\in[s_0,T]}\|u(\cdot,t)\|_\infty$$

for any $T\in(s_0,T_{\max})$. Use Lemma 2.3 with $p=\infty$, $q=k_0$, which leads to

$$\begin{aligned} & \|I_2(\cdot,t)\|_\infty \\ \le \; & C_4\int_{s_0}^{t}\left(1+(t-s)^{-\frac{1}{2}-\frac{N}{2k_0}}\right)e^{-\mu_1(t-s)}\|u(\cdot,s)\nabla\upsilon(\cdot,s)\|_{k_0}\,ds \end{aligned} \tag{3.18}$$

with $C_4>0$. Let's evaluate the expression $\|u(\cdot,s)\nabla\upsilon(\cdot,s)\|_{k_0}$. Here, we may assume that $\frac{N}{2}<\sigma_0<N$, and then we can fix $k_0>N$ such that $1-\frac{(N-\sigma_0)k_0}{N\sigma_0}>0$, which enables us to pick $\theta\in(1,\infty)$ fulfilling

$$\frac{1}{\theta} < 1-\frac{(N-\sigma_0)k_0}{N\sigma_0},$$

that is, $\frac{k_0\theta}{\theta-1}<\frac{N\sigma_0}{N-\sigma_0}$. Then by Hölder's inequality, we can estimate

$$\begin{aligned} \|u(\cdot,s)\nabla\upsilon(\cdot,s)\|_{k_0} & \le \|u(\cdot,s)\|_{\theta k_0}\|\nabla\upsilon(\cdot,s)\|_{\frac{\theta k_0}{\theta-1}} \\ & \le C_5\|u(\cdot,s)\|_{\theta k_0}\|\nabla\upsilon(\cdot,s)\|_{\frac{N\sigma_0}{N-\sigma_0}} \end{aligned} \tag{3.19}$$

for all $s\in(s_0,T_{\max})$, with some $C_5>0$. The Sobolev embedding theorem and elliptic regularity theory applied to the second equation in (1.1) tell us that $\|\upsilon(\cdot,s)\|_{1,\frac{N\sigma_0}{N-\sigma_0}} \le C_6\|\upsilon(\cdot,s)\|_{2,\sigma_0}\le C_7$ with some $C_6,C_7>0$. Thus by interpolation inequality to (3.19), we obtain

$$\begin{aligned} \|u(\cdot,s)\nabla\upsilon(\cdot,s)\|_{k_0} & \le C_5C_7\|u(\cdot,s)\|_{\theta k_0} \le C_8\|u(\cdot,s)\|_1^{1-\theta_0}\|u(\cdot,s)\|_\infty^{\theta_0} \\ & \le C_8D^{1-\theta_0}\|u(\cdot,s)\|_\infty^{\theta_0} \end{aligned}$$

with some $\theta_0 \in (0,1)$ and $\|u(\cdot,s)\|_1 \leq D$. Hence, combining this estimate and (3.18), we infer

$$\begin{aligned} \|I_2(\cdot,t)\|_\infty &\leq C_4 \int_{s_0}^t \left(1+(t-s)^{-\frac{1}{2}-\frac{N}{2k_0}}\right) e^{-\mu_1(t-s)} \|u(\cdot,s)\|_{k_0} ds \\ &\leq C_4 C_8 \sup_{t\in[s_0,T]} \|u(\cdot,t)\|_\infty^{\theta_0} ds = C_4 C_8 M^{\theta_0} = C_9 \end{aligned}$$

for any $T \in (s_0, T_{\max})$, where $C_8 = C_7\chi \left(1+\mu_1^{\frac{N}{2k_0}-\frac{1}{2}} \int_0^\infty r^{-\frac{1}{2}-\frac{N}{2k_0}} e^{-r} dr\right) > 0$ is finite, because $-\frac{1}{2}-\frac{N}{2k_0} > -1$. By the similar way, there exists a positive $C_{10} > 0$ such that

$$\|I_3(\cdot,t)\|_\infty \leq C_{10} \tag{3.20}$$

for all $t \in (s_0, T_{\max})$. By the inequality (3.3), we see that

$$au(\cdot,s) - \mu u^2(\cdot,s) \leq a\left(\frac{a}{\mu}\right) = C_{11} > 0.$$

Then there exist positive constants C_{12} and ϵ such that

$$\|I_4(\cdot,t)\|_\infty \leq C_{11} \int_{s_0}^t e^{(t-s)(\triangle-1)} ds \leq C_{12} \int_{s_0}^t e^{-\epsilon(t-s)} ds \leq \frac{C_{12}}{\epsilon} \tag{3.21}$$

for all $t \in (s_0, T_{\max})$. Hence, collecting (3.16) – (3.21) it is easy to see that $\|u(\cdot,t)\|_\infty \leq C$ for all $t \in (s_0, T_{\max})$. By means of (3.1), we derive that $\|u(\cdot,t)\|_\infty \leq C$ for all $t \in (0, T_{\max})$. Thus the Lemma 3.3 is proved. □

Now, we give the proof of Theorem 1.1.

Proof of Theorem 1.1. The statement of global classical solvability and boundedness is a straightforward consequence of Lemmas 2.1 and 3.3. The proof of Theorem 1.1 is completed. □

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