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Rigidity Results on Generalized m-Quasi Einstein Manifolds with Associated Affine Killing Vector Field

Rahul Poddar*, Subramanian Balasubramanian and Ramesh Sharma

(Dedicated to the memory of Prof. Dr. Krishan Lal DUGGAL (1929 - 2022))

ABSTRACT

We study a non-trivial generalized m-quasi Einstein manifold M with finite m and associated divergence-free affine Killing vector field, and show that M reduces to an m-quasi Einstein manifold. In addition, if M is complete, then it splits as the product of a line and an $(n-1)$-dimensional negatively Einstein manifold. Finally, we show that the same result holds for a complete non-trivial m-quasi Einstein manifold M with finite m and associated affine Killing vector field.

Keywords: *Generalized m-quasi Einstein manifold, affine Killing vector field, Einstein manifold, Ricci almost soliton.*

AMS Subject Classification (2020): *Primary: 53C21; Secondary: 53C24; 53C25.*

1. Introduction

In [7], Pigola et al. defined a Ricci almost soliton as an n-dimensional smooth Riemannian manifold (M, g) satisfying the condition

$$\pounds_V g + 2Ric = 2\lambda g, \tag{1.1}$$

where V is a vector field on M, Ric is the Ricci tensor of g, $\pounds$ is the Lie-derivative operator, and λ is a smooth real valued function on M. If the vector field V is the gradient of a smooth function f, up to the addition of a Killing vector field, then (M, g, f, λ) is called a gradient Ricci almost soliton, in which case equation (1.1) assumes the form

$$Ric + Hessf = \lambda g, \tag{1.2}$$

where $Hessf$ is the Hessian of f with respect to g. For λ constant, equation (1.1) defines a Ricci soliton which corresponds to self-similar solutions of the Ricci flow equation, and equation (1.2) is then known as a gradient Ricci soliton. For compact Ricci almost solitons, we state the following rigidity result of Sharma.

Theorem 1.1 ([8]). *If a compact Ricci almost soliton (M, g, V, λ) has divergence free soliton vector field V, then it is Einstein and V is Killing.*

We show that the same conclusion holds if compactness is waived and V is assumed to be an affine Killing vector field. More precisely, we state this as the following proposition.

Proposition 1.1. *Let (M, g, V, λ) be a Ricci almost soliton. If V is an affine Killing vector field and is divergence free, then V is Killing and g is Einstein.*

Here, we recall that, V is an affine Killing vector field (a generalization of a Killing vector field) if it satisfies the condition

$$(\pounds_V \nabla)(X, Y) = 0. \tag{1.3}$$

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*** Corresponding author***

Geodesics along with their affine parameters are preserved by an affine Killing vector field. The set of all affine Killing vector fields on an n-Riemannian manifold forms a Lie algebra of maximum dimension $n^2 + n$, under the usual bracket operation for vector fields. Also, the divergence of an affine Killing vector field is constant (Yano [10]).

As a generalization of Einstein metrics, gradient Ricci solitons, gradient Ricci almost solitons, Catino [4] defined a generalized quasi-Einstein manifold as an n-dimensional smooth Riemannian manifold (M, g) satisfying the equation

$$Ric + Hessf - \mu df \otimes df = \lambda g, \tag{1.4}$$

and showed that a complete (M, g) satisfying (1.4) with harmonic Weyl tensor and zero radial Weyl curvature is locally a warped product with $(n-1)$-dimensional Einstein fiber. In particular, if

$$\mu = \frac{1}{m},$$

for a real m such that $0 < m \leq \infty$, then (1.4) becomes

$$Ric + Hessf - \frac{1}{m} df \otimes df = \lambda g, \tag{1.5}$$

where the left side is called the m-Bakry-Émery Ricci tensor denoted by Ric_f^m. We will call (M, g) satisfying equation (1.5) as a generalized m-quasi Einstein manifold. We note that for $m = \infty$, and λ constant in (1.5), we get a gradient Ricci soliton. For $m = \infty$, and λ non-constant, (1.5) gives a gradient Ricci almost soliton. If λ is constant, then (1.5) is an m-quasi Einstein metric, further, if m is a positive integer, then (1.5) holds on the base of a $(m+n)$-dimensional Einstein warped product (Case, Shu and Wei [3], He, Petersen and Wylie [6] and Barros and Ribeiro, Jr. [1]). An m-quasi Einstein manifold is said to be expanding, steady, or shrinking, if $\lambda < 0$, $\lambda = 0$, or $\lambda > 0$ respectively.

Motivated by Proposition 1.1, we consider the problem of classifying a generalized m-quasi Einstein manifold with finite m and associated affine Killing and divergence free potential vector field Df (D denotes the gradient operator). In this context, we establish the following result.

Theorem 1.2. *Let (M, g, f, λ) be an n-dimensional non-trivial generalized m-quasi Einstein manifold with finite m. If the potential vector field Df is affine Killing and divergence free, then M reduces to an m-quasi Einstein manifold. In addition, if M is complete, then M splits as the product of a line and an $(n-1)$-dimensional complete negatively Einstein manifold.*

In the case of a non-trivial complete m-quasi Einstein manifold with associated affine Killing vector field, we show that the result of Theorem 1.2 holds without the divergence-free condition on the potential vector field, and prove the following result.

Theorem 1.3. *Let (M, g, f, λ) be an n-dimensional $(n > 2)$ non-trivial complete m-quasi Einstein manifold with finite m. If the potential vector field Df is affine Killing, then M splits as the product of a line and an $(n-1)$-dimensional complete negatively Einstein manifold.*

2. Proofs of the Results

Proof of Proposition 1.1 The g-trace of the Ricci almost soliton equation (1.1) and the hypothesis $divV = 0$ provides

$$r = n\lambda. \tag{2.1}$$

Next, using the commutation formula (Yano [10]):

$$\begin{aligned} & (\pounds_V \nabla_X g - \nabla_X \pounds_V g - \nabla_{[V,X]} g)(Y, Z) \\ = \quad & -g((\pounds_V \nabla)(X, Y), Z) - g((\pounds_V \nabla)(X, Z), Y) \end{aligned}$$

and the hypothesis that V is affine Killing (i.e., $\pounds_V \nabla = 0$), we find that $\nabla_X \pounds_V g = 0$. Using this in the Ricci almost soliton equation (1.1) gives

$$(\nabla_X Q)Y = (X\lambda)Y.$$

Contracting this equation at Y and X separately, and using twice contracted Bianchi second identity: $(divQ)Y = \frac{1}{2}Yr$ provides

$$Dr = nD\lambda, \quad Dr = 2D\lambda.$$

As $n > 2$, the above equations imply that λ and r are constant. So M is a Ricci soliton with constant scalar curvature. Using the following integrability equation (Ghosh and Sharma [5]) for a Ricci soliton:

$$\pounds_V r = \Delta r + 2|Q|^2 - 2\lambda r,$$

and noting that r is constant, we get $|Q|^2 = \lambda r$. But $|Q - \frac{r}{n}I|^2 = |Q|^2 - \frac{r^2}{n}$. Combining the two preceding equations yields

$$|Q - \frac{r}{n}I|^2 = -\frac{r}{n}(r - n\lambda).$$

The use of equation (2.1) in the above equation immediately shows that g is Einstein and $Q = \lambda I$. Consequently, equation (1.1) implies that V is Killing and this completes the proof.

Proof of Theorem 1.2 Equation (1.5) can be written as

$$2Ric(Y,Z) + (\pounds_{Df}g)(Y,Z) - \frac{2}{m}(Yf)(Zf) = 2\lambda g(Y,Z),$$

for arbitrary smooth vector fields Y, Z on M. Covariantly differentiating the preceding equation with respect to X gives

$$\begin{aligned} 2(\nabla_X Ric)(Y,Z) \quad &+ \quad (\nabla_X \pounds_{Df}g)(Y,Z) - \frac{2}{m}g(\nabla_X Df, Y)(Zf) \\ &- \quad \frac{2}{m}g(\nabla_X Df, Z)(Yf) = 2(X\lambda)g(Y,Z). \end{aligned} \tag{2.2}$$

Rearranging the terms yields

$$\begin{aligned} (\nabla_X \pounds_{Df}g)(Y,Z) \quad &= \quad 2(X\lambda)g(Y,Z) + \frac{2}{m}g(\nabla_X Df, Y)(Zf) \\ &+ \quad \frac{2}{m}g(\nabla_X Df, Z)(Yf) - 2(\nabla_X Ric)(Y,Z). \end{aligned} \tag{2.3}$$

Next, using the commutation formula (Yano [10]) mentioned and used in the proof of Proposition 1.1, taking $V = Df$ and the fact that g is parallel with respect to ∇, along with equations (1.3) and (2.3) leads us to

$$2(X\lambda)Y + \frac{2}{m}g(\nabla_X Df, Y)Df + \frac{2}{m}(Yf)\nabla_X Df - 2(\nabla_X Q)Y = 0. \tag{2.4}$$

Contracting equation (2.4) with respect to Y gives

$$nD\lambda + \frac{2}{m}\nabla_{Df}Df - Dr = 0. \tag{2.5}$$

Similarly, contracting equation (2.4) at X, and using the twice contracted Bianchi's identity, we arrive at

$$2D\lambda + \frac{2}{m}\nabla_{Df}Df + \frac{2}{m}(\Delta f)Df - Dr = 0. \tag{2.6}$$

Equations (2.5) and (2.6) along with the hypothesis that Df is divergence free, at once gives

$$(n-2)D\lambda = 0. \tag{2.7}$$

This implies that λ is constant, i.e., M becomes an m-quasi Einstein manifold. This reduces equation (2.5) to

$$Dr = \frac{2}{m}\nabla_{Df}Df. \tag{2.8}$$

Now, equation (1.5) can be written as

$$\nabla_X Df = \lambda X - QX + \frac{1}{m}(Xf)Df. \tag{2.9}$$

Contracting (2.9) at X and using the hypothesis that Df is divergence free, we get

$$r = n\lambda + \frac{1}{m}|Df|^2. \tag{2.10}$$

Next, we indite Lemma 2(2) of Barros and Ribeiro, Jr. [2] for an m-quasi Einstein manifold,

$$\frac{1}{2}Dr = \left(1 - \frac{1}{m}\right)QDf + \frac{1}{m}(r - (n-1)\lambda)Df. \tag{2.11}$$

Using equation (2.9) with $X = Df$, (2.8) and (2.10) in (2.11) provides

$$QDf = 0. \tag{2.12}$$

As λ was shown to be a constant, equation (2.4) reduces to

$$(\nabla_X Q)Y = \frac{1}{m}g(\nabla_X Df, Y)Df + \frac{1}{m}(Yf)\nabla_X Df. \tag{2.13}$$

Using (2.9) and (2.13), we compute $R(X,Y)Df$ and obtain the equation

$$R(X,Y)Df = 0. \tag{2.14}$$

Differentiating (2.14) along an arbitrary smooth vector field Z and using (2.9) entails

$$(\nabla_Z R)(X,Y)Df - R(X,Y)QZ + \lambda R(X,Y)Z = 0.$$

Contracting the foregoing equation at X gives

$$(\nabla_Z Ric)(Y, Df) - Ric(Y, QZ) + \lambda Ric(Y, Z) = 0. \tag{2.15}$$

Using (2.9) and (2.13) we find that

$$\begin{aligned}(\nabla_Z Ric)(Y, Df) &= \frac{1}{m}\Big[\lambda < Z, Y > |Df|^2 - Ric(Z,Y)|Df|^2 \\ &+ \left(\frac{2}{m}|Df|^2 + \lambda\right)(Zf)(Yf)\Big].\end{aligned}$$

In view of (2.15), the foregoing equation takes the form

$$\begin{aligned}Ric(Y, QZ) - \lambda Ric(Y,Z) &= \frac{1}{m}\Big[\lambda < Z, Y > |Df|^2 - Ric(Z,Y)|Df|^2 \\ &+ \left(\frac{2}{m}|Df|^2 + \lambda\right)(Zf)(Yf)\Big].\end{aligned}$$

Substituting Df for Z in the above equation and using (2.12) we find that

$$\left(\lambda + \frac{1}{m}|Df|^2\right)Yf = 0. \tag{2.16}$$

At this point, we show that $|Df|$ is constant on M. For this, we compute the covariant derivative of $|Df|^2$ along an arbitrary vector field Y as follows:

$$Y|Df|^2 = 2g(\nabla_Y Df, Df) = 2g(\lambda Y - QY + \frac{1}{m}(Yf)Df, Df) = 0,$$

where we used equations (2.12) and (2.16). Hence Df is everywhere zero on M (in which case M is trivial, contradicting our hypothesis), or everywhere non-zero on M. Thus Df is nowhere zero on M, and hence (2.16) implies

$$\lambda = -\frac{1}{m}|Df|^2 \tag{2.17}$$

and hence λ is negative. Now, substituting Df for X in equation (2.9) and using (2.12) and (2.17) shows that

$$\nabla_{Df} Df = 0. \tag{2.18}$$

From (2.8) and (2.18), it follows that $Dr = 0$, i.e., the scalar curvature is constant. Using this consequence, equation (2.12), and $Df \neq 0$ on M, in equation (2.11), shows that

$$r = (n-1)\lambda. \tag{2.19}$$

Hence r is negative, because λ was found negative earlier. At this stage, we indite Lemma 3.2 of Case et al. [3] for an m-quasi Einstein manifold

$$\frac{1}{2}\Delta|Df|^2 = |Hessf|^2 - Ric(Df, Df) + \frac{2}{m}|Df|^2\Delta f. \tag{2.20}$$

Using (2.18), (2.20), (2.12) and the hypothesis that the potential vector field is divergence free (i.e., $\Delta f = 0$), we get

$$Hessf = 0, \tag{2.21}$$

i.e., Df is parallel on M. As M is complete, by a result [Theorem 2 (I,A)] of Tashiro [9], M is the Riemannian product of a line $\mathcal{R}$ and an $(n-1)$-dimensional complete Riemannian manifold N. Here Df is tangent to $\mathcal{R}$. As $Hessf = 0$, equation (1.5) reduces to $Ric - \frac{1}{m} df \otimes df = \lambda g$. From this, it follows that, if E is an eigenvector of the Ricci operator Q, then E is point-wise collinear with Df, because $\lambda \neq 0$. It also follows that N is Einstein and hence negatively Einstein, because $r < 0$. This completes the proof.

Proof of Theorem 1.3 For an m-quasi Einstein manifold, we know that λ is constant. Also, as Df is affine Killing, $divDf = constant$, i.e., $\Delta f = constant$ (Yano [10], Pg. 45). Using this in (2.5) and (2.6) yields

$$(\Delta f)Df = 0.$$

Since M is non-trivial, we have $\Delta f = 0$. The rest of the proof follows from Theorem 1.2.

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Author's contributions

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Affiliations

RAHUL PODDAR
ADDRESS: Sri Sathya Sai Institute of Higher Learning, Dept. of Mathematics and Computer Science, Prasanthi Nilayam, 515134, Andhra Pradesh, India.
E-MAIL: rahulpoddar@sssihl.edu.in
ORCID ID: 0000-0002-5671-2774

S. BALASUBRAMANIAN
ADDRESS: Sri Sathya Sai Institute of Higher Learning, Dept. of Mathematics and Computer Science, Prasanthi Nilayam, 515134, Andhra Pradesh, India.
E-MAIL: sbalasubramanian@sssihl.edu.in
ORCID ID: 0000-0001-8947-4840

RAMESH SHARMA
ADDRESS: University of New Haven, Dept. of Mathematics, West Haven, 06516, CT, USA.
E-MAIL: rsharma@newhaven.edu
ORCID ID: 0000-0001-5026-8564