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Research Article

Finite-time property of a mechanical viscoelastic system with nonlinear boundary conditions on corner-Sobolev spaces

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Abstract

In this article, we deal with the initial boundary value problem for a viscoelastic system related to the quasilinear parabolic equation with nonlinear boundary source term on a manifold $\mathbb{M}$ with corner singularities. We prove that, under certain conditions on relaxation function g, any solution u in the corner-Sobolev space $\mathcal{H}^{1,(\frac{N-1}{2},\frac{N}{2})}_{\partial^0\mathbb{M}}(\mathbb{M})$ blows up in finite time. The estimates of the life-span of solutions are also given.

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1. Introduction

Many of the problems in fracture and contact mechanics may be formulated as mixed boundary value problems which, in turn, may be reduced to integral equations of the general form [29, 30]. The main important aim of the present manuscript is the investigation blow up and its life-span of the viscosity solutions of a nonlinear system in space with corner singularity points. Let us recall some background and applications from the such situations in the real world. The singularities of the viscosity equations occur when some derivatives of the velocity field is infinite at any point of a field of flow or, in an evolving flow, becomes infinite at any point within a finite time. In view of mathematics, these singularities can be formulated, for example, in two-dimensional flow near a sharp corner onto a wire boundary in which case they can be resolved by refining the geometrical description. On the other hand, one can consider them in physical form, for instance, in the case of cusp singularities of a fluid in which case the resolution of the singularity involves incorporation of additional physical effects. Two-dimensional flow near a sharp corner exhibits a curious singularity that has been the subject of many investigations [13, 28, 32]. From a mechanical point of view, there are many investigations about the finite-time blow up of the singularity problem that we can provide only some significant here. It is well-known that the configuration most likely to lead to a singularity consists of two interacting non-parallel vortex tubes [5]. In 1996 [12], Constantin, Fefferman and Majda proved the direction of vorticity of the Euler equations should be indeterminate in the limit as the

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singularity is approached. For more details in the mechanical point of view we refer to [28] and the references therein. Now, We present a viscosity system with difficult conditions on the boundary of the configuration space which has the corner singularities and provide a background of these problems from mathematical point of view. More precisely, in this paper, we study the blow up and life-span results of the following viscoelastic problem with nonlinear damping and boundary source terms

$$\begin{cases} |\partial_t u|^{k-1}\partial_{tt}u - \Delta_{\mathbb{M}}u + \int_0^t g(t-s)\Delta_{\mathbb{M}}u(s)ds = |u|^{p-1}u & in \ \ \mathbb{M}\times(0,\infty) \\ u(x,t) = 0 & on \ \ x\in\partial^0\mathbb{M}\times(0,\infty), \\ \partial_\nu u - \int_0^t g(t-s)\partial_\nu u(s)ds + \partial_t u = |u|^{m-1}u, & on \ \ \partial^1\mathbb{M}\times(0,\infty), \\ u(x,0) = u_0, \qquad \partial_t u(x,0) = u_1, & x\in\mathbb{M}. \end{cases} \tag{1.1}$$

where $k \geq 1$ and $\mathbb{M}$ is a corner manifold with finite corner measure, which is a local model of stretched corner-manifolds, i.e. the manifolds with corner singularities of dimension $N = n+2 \geq 3$ with boundary $\partial\mathbb{M} = \partial^0\mathbb{M} \cup \partial^1\mathbb{M}$. Here, let $\{\partial^0\mathbb{M}, \partial^1\mathbb{M}\}$ be a partition of its boundary $\partial\mathbb{M}$ such that $\overline{\partial^0\mathbb{M}} \cap \overline{\partial^1\mathbb{M}} = \emptyset$ and $meas(\partial^0\mathbb{M}) > 0$. Moreover, ν is the unit outward normal to $\partial\mathbb{M}$, $1 \leq m < \frac{N-1}{N-2}$ and $1 < p \leq \frac{N}{N-2}$. The relaxation function g is satisfying certain conditions to be specified later. The author in [23] studied existence and invariance results of weak solutions of problem 1.1.

It is well-known that,elasticity is the tendency of solid materials to return to their original shape after forces are applied to them. When the forces are removed, the object will return to its initial shape and size if the material is elastic. Viscosity is a measure of a fluid resistance to flow. A fluid with large viscosity resists motion. A fluid with low viscosity flows. For example, water flows more easily than syrup because it has a lower viscosity. Viscoelasticity is the property of materials that exhibit both viscous and elastic characteristics when undergoing deformation. Synthetic polymers, wood, and human tissue, as well as metals at high temperature, display significant viscoelastic effects. In some applications, even a small viscoelastic response can be significant. For the fundamental modeling, development of linear viscoelasticity see [11] and we refer the interested reader to the monograph [15] for surveys regarding the mathematical aspect of the theory of viscoelasticity.

In the setting of $\Omega \subset \mathbb{R}^n$, when $k = 1$ and $g = 0$, the problem 1.1 reduces to a hyperbolic system which can be considered under Dirichlet or Neumann boundary conditions. There have been extensive studied on some special cases of these systems and the physical background [4,9,17,18,22,37]. In the presence of the viscoelastic term, Kim and Han [24] proved that any weak solution with negative initial energy blows up in finite time under suitable conditions on the relaxation function g for the equation

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds = |u|^{p-2}u \quad in \ \ (x,t)\in\Omega\times(0,\infty). \tag{1.2}$$

Concerning Cauchy problems, Kafini and Messaoudi [22] established a blow up result for the problem

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds + u_t = |u|^{p-2}u \quad in \ \ (x,t)\in\mathbb{R}^n\times(0,\infty). \tag{1.3}$$

where g satisfied $\int_0^\infty g(s)ds < \frac{2p-4}{2p-3}$ and the initial data were compactly supported with negative energy such that $\int u_0 u_1 dx \geq 0$. Maxim Korpusov [25] studied the initial-boundary value problem for the generalized dissipative high-order equation of Klein-Gordon type with arbitrary positive initial energy. He established a blow-up result using the modified concavity method of Levine developed in [3]. More and new results about the blow up properties with arbitrary positive initial energy can be found, for instance [20,21,27,31,35]. However, there are a few investigations about this type of equations on the manifolds with singularities. For instance, Cavalcanti et al. [2] considered a nonlinear viscoelastic evolution equation as

$$u_{tt} + Au + F(x,t,u,u_t) - \int_0^t g(t-\tau)Au(\tau)d\tau = 0 \qquad on \;\; \Gamma \times (0,\infty)$$

where Γ is a compact manifold. In [7] the initial boundary value problem of the viscoelastic equation with a nonlinear boundary damping term

$$\begin{cases} u_{tt} - \Delta u = 0 & in \;\; x \in \Omega \times (0,\infty) \\ u(x) = 0 & on \;\; x \in \Gamma_0 \times (0,\infty), \\ u_\nu - \int_0^t g(t-s)u_\nu(s)ds + h(u_t) = 0, & on \;\; \Gamma_1 \times (0,\infty), \\ u(x,0) = u_0, u_t(x,0) = u_1, & x \in \Omega. \end{cases} \tag{1.4}$$

studied and the authors obtained a global existence result for strong and weak solutions under the classical assumptions on g. In the setting of the manifolds with singularities such as conical singularities, the authors in [1] studied the initial-boundary value problem for semilinear hyperbolic equations

$$\begin{cases} u_{tt} - \Delta_{\mathbb{B}} u + V(x)u + \gamma u_t = f(x,u), & x \in int\mathbb{B}, t > 0, \\ u(x,0) = u_0(x), \qquad \partial_t u(x,0) = u_1(x), & x \in int\mathbb{B} \\ u(t,x) = 0, & x \in \partial\mathbb{B}, t \geq 0, \end{cases} \tag{1.5}$$

where, γ is a non-negative parameter and V is a potential function. Here the domain $\mathbb{B}$ is $[0,1) \times X$, X is an $(n-1)$-dimensional closed compact manifold, which is regarded as the local model near the conical points on manifolds with conical singularities, and $\partial\mathbb{B} = \{0\} \times X$. Moreover, in [36], the authors obtained the upper bounds of blow up time and the blow up rate for a semilinear edge-degenerate parabolic equation. To our best knowledge, there are no or few investigations of the viscoelastic problem on the manifolds with singularities. But in the Euclidean domain $\Omega \subset \mathbb{R}^n$, Cavalcanti et al. [6] considered the nonlinear viscoelastic equation without source term and weak damping term

$$|u_t|^\rho u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds - \gamma\Delta u_t - \Delta u_{tt} = 0, \qquad in \;\; \Omega \times (0,\infty).$$

They obtained the global existence of weak solutions and uniform decay rates of the energy by assuming that the relaxation g has an exponential decay. In [19], the authors considered an initial-boundary value problem for a nonlinear viscoelastic wave equation with strong damping, nonlinear damping and source terms. They proved a blow up result for the solution with negative initial energy. Our study is in fact provoked by the study of [14] and by modifying the method, which is put forward by Li, Tasi [26] and Vitillaro [34], we proved that, under certain conditions, any solution blows up in finite time. The estimates of the life-span of solutions are also given. In this manuscript, we consider the nonlinear viscoelastic wave equation with an internal nonlinear term $|\partial_t u|^{k-1}\partial_{tt}u$, and a nonlinear boundary source term $|u|^{m-1}u$, on the corner manifold $\mathbb{M}$, and we obtain some blow up results about the problem 1.1.

2. Preliminaries

In this section, we consider the stretched corner manifold $\mathbb{M} = [0,1) \times X \times [0,1)$ with smooth boundary $\partial\mathbb{M}$ [10, 23, 33].

We take $X \subset S^n$ be a bounded open set in the unit sphere of $\mathbb{R}^{n+1}$. As mentioned in [10], one can consider the straight cone as $X^\Delta := \left\{x \in \mathbb{R}^{n+1} \;\mid\; x = 0 \; or \; \frac{x}{|x|} \in X\right\}$. Then, an infinite cone in $\mathbb{R}^{n+1}$ can be defined as the following quotient space

$$X^\Delta = \frac{(\bar{\mathbb{R}}_+ \times X)}{\{0\} \times X},$$

with base X. The coordinates $(r,\varphi) \in X^\Delta - \{0\}$ are the standard coordinates in this quotient space by using the cylindrical coordinates in $\mathbb{R}^{n+1}$. So we can describe $X^\Delta - \{0\}$

in the form $\mathbb{R}_+ \times X$. Therefore, the stretched cone is defined by $X^\wedge := \bar{\mathbb{R}}_+ \times X$. Set the coordinates in $X^\wedge$ as (r_1, x) such that in the case $0 \le r_1 < 1$ one can consider a finite cone

$$E = \frac{([0,1) \times X)}{\{0\} \times X}.$$

Then, the finite stretched cone corresponding to E is defined as $\mathbb{E} := [0,1) \times X$, with a smooth boundary $\partial\mathbb{E} = \{0\} \times X$. By the similar way, one can define an infinite corner as

$$E^\wedge := \frac{(E \times \bar{\mathbb{R}}_+)}{E \times \{0\}},$$

where the base E is a finite cone with base X as above. Hence, the stretched corner is $E^\wedge = \mathbb{E} \times \bar{\mathbb{R}}_+$. Take $(r_1, x, r_2) \in E^\wedge$, we concentrate on the case $0 \le r_2 < 1$, then the finite corner is $M = \frac{(E \times [0,1))}{E \times \{0\}}$. Therefore, $\mathbb{M} = \mathbb{E} \times [0,1) = [0,1) \times X \times [0,1)$ is a finite stretched corner with the smooth boundary $\partial\mathbb{M} = \partial\mathbb{E} \times \{0\}$. The typical degenerate differential operator A on the stretched corner $\mathbb{M}$ is of the following form

$$A = r_2^{-\nu} \sum_{l \le \nu} a_{2,l}(r_2)(r_2\partial_{r_2})^l,$$

where $a_{2,l}(r_2) \in C^\infty(\bar{\mathbb{R}}_+, Diff_{deg}^{\nu-l}(\mathbb{E}))$ that is

$$a_{2,l}(r_2) = r_1^{-(\nu-l)} \sum_{j \le (\nu-l)} a_{1,jl}(r_1, r_2)(r_1\partial_1)^j,$$

such that $a_{1,jl}(r_1, r_2) \in C^\infty(\bar{\mathbb{R}}_+, Diff^{\nu-l-j}(X))$. Then, it follows that

$$A = (r_1r_2)^{-\nu} \sum_{j+l \le \nu} a_{jl}(r_1, r_2)(r_1\partial_{r_1})^j (r_1r_2\partial_{r_2})^l = (r_1r_2)^{-\nu} A_{\mathbb{M}},$$

where $a_{jl}(r_1, r_2) \in C^\infty(\bar{\mathbb{R}}_+, Diff^{\nu-l-j}(X))$ and $A_{\mathbb{M}}$ is called as a degenerate corner operator [8, 10, 23]. We can consider the following Riemannian metric on the corner manifold M

$$g_M = dr_2^2 + r_2^2(dr_1^2 + r_1^2 g_X),$$

where g_X is a Riemannian metric on X. Therefore, the corresponding gradient operator with corner degeneracy is $\nabla_{\mathbb{M}} = (r_1\partial_{r_1}, \partial_{x_1}, ..., \partial_{x_n}, r_1r_2\partial_{r_2})$.

Now, we recall some definitions of the weighted $p-$Sobolev spaces $L_p^{\gamma_1,\gamma_2}$ on $\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}_+$.

Definition 2.1. Let $(r_1, x, r_2) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}_+$, weights $\gamma_1, \gamma_2 \in \mathbb{R}$ and $1 \le p < \infty$. Then,

$$L_p^{\gamma_1,\gamma_2}\Big(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}_+; \frac{dr_1}{r_1} dx \frac{dr_2}{r_1r_2}\Big)$$
$$:= \Big\{ u(r_1, x, r_2) \in \mathcal{D}'(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}_+) \mid \|u\|_{L_p^{\gamma_1,\gamma_2}} < +\infty, \Big\}$$

where

$$\|u\|_{L_p^{\gamma_1,\gamma_2}} = \Big(\int_{\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}_+} |r_1^{\frac{N}{p}-\gamma_1} r_2^{\frac{N}{p}-\gamma_2} u(r_1, x, r_2)|^p \frac{dr_1}{r_1} dx \frac{dr_2}{r_1r_2} \Big)^{\frac{1}{p}}.$$

Definition 2.2. Let $m \in \mathbb{N}$, $\gamma_1, \gamma_2 \in \mathbb{R}$ and $N = n+2$. Then $\mathcal{H}_p^{m,(\gamma_1,\gamma_2)}(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}_+)$ contains those of the functions $u \in \mathcal{D}'(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}_+)$ such that

$$(r_1\partial_{r_1})^l \partial_x^\alpha (r_1r_2\partial_{r_2})^k u(r_1, x, r_2) \in L_p^{\gamma_1,\gamma_2}\Big(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}_+; \frac{dr_1}{r_1} dx \frac{dr_2}{r_1r_2}\Big)$$

for all $k, l \in \mathbb{N}$ and any multi-index $\alpha \in \mathbb{N}^n$ with $k + l + |\alpha| \le m$.

We denote the closure of C_0^∞ functions in $\mathcal{H}_p^{m,(\gamma_1,\gamma_2)}(\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}_+)$ by $\mathcal{H}_{p,0}^{m,(\gamma_1,\gamma_2)}(\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}_+)$. Now, we can define the weighted $p-$Sobolev spaces on an open stretched corner $\mathbb{R}_+\times X\times\mathbb{R}_+$ as following :

$$\mathcal{H}_p^{m,(\gamma_1,\gamma_2)}(\mathbb{R}_+\times X\times\mathbb{R}_+) := \Big\{u(r_1,x,r_2)\in\mathcal{D}'(\mathbb{R}_+\times X\times\mathbb{R}_+) \quad | \\ (r_1\partial_{r_1})^l\partial_x^\alpha(r_1r_2\partial_{r_2})^k u(r_1,x,r_2)\in L_p^{\gamma_1,\gamma_2}\Big(\mathbb{R}_+\times X\times\mathbb{R}_+;\frac{dr_1}{r_1}dx\frac{dr_2}{r_1r_2}\Big)\Big\}$$

for all $k,l\in\mathbb{N}$ and any multi-index $\alpha\in\mathbb{N}^n$ with $k+l+|\alpha|\leq m$, which is a Banach space with norm

$$\Big\|u\Big\|_{\mathcal{H}_p^{m,(\gamma_1,\gamma_2)}} = \Bigg\{\sum_{k+l+|\alpha|\leq m}\int_{\mathbb{R}_+\times X\times\mathbb{R}_+}\Big|r_1^{\frac{N}{p}-\gamma_1}r_2^{\frac{N}{p}-\gamma_2}(r_1\partial_{r_1})^l\partial_x^\alpha(r_1r_2\partial_{r_2})^k u(r_1,x,r_2)\Big|^p\frac{dr_1}{r_1}dx\frac{dr_2}{r_1r_2}\Bigg\}^{\frac{1}{p}}.$$

The subspace $\mathcal{H}_{p,0}^{m,(\gamma_1,\gamma_2)}(\mathbb{R}_+\times X\times\mathbb{R}_+)$ indicates the closure of C_0^∞ functions in $\mathcal{H}_p^{m,(\gamma_1,\gamma_2)}(\mathbb{R}_+\times X\times\mathbb{R}_+)$. Now, we express the weighted $p-$Sobolev space on the finite stretched corner manifold $\mathbb{M}$, see [8, 10, 23].

Definition 2.3. Let $m\in\mathbb{N}$, $\gamma_1,\gamma_2\in\mathbb{R}$, $1\leq p<\infty$ and $W_{loc}^{m,p}(int\ \mathbb{M})$ is the classical local Sobolev space. Then

$$\mathcal{H}_p^{m,(\gamma_1,\gamma_2)}(\mathbb{M}) = \Big\{u(r_1,x,r_2)\in W_{loc}^{m,p}(int\ \mathbb{M}) \ | \ \omega_1\omega_2 u(r_1,x,r_2)\in\mathcal{H}_p^{m,(\gamma_1,\gamma_2)}(\mathbb{R}_+\times X\times\mathbb{R}_+)\Big\}$$

for every cut-off functions $\omega_1=\omega(r_1,x)$ and $\omega_2=\omega(r_2,x)$ supported by a collar neighborhoods of $(0,1)\times\partial\mathbb{M}$ and $\partial\mathbb{M}\times(0,1)$ respectively.

We know that, in differential geometry, one can attach to any point $\tilde{x}=(r_1,x,r_2)\in\mathbb{M}$ a tangent space $T_{\tilde{x}}\mathbb{M}$ which is a real vector space that intuitively contains the possible directions in which one can tangentially pass through $\tilde{x}$. Then up to isomorphism, for arbitrary and fixed point $\tilde{x}_0\in\mathbb{M}$ we define $\eta(\tilde{x})=\tilde{x}-\tilde{x}_0$ and give a partition of the boundary ∂M such that

$$\partial^0\mathbb{M}=\Big\{\tilde{x}\in\partial\mathbb{M} \quad | \quad \eta(\tilde{x}).\nu(\tilde{x})\leq 0\Big\} \quad \textit{and} \quad \partial^1\mathbb{M}=\Big\{\tilde{x}\in\partial\mathbb{M} \quad | \quad \eta(\tilde{x}).\nu(\tilde{x})>0\Big\}.$$

For the weights $\gamma_1=\frac{N-1}{2}$, $\gamma_2=\frac{N}{2}$ and $1\leq p<\infty$, we take the following inner products and norms [23]

$$\Big(u,v\Big)_{L_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})} = \Big(u,v\Big)_{\mathbb{M}} := \int_{\mathbb{M}} r_1 u(\tilde{x})v(\tilde{x})\frac{dr_1}{r_1}dx\frac{dr_2}{r_1r_2},$$
$$\Big(u,v\Big)_{L_2^{\frac{N-1}{2},\frac{N}{2}}(\partial^1\mathbb{M})} = \Big(u,v\Big)_{\partial^1\mathbb{M}} := \int_{\partial^1\mathbb{M}} u(\tilde{x})v(\tilde{x})d(\partial\mathbb{M}),$$
$$\Big\|u\Big\|^p_{L_p^{\frac{N-1}{p},\frac{N}{p}}(\mathbb{M})} = \Big\|u\Big\|^p_{\frac{N-1}{p},\frac{N}{p}} := \int_{\mathbb{M}} r_1\Big|u(\tilde{x})\Big|^p\frac{dr_1}{r_1}dx\frac{dr_2}{r_1r_2},$$
$$\Big\|u\Big\|^p_{L_p^{\frac{N-1}{p},\frac{N}{p}}(\partial^1\mathbb{M})} = \Big\|u\Big\|^p_{\frac{N-1}{p},\frac{N}{p},\partial^1\mathbb{M}} := \int_{\partial^1\mathbb{M}}\Big|u(\tilde{x})\Big|^p d(\partial\mathbb{M}),$$
$$\Big\|u\Big\|_\infty := ess\sup_{\tilde{x}\in\mathbb{M}}\Big|u(\tilde{x})\Big|.$$

Now, we consider the set

$$\mathcal{H}^{1,(\frac{N-1}{2},\frac{N}{2})}_{\partial^0\mathbb{M}}(\mathbb{M}) := \left\{ u \in \mathcal{H}^{1,(\frac{N-1}{2},\frac{N}{2})}_{2}(\mathbb{M}) \quad | \quad u = 0 \ \ on \ \ \partial^0\mathbb{M} \right\}$$

and endow $\mathcal{H}^{1,(\frac{N-1}{2},\frac{N}{2})}_{\partial^0\mathbb{M}}$ with the Hilbert structure induced by $\mathcal{H}^{1,(\frac{N-1}{2},\frac{N}{2})}_{2}(\mathbb{M})$, we have that $\mathcal{H}^{1,(\frac{N-1}{2},\frac{N}{2})}_{\partial^0\mathbb{M}}$ is a Hilbert space. Since $N = 1+n+1 > 2$, $1 \leq p < \frac{N}{N-2}$ and $1 \leq m < \frac{N-1}{N-2}$, we have the embedding $\mathcal{H}^{1,(\frac{N-1}{2},\frac{N}{2})}_{\partial^0\mathbb{M}}(\mathbb{M}) \hookrightarrow L^{\frac{N-1}{p+1},\frac{N}{p+1}}_{p+1}(\mathbb{M})$. Suppose that $C_* > 0$ is the optimal constant of weighted corner Sobolev embedding which satisfies the inequality

$$\left\|u\right\|_{\frac{N-1}{p+1},\frac{N}{p+1}} \leq C_* \left\|\nabla_{\mathbb{M}} u\right\|_{\frac{N-1}{2},\frac{N}{2}} \qquad \forall\ u \in \mathcal{H}^{1,(\frac{N-1}{2},\frac{N}{2})}_{\partial^0\mathbb{M}}. \tag{2.1}$$

Moreover, we use the corner trace-Sobolev type embedding $\mathcal{H}^{1,(\frac{N-1}{2},\frac{N}{2})}_{\partial^0\mathbb{M}} \hookrightarrow L^{\frac{N-1}{m+1},\frac{N}{m+1}}_{m+1}(\partial^1\mathbb{M})$, $1 \leq m < \frac{N}{N-2}$. In this case, the embedding constant is denoted by $B_* > 0$, i.e.

$$\left\|u\right\|_{\frac{N-1}{m+1},\frac{N}{m+1},\partial^1\mathbb{M}} \leq B_* \left\|\nabla_{\mathbb{M}} u\right\|_{\frac{N-1}{2},\frac{N}{2}}. \tag{2.2}$$

Since $\partial^0\mathbb{M}$ has positive $(N-1)-$dimensional Lebesgue measure, using of Poincaré inequality, we can endow $\mathcal{H}^{1,(\frac{N-1}{2},\frac{N}{2})}_{\partial^0\mathbb{M}}(\mathbb{M})$ with the following equivalent norm

$$\left\|u\right\|_{\mathcal{H}^{1,(\frac{N-1}{2},\frac{N}{2})}_{\partial^0\mathbb{M}}(\mathbb{M})} = \left\|\nabla_{\mathbb{M}} u\right\|_{\frac{N-1}{2},\frac{N}{2}} = \left(\int_{\mathbb{M}} r_1 \left|\nabla_{\mathbb{M}} u\right|^2 \frac{dr_1}{r_1} dx \frac{dr_2}{r_1 r_2}\right)^{\frac{1}{2}}. \tag{2.3}$$

Next, we express the assumptions for problem 1.1:

(A) : Assume that the relaxation function $g : [0,\infty) \to [0,\infty)$ is a C^1 function satisfying

$$g'(t) \leq 0, \qquad 1 - \int_0^t g(s)ds = l > 0,$$

and $m + 2 < p$.

To obtain our main results we need to define the following energy functionals corresponding to our problem 1.1 for every $u \in \mathcal{H}^{1,(\frac{N-1}{2},\frac{N}{2})}_{\partial^0\mathbb{M}}(\mathbb{M})$:

$$\begin{aligned} J(u) =& \frac{1}{2}(1 - \int_0^t g(s)ds) \left\|\nabla_{\mathbb{M}} u\right\|^2_{\frac{N-1}{2},\frac{N}{2}} + \frac{1}{2}(g \circ \nabla_{\mathbb{M}} u)(t) \\ & - \frac{1}{p+1}\left\|u\right\|^{p+1}_{\frac{N-1}{p+1},\frac{N}{p+1}} - \frac{1}{m+1}\left\|u\right\|^{m+1}_{\frac{N-1}{m+1},\frac{N}{m+1},\partial^1\mathbb{M}}, \end{aligned} \tag{2.4}$$

$$\begin{aligned} K(u) =& (1 - \int_0^t g(s)ds) \left\|\nabla_{\mathbb{M}} u\right\|^2_{\frac{N-1}{2},\frac{N}{2}} + (g \circ \nabla_{\mathbb{M}} u)(t) \\ & - \left\|u\right\|^{p+1}_{\frac{N-1}{p+1},\frac{N}{p+1}} - \left\|u\right\|^{m+1}_{\frac{N-1}{m+1},\frac{N}{m+1},\partial^1\mathbb{M}}, \end{aligned} \tag{2.5}$$

and the energy function

$$\begin{aligned} E(t) =& \frac{1}{k+1}\left\|\partial_t u\right\|^{k+1}_{\frac{N-1}{k+1},\frac{N}{k+1}} + \frac{1}{2}(1 - \int_0^t g(s)ds) \left\|\nabla_{\mathbb{M}} u\right\|^2_{\frac{N-1}{2},\frac{N}{2}} + \frac{1}{2}(g \circ \nabla_{\mathbb{M}} u)(t) \\ & - \frac{1}{p+1}\left\|u\right\|^{p+1}_{\frac{N-1}{p+1},\frac{N}{p+1}} - \frac{1}{m+1}\left\|u\right\|^{m+1}_{\frac{N-1}{m+1},\frac{N}{m+1},\partial^1\mathbb{M}} \\ =& \frac{1}{k+1}\left\|\partial_t u\right\|^{k+1}_{\frac{N-1}{k+1},\frac{N}{k+1}} + J(u(t)), \end{aligned} \tag{2.6}$$

where, $(g\circ\nabla_{\mathbb{M}}u)(t)=\int_0^t g(t-s)\|\nabla_{\mathbb{M}}u(t)-\nabla_{\mathbb{M}}u(s)\|^2_{\frac{N-1}{2},\frac{N}{2}}ds$. Now, we introduce

$$\mathcal{N}=\left\{u\in\mathcal{H}^{1,(\frac{N-1}{2},\frac{N}{2})}_{\partial^0\mathbb{M}}(\mathbb{M})\ \mid\ K(u)=0,\quad \left\|\nabla_{\mathbb{M}}u\right\|^2_{\frac{N-1}{2},\frac{N}{2}}\neq 0\right\},$$

$$d=\inf\left\{\sup_{\lambda\geq 0}J(\lambda u),\ \ u\in\mathcal{H}^{1,(\frac{N-1}{2},\frac{N}{2})}_{\partial^0\mathbb{M}}(\mathbb{M}),\quad \left\|\nabla_{\mathbb{M}}u\right\|^2_{\frac{N-1}{2},\frac{N}{2}}\neq 0\right\}.$$

The similar results in [16] one can get $d=\inf_{u\in\mathcal{N}}J(u)$.

Before going on our task in this section, let us conclude some facts about the functional $J(u)$ for certain solutions of problem 1.1. We consider two cases about the solutions $u\in\mathcal{H}^{1,(\frac{N-1}{2},\frac{N}{2})}_{\partial^0\mathbb{M}}(\mathbb{M})$.

I) If $\left\|\nabla_{\mathbb{M}}u\right\|_{\frac{N-1}{2},\frac{N}{2}}\geq 1$, then by Sobolev inequality and trace inequality on the boundary we obtain

$$\begin{aligned}J(u)&=\frac{l}{2}\left\|\nabla_{\mathbb{M}}u\right\|^2_{\frac{N-1}{2},\frac{N}{2}}+\frac{1}{2}(g\circ\nabla_{\mathbb{M}}u)(t)-\frac{1}{p+1}\left\|u\right\|^{p+1}_{\frac{N-1}{p+1},\frac{N}{p+1}}-\frac{1}{m+1}\left\|u\right\|^{m+1}_{\frac{N-1}{m+1},\frac{N}{m+1},\partial^1(\mathbb{M})}\\&\geq\frac{l}{2}\left\|\nabla_{\mathbb{M}}u\right\|^2_{\frac{N-1}{2},\frac{N}{2}}-\frac{C_*^{p+1}}{m+1}\left\|\nabla_{\mathbb{M}}u\right\|^{p+1}_{\frac{N-1}{2},\frac{N}{2}}-\frac{B_*^{m+1}}{m+1}\left\|\nabla_{\mathbb{M}}u\right\|^{m+1}_{\frac{N-1}{2},\frac{N}{2}}\\&\geq\frac{l}{2}\left\|\nabla_{\mathbb{M}}u\right\|^2_{\frac{N-1}{2},\frac{N}{2}}-[\frac{C_*^{p+1}}{m+1}+\frac{B_*^{m+1}}{m+1}]\left\|\nabla_{\mathbb{M}}u\right\|^{p+1}_{\frac{N-1}{2},\frac{N}{2}}\\&=\frac{l}{2}\left\|\nabla_{\mathbb{M}}u\right\|^2_{\frac{N-1}{2},\frac{N}{2}}-\frac{\alpha}{m+1}\left\|\nabla_{\mathbb{M}}u\right\|^{p+1}_{\frac{N-1}{2},\frac{N}{2}}\end{aligned}\tag{2.7}$$

where $\alpha=C_*^{p+1}+B_*^{m+1}$. Now, we define $P(\lambda):=\frac{l}{2}\lambda^2-\frac{\alpha}{m+1}\lambda^{p+1}$ for all $\lambda\geq 1$. Then there exists a $\bar{\lambda}=\left(\frac{l(m+1)}{\alpha(p+1)}\right)^{\frac{1}{p-1}}$ which admits $P(\lambda)$ its maximum at this point and

$$\begin{aligned}\bar{d}=P(\bar{\lambda})&=\frac{l}{2}\left(\frac{l(m+1)}{\alpha(p+1)}\right)^{\frac{2}{p-1}}-\frac{\alpha}{m+1}\left(\frac{l(m+1)}{\alpha(p+1)}\right)^{\frac{p+1}{p-1}}\\&=\frac{(p+1)}{2(m+1)}l^{\frac{p+1}{p-1}}\left(\frac{m+1}{p+1}\right)^{\frac{p+1}{p-1}}\alpha^{\frac{-2}{p-1}}-\frac{1}{m+1}l^{\frac{p+1}{p-1}}\left(\frac{m+1}{p+1}\right)^{\frac{p+1}{p-1}}\alpha^{\frac{-2}{p-1}}\\&=\left(\frac{l(m+1)}{\alpha^{\frac{2}{p+1}}(p+1)}\right)\frac{p+1}{p-1}\left[\frac{p+1}{2(m+1)}-\frac{2}{m+1}\right]\\&=\frac{(p-1)}{2(m+1)}\alpha^{\frac{-2}{p-1}}\left(\frac{l(m+1)}{p+1}\right)^{\frac{p+1}{p-1}}.\end{aligned}\tag{2.8}$$

II) If $\left\|\nabla_{\mathbb{M}}u\right\|_{\frac{N-1}{2},\frac{N}{2}}\leq 1$, then

$$\begin{aligned}J(u)&=\frac{l}{2}\left\|\nabla_{\mathbb{M}}u\right\|^2_{\frac{N-1}{2},\frac{N}{2}}+\frac{1}{2}(g\circ\nabla_{\mathbb{M}}u)(t)-\frac{1}{p+1}\left\|u\right\|^{p+1}_{\frac{N-1}{p+1},\frac{N}{p+1}}-\frac{1}{m+1}\left\|u\right\|^{m+1}_{\frac{N-1}{m+1},\frac{N}{m+1},\partial^1(\mathbb{M})}\\&\geq\frac{l}{2}\left\|\nabla_{\mathbb{M}}u\right\|^2_{\frac{N-1}{2},\frac{N}{2}}-\frac{\alpha}{m+1}\left\|\nabla_{\mathbb{M}}u\right\|^{m+1}_{\frac{N-1}{2},\frac{N}{2}}.\end{aligned}\tag{2.9}$$

Again we define $P(\lambda):=\frac{l}{2}\lambda^2-\frac{\alpha}{m+1}\lambda^{m+1}$ for all $0<\lambda<1$. Then the function $P(\lambda)$ in this case admits its maximum at $\tilde{\lambda}=\left(\frac{l}{\alpha}\right)^{\frac{1}{m-1}}$. Therefore,

$$\tilde{d}=P(\tilde{\lambda})=\frac{l}{2}\Big(\frac{l}{\alpha}\Big)^{\frac{2}{m-1}}-\frac{\alpha}{m+1}\Big(\frac{l}{\alpha}\Big)^{\frac{m+1}{m-1}}=\frac{1}{2}\alpha^{\frac{-2}{m-1}}l^{\frac{m+1}{m-1}}-\frac{1}{m+1}\alpha^{\frac{-2}{m-1}}l^{\frac{m+1}{m-1}}$$

$$= \frac{(m-1)}{2(m+1)}\Big(\frac{l}{\alpha^{\frac{2}{m-1}}}\Big)^{\frac{m+1}{m-1}}. \tag{2.10}$$

Proposition 2.4. *Suppose that the assumptions* (A) *are satisfied and for every* $0 < \beta < \frac{m+1}{p-1}\Big(\frac{m+1}{p+1}\Big)^{-\frac{p+1}{p-1}} < 1$ *assume that* $K(u_0) < 0$ *and* $E(0) < \bar{d}\beta$. *Then for any solution* $u \in \mathcal{H}^{1,(\frac{N-1}{2},\frac{N}{2})}_{\partial^0\mathbb{M}}$ *such that* $\Big\|\nabla_{\mathbb{M}}u\Big\|_{\frac{N-1}{2},\frac{N}{2}} \geq 1$ *and for every* $t \in [0,\infty)$, $K(u(t)) < 0$ *and also*

$$\begin{aligned}\bar{d} &< \Big(\frac{m+1}{p+1}\Big)^{\frac{2}{p-1}}\Big[\frac{p-1}{2(p+1)}\Big]\Big(l\Big\|\nabla_{\mathbb{M}}u\Big\|^2_{\frac{N-1}{2},\frac{N}{2}} + (g\circ\nabla_{\mathbb{M}}u)(t)\Big)\\ &< \alpha\Big(\frac{m+1}{p+1}\Big)^{\frac{2}{p-1}}\Big[\frac{p-1}{(p+1)}\Big]\Big\|\nabla_{\mathbb{M}}u\Big\|^{p+1}_{\frac{N-1}{2},\frac{N}{2}}.\end{aligned} \tag{2.11}$$

Proof. Arguing by contradiction, one can get $K(u(t)) < 0$ for all $t \in [0,\infty)$. In other words, if one supposes that this is not true, then there exists $t_0 > 0$ such that $K(u(t_0)) = 0$ and $K(u(t)) < 0$ for every $0 \leq t < t_0$. Thus by (2.8) one obtains

$$\begin{aligned}\bar{d} &= \frac{(p-1)}{2(m+1)}\alpha^{\frac{-2}{p-1}}\Big(\frac{l(m+1)}{p+1}\Big)^{\frac{p+1}{p-1}}\\ &\leq \frac{(p-1)}{2(m+1)}\Big(\frac{m+1}{p+1}\Big)^{\frac{p+1}{p-1}} \times \Bigg[\frac{l\Big\|\nabla_{\mathbb{M}}u(t_0)\Big\|^2_{\frac{N-1}{2},\frac{N}{2}} + (g\circ\nabla_{\mathbb{M}}u)(t_0)}{\Big(\Big\|u(t_0)\Big\|^{p+1}_{\frac{N-1}{p+1},\frac{N}{p+1}} + \Big\|u(t_0)\Big\|^{m+1}_{\frac{N-1}{m+1},\frac{N}{m+1},\partial^0(\mathbb{M})}\Big)^{\frac{2}{m+1}}}\Bigg]^{\frac{p+1}{p-1}}\\ &\leq \frac{(p-1)}{2(m+1)}\Big(\frac{m+1}{p+1}\Big)^{\frac{p+1}{p-1}} \times \Bigg[\frac{l\Big\|\nabla_{\mathbb{M}}u(t_0)\Big\|^2_{\frac{N-1}{2},\frac{N}{2}} + (g\circ\nabla_{\mathbb{M}}u)(t_0)}{\Big(l\Big\|\nabla_{\mathbb{M}}u(t_0)\Big\|^2_{\frac{N-1}{2},\frac{N}{2}} + (g\circ\nabla_{\mathbb{M}}u)(t_0)\Big)^{\frac{2}{m+1}}}\Bigg]^{\frac{p+1}{p-1}}\\ &= \frac{(p-1)}{2(m+1)}\alpha^{\frac{-2}{p-1}}\Big(\frac{l(m+1)}{p+1}\Big)^{\frac{p+1}{p-1}}\Big(l\Big\|\nabla_{\mathbb{M}}u(t_0)\Big\|^2_{\frac{N-1}{2},\frac{N}{2}} + (g\circ\nabla_{\mathbb{M}}u)(t_0)\Big)\\ &= \Bigg[\frac{(p+1)}{2(m+1)}\Big(\frac{m+1}{p+1}\Big)^{\frac{p+1}{p-1}} - \frac{1}{m+1}\Big(\frac{m+1}{p+1}\Big)^{\frac{p+1}{p-1}}\Bigg]\Big(l\Big\|\nabla_{\mathbb{M}}u(t_0)\Big\|^2_{\frac{N-1}{2},\frac{N}{2}} + (g\circ\nabla_{\mathbb{M}}u)(t_0)\Big)\\ &= \Big(\frac{m+1}{p+1}\Big)^{\frac{2}{p-1}}\Big[\frac{1}{2} - \frac{1}{p+1}\Big]\Big(l\Big\|\nabla_{\mathbb{M}}u(t_0)\Big\|^2_{\frac{N-1}{2},\frac{N}{2}} + (g\circ\nabla_{\mathbb{M}}u)(t_0)\Big)\\ &\leq \frac{1}{2}\Big(l\Big\|\nabla_{\mathbb{M}}u(t_0)\Big\|^2_{\frac{N-1}{2},\frac{N}{2}} + (g\circ\nabla_{\mathbb{M}}u)(t_0)\Big) - \frac{1}{p+1}\Big(\frac{m+1}{p+1}\Big)^{\frac{2}{p-1}}\Big[\Big\|u(t_0)\Big\|^{p+1}_{\frac{N-1}{p+1},\frac{N}{p+1}}\\ &\quad + \Big\|u(t_0)\Big\|^{m+1}_{\frac{N-1}{m+1},\frac{N}{m+1},\partial^1(\mathbb{M})}\Big] \leq \frac{l}{2}\Big\|\nabla_{\mathbb{M}}u(t_0)\Big\|^2_{\frac{N-1}{2},\frac{N}{2}} + \frac{1}{2}(g\circ\nabla_{\mathbb{M}}u)(t_0)\\ &\quad - \frac{1}{p+1}\Big\|u(t_0)\Big\|^{p+1}_{\frac{N-1}{p+1},\frac{N}{p+1}} - \frac{1}{m+1}\Big\|u(t_0)\Big\|^{m+1}_{\frac{N-1}{m+1},\frac{N}{m+1},\partial^1(\mathbb{M})} = J(u(t_0)).\end{aligned} \tag{2.12}$$

But, this is impossible since $J(u(t_0)) \leq E(t_0) \leq E(0) < \bar{d}$. Therefore, for every $t \in [0,\infty)$ one has $K(u(t)) < 0$. Furthermore, by inequality (2.12) one can get

$$\begin{aligned}\bar{d} &< \Big(\frac{m+1}{p+1}\Big)^{\frac{2}{p-1}}\Big[\frac{1}{2} - \frac{1}{p+1}\Big]\Big(l\Big\|\nabla_{\mathbb{M}}u\Big\|^2_{\frac{N-1}{2},\frac{N}{2}} + (g\circ\nabla_{\mathbb{M}}u)(t)\Big)\\ &< \Big(\frac{m+1}{p+1}\Big)^{\frac{2}{p-1}}\Big(\frac{p-1}{2(p+1)}\Big)\Big[\Big\|u\Big\|^{p+1}_{\frac{N-1}{p+1},\frac{N}{p+1}} + \Big\|u\Big\|^{m+1}_{\frac{N-1}{m+1},\frac{N}{m+1},\partial^1(\mathbb{M})}\Big]\end{aligned}$$

$$< \alpha\Big(\frac{m+1}{p+1}\Big)^{\frac{2}{p-1}}\Big(\frac{p-1}{(p+1)}\Big)\Big\|\nabla_{\mathbb{M}}u\Big\|^{p+1}_{\frac{N-1}{2},\frac{N}{2}}. \tag{2.13}$$

Hence, the proof is completed. □

Proposition 2.5. *Suppose that the assumptions (A) hold and for every $0 < \beta < \frac{m+1}{p-1} < 1$ we have $K(u_0) < 0$ and $E(0) < \tilde{d}\beta$. Then for any solution of problem 1.1 such that $\Big\|\nabla_{\mathbb{M}}u\Big\|_{\frac{N-1}{2},\frac{N}{2}} \leq 1$ and for every $t \in [0,\infty)$, $K(u(t)) < 0$ and also*

$$\tilde{d} < \frac{(m-1)}{2(m+1)}\Big[l\Big\|\nabla_{\mathbb{M}}u\Big\|^{2}_{\frac{N-1}{2},\frac{N}{2}} + (g\circ\nabla_{\mathbb{M}}u)(t)\Big] < \frac{\alpha(m-1)}{(m+1)}\Big\|\nabla_{\mathbb{M}}\Big\|^{m+1}_{\frac{N-1}{2},\frac{N}{2}}. \tag{2.14}$$

Proof. By the similar way in the proof of Proposition 2.4, we suppose that there exists $t_0 > 0$ such that $K(u(t_0)) = 0$ and $K(u(t)) < 0$ for every $0 \leq t < t_0$. Then

$$\begin{aligned}
\tilde{d} &< \frac{(m-1)}{2(m+1)}\Big(\frac{l}{\alpha^{\frac{2}{m+1}}}\Big)^{\frac{m+1}{m-1}} \leq \frac{(m-1)}{2(m+1)}\left\{\frac{l\Big\|\nabla_{\mathbb{M}}u\Big\|^{2}_{\frac{N-1}{2},\frac{N}{2}} + (g\circ\nabla_{\mathbb{M}}u)(t)}{\Big(\Big\|u\Big\|^{p+1}_{\frac{N-1}{p+1},\frac{N}{p+1}} + \Big\|u\Big\|^{m+1}_{\frac{N-1}{m+1},\frac{N}{m+1},\partial^1(\mathbb{M})}\Big)^{\frac{2}{m+1}}}\right\}^{\frac{m+1}{m-1}} \\
&\leq \frac{(m-1)}{2(m+1)}\left\{\frac{l\Big\|\nabla_{\mathbb{M}}u\Big\|^{2}_{\frac{N-1}{2},\frac{N}{2}} + (g\circ\nabla_{\mathbb{M}}u)(t)}{\Big(l\Big\|\nabla_{\mathbb{M}}u\Big\|^{2}_{\frac{N-1}{2},\frac{N}{2}} + (g\circ\nabla_{\mathbb{M}}u)(t)\Big)^{\frac{2}{m+1}}}\right\}^{\frac{m+1}{m-1}} \\
&= \frac{(m-1)}{2(m+1)}\Big[l\Big\|\nabla_{\mathbb{M}}u\Big\|^{2}_{\frac{N-1}{2},\frac{N}{2}} + (g\circ\nabla_{\mathbb{M}}u)(t)\Big] \\
&< \frac{(m-1)}{2(m+1)}\Big[\Big\|u\Big\|^{p+1}_{\frac{N-1}{p+1},\frac{N}{p+1}} + \Big\|u\Big\|^{m+1}_{\frac{N-1}{m+1},\frac{N}{m+1},\partial^1(\mathbb{M})}\Big].
\end{aligned}$$

Thus for this t_0 we get

$$\begin{aligned}
\tilde{d} &< (\frac{1}{2} - \frac{1}{m+1})\Big[\Big\|u(t_0)\Big\|^{p+1}_{\frac{N-1}{p+1},\frac{N}{p+1}} + \Big\|u(t_0)\Big\|^{m+1}_{\frac{N-1}{m+1},\frac{N}{m+1},\partial^1(\mathbb{M})}\Big] \leq \frac{l}{2}\Big\|\nabla_{\mathbb{M}}u(t_0)\Big\|^{2}_{\frac{N-1}{2},\frac{N}{2}} \\
&+ \frac{1}{2}(g\circ\nabla_{\mathbb{M}}u)(t_0) - \frac{1}{p+1}\Big\|u(t_0)\Big\|^{p+1}_{\frac{N-1}{p+1},\frac{N}{p+1}} - \frac{1}{m+1}\Big\|u(t_0)\Big\|^{m+1}_{\frac{N}{m+1},\frac{N}{m+1},\partial^1(\mathbb{M})} = J(u(t_0)).
\end{aligned}$$

But, this is impossible since $J(u(t_0)) \leq E(t_0) \leq E(0) < \tilde{d}$. Therefore, for every $t \in [0,\infty)$ we have $K(u(t)) < 0$. Hence,

$$\tilde{d} < \frac{(m-1)}{2(m+1)}\Big[l\Big\|\nabla_{\mathbb{M}}u\Big\|^{2}_{\frac{N-1}{2},\frac{N}{2}} + (g\circ\nabla_{\mathbb{M}}u)(t)\Big] < \frac{\alpha(m-1)}{(m+1)}\Big\|\nabla_{\mathbb{M}}\Big\|^{m+1}_{\frac{N-1}{2},\frac{N}{2}}.$$

□

3. Blow up of solutions and the life-span

In this section we prove the finite time blow up phenomena of the solution for the problem 1.1 for negative initial energy and obtain estimates for the blow up time T^* as a life-span of our problem.

Theorem 3.1. *Suppose that the assumption (A) hold and for every positive and fixed constant $\beta < \frac{m+1}{p-1}\Big(\frac{m+1}{p+1}\Big)^{-\frac{p+1}{p-1}} < 1$, let us consider $u_0 \in \mathcal{H}^{1,(\frac{N-1}{2},\frac{N}{2})}_{\partial^0(\mathbb{M})}(\mathbb{M})$, $u_1 \in L^{\frac{N-1}{k+1},\frac{N}{k+1}}_{k+1}(\mathbb{M})$ such that $K(u_0) < 0$ and $E(0) < \beta\bar{d}$. Moreover, Let $1 < k \leq \frac{N+2}{N-2}$ and relaxation function* *g satisfies the following inequality :*

$$\int_0^\infty g(s)ds < \frac{\Big(m+1-\gamma-\beta(p-1)(\frac{m+1}{p+1})^{\frac{p+1}{p-1}}\Big)\Big(\frac{m-1-\gamma}{2}-\frac{\beta(p-1)}{2}(\frac{m+1}{p+1})^{\frac{p+1}{p-1}}\Big)}{\frac{1}{2}\Big(m+1-\gamma-\beta(p-1)(\frac{m+1}{p+1})^{\frac{p+1}{p-1}}\Big)^2+1}, \tag{3.1}$$

where $0 < m+1-\beta(p-1)(\frac{m+1}{p+1})^{\frac{p+1}{p-1}} < \gamma$. *Then, a solution* u *of problem 1.1 with* $\Big\|\nabla_{\mathbb{M}}u\Big\|_{\frac{N-1}{2},\frac{N}{2}} \geq 1$ *blows up in finite time, that is, the maximum of the existence time* T_{max} *of* $u(t)$ *is finite and*

$$\lim_{t\to T_{\max}}\Big[\Big\|\partial_t u\Big\|^{k+1}_{\frac{N-1}{k+1},\frac{N}{k+1}}+\Big\|\nabla_{\mathbb{M}}u\Big\|^{p+1}_{\frac{N-1}{2},\frac{N}{2}}\Big]=+\infty. \tag{3.2}$$

Proof. From Proposition 2.4 we have that if $K(u_0) < 0$ then $K(u) < 0$ for any $t \in [0, T_{max})$ in the case of $E(0) < \beta\bar{d}$. By contradiction, we assume that the solution of problem 1.1 is global, that is $T_{\max} = +\infty$. Thus for any $T > 0$ we define the functional $F : [0, T] \to \mathbb{R}_+$ as follows :

$$F(t) := \Big\|\partial_t u\Big\|^{k+1}_{\frac{N-1}{k+1},\frac{N}{k+1}}+\Big\|\nabla_{\mathbb{M}}u\Big\|^{2}_{\frac{N-1}{2},\frac{N}{2}}+\Big\|u\Big\|^{p+1}_{\frac{N-1}{p+1},\frac{N}{p+1}}+\Big\|u\Big\|^{m+1}_{\frac{N-1}{m+1},\frac{N}{m+1},\partial^1(\mathbb{M})}. \tag{3.3}$$

From the continuity of the function F on $[0, T]$, there exist two positive constants δ_1, δ_2 such that $\delta_1 \leq F(t) \leq \delta_2$. Now, we take

$$N(t) = \beta\bar{d} - E(t), \qquad \forall\, t \in [0, T]. \tag{3.4}$$

Differentiating identity (3.4) with respect to t, we obtain

$$\begin{aligned} N^{'}(t) = -E^{'}(t) &= -\frac{1}{2}\int_{\mathbb{M}}\int_0^t g(t-s)\Big[\nabla_{\mathbb{M}}u(s)-\nabla_{\mathbb{M}}u(t)\Big]^2 ds\frac{dr_1}{r_1}dx\frac{dr_2}{r_1r_2} \\ &+\frac{1}{2}g(t)\Big\|\nabla_{\mathbb{M}}u(t)\Big\|^2_{\frac{N-1}{2},\frac{N}{2}} \geq 0. \end{aligned} \tag{3.5}$$

Therefore, $N(t) \geq N(0) = \beta\bar{d} - E(0) > 0$. From Proposition 2.4 we conclude

$$\begin{aligned} N(t) = \beta\bar{d} - E(t) &\leq \beta\bar{d} + \frac{1}{p+1}\Big\|u\Big\|^{p+1}_{\frac{N-1}{p+1},\frac{N}{p+1}} + \frac{1}{m+1}\Big\|u\Big\|^{m+1}_{\frac{N-1}{m+1},\frac{N}{m+1},\partial^1(\mathbb{M})} \\ &\leq \beta\Big(\frac{m+1}{p+1}\Big)^{\frac{2}{p-1}}\Big(\frac{\alpha(p-1)}{p+1}\Big)\Big\|\nabla_{\mathbb{M}}u\Big\|^{p+1}_{\frac{N-1}{2},\frac{N}{2}} + \frac{\alpha}{m+1}\Big\|\nabla_{\mathbb{M}}u\Big\|^{p+1}_{\frac{N-1}{2},\frac{N}{2}} \\ &= \alpha\Big[\frac{\beta(p-1)}{p+1}\Big(\frac{m+1}{p+1}\Big)^{\frac{2}{p-1}} + \frac{1}{m+1}\Big]\Big\|\nabla_{\mathbb{M}}u\Big\|^{p+1}_{\frac{N-1}{2},\frac{N}{2}}, \quad \forall\, t \in [0,T]. \end{aligned} \tag{3.6}$$

Now we define

$$G(t) := N^{1-\sigma}(t) + \frac{\epsilon}{k}\int_{\mathbb{M}} r_1 u|\partial_t u|^{k-1}\frac{dr_1}{r_1}dx\frac{dr_2}{r_1r_2}, \tag{3.7}$$

for every $t \geq 0$, and $0 < \epsilon \ll 1$ to be chosen later and $0 < \sigma < \frac{1}{k+1}$. Differentiating the equality 3.7 with respect to t and using equation (1.1), we get

$$\begin{aligned} G^{'}(t) &= (1-\sigma)N^{-\sigma}(t)N^{'}(t) + \frac{\epsilon}{k}\Big\|\partial_t u\Big\|^{k+1}_{\frac{N-1}{k+1},\frac{N}{k+1}} + \epsilon\Big(|\partial_t u|^{k-1}\partial_{tt}u, u\Big)_{\mathbb{M}} \\ &= (1-\sigma)N^{-\sigma}(t)N^{'}(t) + \frac{\epsilon}{k}\Big\|\partial_t u\Big\|^{k+1}_{\frac{N-1}{k+1},\frac{N}{k+1}} - \epsilon\Big\|\nabla_{\mathbb{M}}u\Big\|^2_{\frac{N-1}{2},\frac{N}{2}} \\ &+ \epsilon\int_{\mathbb{M}}\nabla_{\mathbb{M}}u(t)\int_0^t g(t-s)\nabla_{\mathbb{M}}u(s)ds\frac{dr_1}{r_1}dx\frac{dr_2}{r_1r_2} \end{aligned}$$

$$+\epsilon\Big\|u\Big\|^{p+1}_{\frac{N-1}{p+1},\frac{N}{p+1}}+\epsilon\Big\|u\Big\|^{m+1}_{\frac{N-1}{m+1},\frac{N}{m+1},\partial^1(\mathbb{M})}. \tag{3.8}$$

On the other hand,

$$\begin{aligned}(m+1-\gamma)N(t)&=(m+1-\gamma)\beta\bar{d}-\frac{(m+1-\gamma)}{k+1}\Big\|\partial_t u\Big\|^{k+1}_{\frac{N-1}{k+1},\frac{N}{k+1}}\\&-\frac{(m+1-\gamma)l}{2}\Big\|\nabla_{\mathbb{M}}u\Big\|^2_{\frac{N-1}{2},\frac{N}{2}}+\frac{(m+1-\gamma)}{2}\Big(go\nabla_{\mathbb{M}}u\Big)(t)\\&+\frac{(m+1-\gamma)}{p+1}\Big\|u\Big\|^{p+1}_{\frac{N-1}{p+1},\frac{N}{p+1}}+\frac{(m+1-\gamma)}{m+1}\Big\|u\Big\|^{m+1}_{\frac{N-1}{m+1},\frac{N}{m+1},\partial^1(\mathbb{M})}.\end{aligned} \tag{3.9}$$

By making use the Young inequality we conclude

$$\begin{aligned}&\int_{\mathbb{M}}\nabla_{\mathbb{M}}u(t)\int_0^t g(t-s)\Big[\nabla_{\mathbb{M}}u(s)-\nabla_{\mathbb{M}}u(t)\Big]ds\frac{dr_1}{r_1}dx\frac{dr_2}{r_1r_2}\\&\quad\le\frac{1}{4\xi}\int_0^t g(s)ds\Big\|\nabla_{\mathbb{M}}u(t)\Big\|^2_{\frac{N-1}{2},\frac{N}{2}}\\&\quad+\xi\int_0^t g(t-s)\int_{\mathbb{M}}\Big[\nabla_{\mathbb{M}}u(s)-\nabla_{\mathbb{M}}u(t)\Big]^2ds\frac{dr_1}{r_1}dx\frac{dr_2}{r_1r_2},\end{aligned} \tag{3.10}$$

where $\gamma,\xi>0$ to be determined later, we get from (3.8) and (3.10)

$$\begin{aligned}&G^{'}(t)=(1-\sigma)N^{-\sigma}(t)N^{'}(t)+\epsilon(m+1-\gamma)N(t)-\epsilon(m+1-\gamma)N(t)+\frac{\epsilon}{k}\Big\|\partial_t u\Big\|^{k+1}_{\frac{N-1}{k+1},\frac{N}{k+1}}\\&-\epsilon\Big\|\nabla_{\mathbb{M}}u\Big\|^2_{\frac{N-1}{2},\frac{N}{2}}+\epsilon\int_{\mathbb{M}}\nabla_{\mathbb{M}}u(t)\int_0^t g(t-s)\nabla_{\mathbb{M}}u(s)ds\frac{dr_1}{r_1}dx\frac{dr_2}{r_1r_2}+\epsilon\Big\|u\Big\|^{p+1}_{\frac{N-1}{p+1},\frac{N}{p+1}}\\&+\epsilon\Big\|u\Big\|^{m+1}_{\frac{N-1}{m+1},\frac{N}{m+1},\partial^1(\mathbb{M})}=(1-\sigma)N^{-\sigma}(t)N^{'}(t)+\epsilon(m+1-\gamma)N(t)-\epsilon(m+1-\gamma)\beta\bar{d}\\&+\frac{\epsilon(m+1-\gamma)}{k+1}\Big\|\partial_t u\Big\|^{k+1}_{\frac{N-1}{k+1},\frac{N}{k+1}}+\frac{\epsilon(m+1-\gamma)l}{2}\Big\|\nabla_{\mathbb{M}}u\Big\|^2_{\frac{N-1}{2},\frac{N}{2}}+\frac{\epsilon(m+1-\gamma)}{2}\Big(go\nabla_{\mathbb{M}}u\Big)(t)\\&-\frac{\epsilon(m+1-\gamma)}{p+1}\Big\|u\Big\|^{p+1}_{\frac{N-1}{p+1},\frac{N}{p+1}}-\frac{\epsilon(m+1-\gamma)}{m+1}\Big\|u\Big\|^{m+1}_{\frac{N-1}{m+1},\frac{N}{m+1},\partial^1(\mathbb{M})}+\frac{\epsilon}{k}\Big\|\partial_t u\Big\|^{k+1}_{\frac{N-1}{k+1},\frac{N}{k+1}}\\&-\epsilon\Big\|\nabla_{\mathbb{M}}u\Big\|^2_{\frac{N-1}{2},\frac{N}{2}}+\epsilon\int_{\mathbb{M}}r_1\nabla_{\mathbb{M}}u(t)\int_0^t g(t-s)\nabla_{\mathbb{M}}u(s)ds\frac{dr_1}{r_1}dx\frac{dr_2}{r_1r_2}+\epsilon\Big\|u\Big\|^{p+1}_{\frac{N-1}{p+1},\frac{N}{p+1}}\\&+\epsilon\Big\|u\Big\|^{m+1}_{\frac{N-1}{m+1},\frac{N}{m+1},\partial_1(\mathbb{M})}\ge(1-\sigma)N^{-\sigma}(t)N^{'}(t)+\epsilon(m+1-\gamma)N(t)-\epsilon(m+1-\gamma)\beta\bar{d}\\&+\epsilon\Big[\frac{m+1-\gamma}{k+1}+\frac{1}{k}\Big]\Big\|\partial_t u\Big\|^{k+1}_{\frac{N-1}{k+1},\frac{N}{k+1}}+\epsilon\Big[\frac{l(m+1-\gamma)}{2}-1\Big]\Big\|\nabla_{\mathbb{M}}u\Big\|^2_{\frac{N-1}{2},\frac{N}{2}}\\&+\frac{\epsilon(m+1-\gamma)}{2}\Big(go\nabla_{\mathbb{M}}\Big)(t)+\epsilon\Big[\frac{m+1-\gamma}{p+1}+1\Big]\Big\|u\Big\|^{p+1}_{\frac{N-1}{p+1},\frac{N}{p+1}}\\&+\epsilon\Big[\frac{m+1-\gamma}{m+1}+1\Big]\Big\|u\Big\|^{m+1}_{\frac{N-1}{m+1},\frac{N}{m+1},\partial^1(\mathbb{M})}-\frac{1}{4\xi}\int_0^t g(s)ds\Big\|\nabla_{\mathbb{M}}u(t)\Big\|^2_{\frac{N-1}{2},\frac{N}{2}}\\&+\xi\int_0^t g(t-s)\int_{\mathbb{M}}\Big[\nabla_{\mathbb{M}}u(s)-\nabla_{\mathbb{M}}u(t)\Big]^2ds\frac{dr_1}{r_1}dx\frac{dr_2}{r_1r_2}\ge(1-\sigma)N^{-\sigma}(t)N^{'}(t)\\&+\epsilon(m+1-\gamma)N(t)+\epsilon\Big[\frac{m+1-\gamma}{k+1}+\frac{1}{k}\Big]\Big\|\partial_t u\Big\|^{k+1}_{\frac{N-1}{k+1},\frac{N}{k+1}}\\&+\epsilon\Big[\Big(\frac{m+1-\gamma}{2}-1\Big)-\Big(\frac{m+1-\gamma}{2}+\frac{1}{4\xi}\Big)\int_0^t g(s)ds\Big]\Big\|\nabla_{\mathbb{M}}u\Big\|^2_{\frac{N-1}{2},\frac{N}{2}}\end{aligned}$$

$$+\epsilon\Big[\frac{m+1-\gamma}{2}-\xi\Big]\Big(go\nabla_{\mathbb{M}}u\Big)(t)-\epsilon(m+1-\gamma)\beta\bar{d}+\epsilon\Big[\frac{m+1-\gamma}{p+1}+1\Big]\Big\|u\Big\|^{p+1}_{\frac{N-1}{p+1},\frac{N}{p+1}}$$
$$+\epsilon\Big[\frac{m+1-\gamma}{m+1}+1\Big]\Big\|u\Big\|^{m+1}_{\frac{N-1}{m+1},\frac{N}{m+1},\partial^1(\mathbb{M})}. \tag{3.11}$$

Taking into account Proposition 2.4 we obtain

$$-\epsilon(m+1-\gamma)\beta\bar{d}\geq-\epsilon(m+1)\beta\Big(\frac{m+1}{p+1}\Big)^{\frac{2}{p1-1}}(\frac{1}{2}-\frac{1}{p+1})\Big[l\Big\|\nabla_{\mathbb{M}}u\Big\|^2_{\frac{N-1}{2},\frac{N}{2}}+\Big(go\nabla_{\mathbb{M}}u\Big)(t)\Big]$$
$$=-\epsilon(m+1)\beta\Big(\frac{m+1}{p+1}\Big)^{\frac{2}{p-1}}\Big(\frac{p-1}{2(p+1)}\Big)\Big[l\Big\|\nabla_{\mathbb{M}}u\Big\|^2_{\frac{N-1}{2},\frac{N}{2}}+\Big(go\nabla_{\mathbb{M}}u\Big)(t)\Big]$$
$$\frac{-\epsilon\beta(p-1)}{2}\Big(\frac{m+1}{p+1}\Big)^{\frac{p+1}{p-1}}\Big[l\Big\|\nabla_{\mathbb{M}}u\Big\|^2_{\frac{N-1}{2},\frac{N}{2}}+\Big(go\nabla_{\mathbb{M}}u\Big)(t)\Big]. \tag{3.12}$$

Now, by making use inequalities (3.11) and (3.12) we conclude

$$G^{'}(t)\geq(1-\sigma)N^{-\sigma}(t)N^{'}(t)+\epsilon(m+1-\gamma)N(t)$$
$$+\epsilon\Big[\frac{m+1-\gamma}{k+1}-\frac{1}{k}\Big]\Big\|\partial_t u\Big\|^{k+1}_{\frac{N-1}{k+1},\frac{N}{k+1}}+\epsilon\Bigg[\frac{m-1-\gamma}{2}-\frac{\beta(p-1)}{2}(\frac{m+1}{p+1})^{\frac{p+1}{p-1}}$$
$$-\Big(\frac{m+1-\gamma}{2}-\frac{\beta(p-1)}{2}(\frac{m+1}{p+1})^{\frac{p+1}{p-1}}+\frac{1}{4\xi}\Big)\int_0^t g(s)ds\Bigg]\Big\|\nabla_{\mathbb{M}}u\Big\|^2_{\frac{N-1}{2},\frac{N}{2}}$$
$$+\epsilon\Bigg[\frac{m+1-\gamma}{2}-\frac{\beta(p-1)}{2}\Big(\frac{m+1}{p+1}\Big)^{\frac{p+1}{p-1}}-\xi\Bigg]\Big(go\nabla_{\mathbb{M}}u\Big)(t)$$
$$+\epsilon\Big[\frac{m+1-\gamma}{p+1}+1\Big]\Big\|u\Big\|^{p+1}_{\frac{N-1}{p+1},\frac{N}{p+1}}+\epsilon\Big[\frac{m+1-\gamma}{m+1}+1\Big]\Big\|u\Big\|^{m+1}_{\frac{N-1}{m+1},\frac{N}{m+1},\partial^1(\mathbb{M})}. \tag{3.13}$$

According to assumption 3.1 we take $0<\xi<\frac{m+1-\gamma}{2}-\frac{\beta(p-1)}{2}(\frac{m+1}{p+1})^{\frac{p+1}{p-1}}$, then we obtain

$$G^{'}(t)\geq(1-\sigma)N^{-\sigma}(t)N^{'}(t)+\epsilon(m+1-\gamma)N(t)$$
$$+\epsilon\Big[\frac{m+1-\gamma}{k+1}-\frac{1}{k}\Big]\Big\|\partial_t u\Big\|^{k+1}_{\frac{N-1}{k+1},\frac{N}{k+1}}+\epsilon\theta_1\Big\|\nabla_{\mathbb{M}}u\Big\|^2_{\frac{N-1}{2},\frac{N}{2}}+\epsilon\theta_2\Big(go\nabla_{\mathbb{M}}u\Big)(t)$$
$$+\epsilon\Big[\frac{m+1-\gamma}{2}+1\Big]\Big\|u\Big\|^{p+1}_{\frac{N-1}{p+1},\frac{N}{p+1}}+\epsilon\Big[\frac{m+1-\gamma}{2}+1\Big]\Big\|u\Big\|^{m+1}_{\frac{N-1}{m+1},\frac{N}{m+1},\partial^1(\mathbb{M})} \tag{3.14}$$

such that

$$\theta_1=\frac{m-1-\gamma}{2}-\frac{\beta(p-1)}{2}(\frac{m+1}{p+1})^{\frac{p+1}{p-1}}-\Big(\frac{m+1-\gamma}{2}-\frac{\beta(p-1)}{2}(\frac{m+1}{p+1})^{\frac{p+1}{p-1}}+\frac{1}{4\xi}\Big)\int_0^t g(s)ds$$

and

$$\theta_2=\frac{m+1-\gamma}{2}-\frac{\beta(p-1)}{2}\Big(\frac{m+1}{p+1}\Big)^{\frac{p+1}{p-1}}-\xi$$

where the positive constant γ satisfies $0<m+1-\beta(p-1)(\frac{m+1}{p+1})^{\frac{p+1}{p-1}}<\gamma$. Therefore, we can estimate the following inequality for small number μ :

$$G^{'}(t)\geq\epsilon\mu\Bigg[N(t)+\Big\|\partial_t u\Big\|^{k+1}_{\frac{N-1}{k+1},\frac{N}{k+1}}+\Big\|\nabla_{\mathbb{M}}u\Big\|^2_{\frac{N-1}{2},\frac{N}{2}}+\Big(go\nabla_{\mathbb{M}}u\Big)(t)$$
$$+\Big\|u\Big\|^{p+1}_{\frac{N-1}{p+1},\frac{N}{p+1}}+\Big\|u\Big\|^{m+1}_{\frac{N-1}{m+1},\frac{N}{m+1},\partial^1(\mathbb{M})}\Bigg]\geq0. \tag{3.15}$$

Here, we choose ϵ small enough such that

$$G(0)=N^{1-\sigma}(0)+\frac{\epsilon}{k}\int_{\mathbb{M}}r_1u_0|u_1|^{k-1}u_1\frac{dr_1}{r_1}dx\frac{dr_2}{r_1r_2}.$$

Hence, for $t \in [0,T]$ we have $G(t) \geq (0) > 0$. Now, by making use of the Hölder and corner Sobolev inequalities, we conclude

$$\begin{aligned} \Big|\int_{\mathbb{M}} r_1 u|\partial_t u|^{k-1}\partial_t u \frac{dr_1}{r_1} dx \frac{dr_2}{r_1 r_2}\Big|^{\frac{1}{1-\sigma}} &\leq \Big\|\partial_t u\Big\|^{k+1}_{\frac{N-1}{k+1},\frac{N}{k+1}} \Big\|u\Big\|^{\frac{1}{1-\sigma}}_{\frac{N-1}{p+1},\frac{N}{p+1}} \\ &\leq C\Big\|\partial_t u\Big\|^{\frac{k}{1-\sigma}}_{\frac{N-1}{p+1},\frac{N}{p+1}} \Big\|\nabla_{\mathbb{M}} u\Big\|^{\frac{1}{1-\sigma}}_{\frac{N-1}{2},\frac{N}{2}} \\ &\leq C\Big(\Big\|\partial_t u\Big\|^{\frac{kq}{1-\sigma}}_{\frac{N-1}{k+1},\frac{N}{k+1}} + \Big\|\nabla_{\mathbb{M}} u\Big\|^{\frac{q'}{1-\sigma}}_{\frac{N-1}{2},\frac{N}{2}}\Big), \end{aligned} \tag{3.16}$$

where $\frac{1}{q} + \frac{1}{q'} = 1$. Furthermore, if we take $q = \frac{(k+1)(1-\sigma)}{k} > 1$, then $\frac{q'}{1-\sigma} = \frac{k+1}{(k+1)(1-\sigma)-k}$. Thus, from inequality (3.16)

$$\Big|\int_{\mathbb{M}} r_1 u|\partial_t u|^{k-1}\partial_t u \frac{dr_1}{r_1} dx \frac{dr_2}{r_1 r_2}\Big|^{\frac{1}{1-\sigma}} \leq C\Big(\Big\|\partial_t u\Big\|^{k+1}_{\frac{N-1}{k+1},\frac{N}{k+1}} + \|\nabla_{\mathbb{M}} u\|^{\frac{k+1}{(k+1)(1-\sigma)-k}}_{\frac{N-1}{2},\frac{N}{2}}\Big). \tag{3.17}$$

Combine Definition 3.7 and inequality (3.17), then

$$\begin{aligned} G^{\frac{1}{1-\sigma}}(t) &= \Big(N^{1-\sigma}(t) + \frac{\epsilon}{k}\int_{\mathbb{M}} r_1 u|\partial_t u|^{k-1}\partial_t u \frac{dr_1}{r_1} dx \frac{dr_2}{r_1 r_2}\Big)^{\frac{1}{1-\sigma}} \\ &\leq C\Big(N(t) + \Big\|\partial_t u\Big\|^{k+1}_{\frac{N-1}{k+1},\frac{N}{k+1}} + \Big\|\nabla_{\mathbb{M}} u\Big\|^{\frac{k+1}{(k+1)(1-\sigma)-k}}_{\frac{N-1}{2},\frac{N}{2}}\Big). \end{aligned} \tag{3.18}$$

From the functional in (3.3) and $N(t) \geq N(0) = \beta\bar{d} - E(0) > 0$ we have

$$\Big\|\nabla_{\mathbb{M}} u\Big\|^{\frac{k+1}{(k+1)(1-\sigma)-k}}_{\frac{N-1}{2},\frac{N}{2}} \leq \delta_2^{\frac{k+1}{2((k+1)(1-\sigma)-k)}} \leq \frac{\delta_2^{\frac{k+1}{2((k+1)(1-\sigma)-k)}}}{N(0)} N(t). \tag{3.19}$$

Now, by making use of relations (3.6), (3.18), and (3.19) we can estimate the following inequality

$$\begin{aligned} G^{\frac{1}{1-\sigma}} &\leq C\Bigg(\alpha\Big[\frac{\beta(p-1)}{p+1}\Big(\frac{m+1}{p+1}\Big)^{\frac{2}{p-1}} + \frac{1}{m+1}\Big]\Big\|\nabla_{\mathbb{M}} u\Big\|^{p+1}_{\frac{N-1}{2},\frac{N}{2}} + \Big\|\partial_t u\Big\|^{k+1}_{\frac{N-1}{k+1},\frac{N}{k+1}} \\ &\quad + \frac{\delta_2^{\frac{k+1}{2((k+1)(1-\sigma)-k)}}}{N(0)}\alpha\Big[\frac{\beta(p-1)}{p+1}\Big(\frac{m+1}{p+1}\Big)^{\frac{2}{p-1}} + \frac{1}{m+1}\Big]\Big\|\nabla_{\mathbb{M}} u\Big\|^{p+1}_{\frac{N-1}{2},\frac{N}{2}}\Bigg) \\ &\leq D\Big(\Big\|\partial_t u\Big\|^{k+1}_{\frac{N-1}{k+1},\frac{N}{k+1}} + \Big\|\nabla_{\mathbb{M}} u\Big\|^{p+1}_{\frac{N-1}{2},\frac{N}{2}}\Big) \end{aligned} \tag{3.20}$$

where $D = D(m,p,k,C)$ is a positive constant such that

$$\begin{aligned} D = C\max\Big\{&\alpha\Big[\frac{\beta(p-1)}{p+1}\Big(\frac{m+1}{p+1}\Big)^{\frac{2}{p-1}} + \frac{1}{m+1}\Big], 1, \\ &\frac{\delta_2^{\frac{k+1}{2((k+1)(1-\sigma)-k)}}}{N(0)}\alpha\Big[\frac{\beta(p-1)}{p+1}\Big(\frac{m+1}{p+1}\Big)^{\frac{2}{p-1}} + \frac{1}{m+1}\Big]\Big\}. \end{aligned}$$

By the combination of (3.15) and (3.20), we obtain

$$G^{'}(t) \geq DG^{\frac{1}{1-\sigma}}(t) \qquad \forall t \in [0,T]. \tag{3.21}$$

By integrating (3.21) on $(0,t)$, it follows that

$$G^{\frac{\sigma}{1-\sigma}}(t) \geq \frac{1}{G^{\frac{-\sigma}{1-\sigma}}(0) - D\frac{\sigma t}{1-\sigma}}, \qquad \forall\, t \in [0,T]. \tag{3.22}$$

Inequality (3.22) shows that $G(t)$ blows up in finite time

$$T^* \le \frac{1-\sigma}{G^{\frac{\sigma}{1-\sigma}}(0)D\sigma}. \tag{3.23}$$

Since the time T is arbitrary, one can choose T such that $T \ge \frac{1-\sigma}{G^{\frac{\sigma}{1-\sigma}}(0)D\sigma}$. Hence, we observe from (3.20) that there exits a time $T^* \in (0, T]$ such that

$$\lim_{t\to T^{*-}} \left(\left\|\partial_t u\right\|^{k+1}_{\frac{N-1}{k+1},\frac{N}{k+1}} + \left\|\nabla_{\mathbb{M}} u\right\|^{p+1}_{\frac{N-1}{2},\frac{N}{2}} \right) = +\infty, \tag{3.24}$$

which contradicts $T_{max} = +\infty$. Therefore, the solution of problem 1.1 blows up in finite time. □

Theorem 3.2. *Suppose that the assumption* (A) *hold and for an arbitrary positive and fixed constant* $\beta < \frac{m+1}{p-1} < 1$, *let us consider* $u_0 \in \mathcal{H}^{1,(\frac{N-1}{2},\frac{N}{2})}_{\partial^0(\mathbb{M})}(\mathbb{M})$, $u_1 \in L^{\frac{N-1}{k+1},\frac{N}{k+1}}_{k+1}(\mathbb{M})$ *such that* $K(u_0) < 0$ *and* $E(0) < \beta\tilde{d}$. *Moreover, assume that* $1 < k \le \frac{N+2}{N-2}$ *and the relaxation function* g *satisfies the following relation:*

$$\int_0^\infty g(s)ds < \frac{\Big[(m-1)(1-\beta)-\gamma\Big]^2 + 2\Big[(m-1)(1-\beta)-\gamma\Big]}{\Big[(m-1)(1-\beta)-\gamma+\Big]^2+1} \tag{3.25}$$

where $0 < \gamma < (m-1)(1-\beta)$. *Then, a solution* u *of the problem 1.1 with* $\left\|\nabla_{\mathbb{M}} u\right\|_{\frac{N-1}{2},\frac{N}{2}} \le 1$ *blows up in finite time, that is, the maximum existence time* T_{max} *of* $u(t)$ *is finite and*

$$\lim_{t\to T_{\max}} \left[\left\|\partial_t u\right\|^{k+1}_{\frac{N-1}{k+1},\frac{N}{k+1}} + \left\|\nabla_{\mathbb{M}} u\right\|^{m+1}_{\frac{N-1}{2},\frac{N}{2}} \right] = +\infty. \tag{3.26}$$

Proof. From Proposition 2.5 we have that if $K(u_0)) < 0$ and for every $t \in [0, T_{max})$, then $K(u(t)) < 0$. Similar to Theorem 3.1, we apply the contradiction method to prove of this theorem. For an arbitrary positive T we consider the functional $F : [0, T] \to \mathbb{R}_+$ as follows

$$F(t) := \left\|\partial_t u\right\|^{k+1}_{\frac{N-1}{k+1},\frac{N}{k+1}} + \left\|\nabla_{\mathbb{M}} u\right\|^{2}_{\frac{N-1}{2},\frac{N}{2}} + \left\|u\right\|^{p+1}_{\frac{N-1}{p+1},\frac{N}{p+1}} + \left\|u\right\|^{m+1}_{\frac{N-1}{m+1},\frac{N}{m+1},\partial^1(\mathbb{M})} \tag{3.27}$$

Because of the continuity of the functional F on $[0, T]$, there exist two positive constants δ_1, δ_2 for which $\delta_1 \le F(t) \le \delta_2$. Now, we set

$$N(t) = \beta\tilde{d} - E(t), \qquad \forall\, t \in [0, T]. \tag{3.28}$$

By making use of (3.28) and Proposition 2.5, we obtain

$$\begin{aligned} N(t) = \beta\tilde{d} - E(t) &\le \frac{\beta\alpha(m-1)}{m+1}\left\|\nabla_{\mathbb{M}} u\right\|^{m+1}_{\frac{N-1}{2},\frac{N}{2}} + \frac{\alpha}{m+1}\left\|\nabla_{\mathbb{M}} u\right\|^{m+1}_{\frac{N-1}{2},\frac{N}{2}} \\ &= \frac{\alpha}{m+1}\Big(1+\beta(m-1)\Big)\left\|\nabla_{\mathbb{M}} u\right\|^{m+1}_{\frac{N-1}{2},\frac{N}{2}} \qquad \forall\, t \in [0, T]. \end{aligned} \tag{3.29}$$

For every $t \ge 0$ we define

$$G(t) := N^{1-\sigma}(t) + \frac{\epsilon}{k}\int_{\mathbb{M}} r_1 u|\partial_t u|^{k-1}\frac{dr_1}{r_1}dx\frac{dr_2}{r_1 r_2}, \tag{3.30}$$

where $0 < \epsilon|||1$ and $0 < \sigma < \frac{1}{k+1}$. Applying Proposition 2.5, we estimate

$$-\epsilon(m+1-\gamma)\beta\tilde{d} \ge \frac{-\epsilon\beta(m-1)}{2}\Big(l\left\|\nabla_{\mathbb{M}} u\right\|^{2}_{\frac{N-1}{2},\frac{N}{2}} + \Big(g\circ\nabla_{\mathbb{M}} u\Big)(t)\Big). \tag{3.31}$$

Now, by the similar way in the proof of Theorem 3.1 and of the assumption (3.25), and also relations (3.30), (3.31) and the Young's inequality we can conclude for any $0<\xi\leq \frac{(m-1)(1-\beta)}{2}+\frac{2-\gamma}{2}$ the following conclusion:

$$\begin{aligned} G^{'}(t) &\geq (1-\sigma)N^{-\sigma}(t)N^{'}(t)+\epsilon(m+1-\gamma)N(t)+\epsilon\Big[\frac{1}{k}+\frac{m+1-\gamma}{k+1}\Big]\Big\|\partial_t u\Big\|^{k+1}_{\frac{N-1}{k+1},\frac{N}{k+1}} \\ &+\epsilon\kappa_1\Big\|\nabla_{\mathbb{M}}u\Big\|^2_{\frac{N-1}{2},\frac{N}{2}}+\epsilon\kappa_2\Big(go\nabla_{\mathbb{M}}u\Big)(t) \\ &+\epsilon\Big[1+\frac{m+1-\gamma}{p+1}\Big]\Big\|u\Big\|^{p+1}_{\frac{N-1}{p+1},\frac{N}{p+1}}+\epsilon\Big[1+\frac{m+1-\gamma}{m+1}\Big]\Big\|u\Big\|^{m+1}_{\frac{N-1}{m+1},\frac{N}{m+1},\partial^1(\mathbb{M})} \end{aligned} \tag{3.32}$$

where

$$\kappa_1=\frac{(m-1)(1-\beta)-\gamma}{2}-\Big(\frac{(m-1)(1-\beta)}{2}+\frac{2-\gamma}{2}+\frac{1}{4\xi}\Big)\int_0^t g(s)ds>0,$$

$$\kappa_2=\frac{(p-1)(1-\beta)}{2}+\frac{2-\gamma}{2}-\xi>0.$$

Then, for any positive and fixed constant μ, we have

$$\begin{aligned} G^{'}(t)\geq \epsilon\mu\Big[&N(t)+\Big\|\partial_t u\Big\|^{k+1}_{\frac{N-1}{k+1},\frac{N}{k+1}}+\Big\|\nabla_{\mathbb{M}}u\Big\|^2_{\frac{N-1}{2},\frac{N}{2}}+\Big(go\nabla_{\mathbb{M}}u\Big)(t) \\ &+\Big\|u\Big\|^{p+1}_{\frac{N-1}{p+1},\frac{N}{p+1}}+\Big\|u\Big\|^{m+1}_{\frac{N-1}{m+1},\frac{N}{m+1},\partial^1(\mathbb{M})}\Big]\geq 0. \end{aligned} \tag{3.33}$$

Hence, by choosing ϵ small enough and positive

$$G(0)=N^{1-\sigma}(0)+\frac{\epsilon}{k}\int_{\mathbb{M}} r_1u_0|u_1|^{k-1}u_1\frac{dr_1}{r_1}dx\frac{dr_2}{r_1r_2}.$$

Thus $G(t)\geq(0)>0$ for every $t\in[0,T]$. Now, by the functional in (3.27) and $N(t)\geq N(0)=\beta\tilde{d}-E(0)>0$ we obtain

$$\Big\|\nabla_{\mathbb{M}}u\Big\|^{\frac{k+1}{(k+1)(1-\sigma)-k}}_{\frac{N-1}{2},\frac{N}{2}}\leq \delta_2^{\frac{k+1}{2((k+1)(1-\sigma)-k)}}\leq \frac{\delta_2^{\frac{k+1}{2((k+1)(1-\sigma)-k)}}}{N(0)}N(t). \tag{3.34}$$

Therefore, similar Theorem 3.1,

$$G^{\frac{1}{1-\sigma}}(t)\leq D\Big(\Big\|\partial_t u\Big\|^{k+1}_{\frac{N-1}{k+1},\frac{N}{k+1}}+\Big\|\nabla_{\mathbb{M}}u\Big\|^{m+1}_{\frac{N-1}{2},\frac{N}{2}}\Big) \tag{3.35}$$

where

$$D=\max\Big\{\alpha\Big[\frac{\beta(m-1)+1}{m+1}\Big],1,\frac{\delta_2^{\frac{k+1}{2((k+1)(1-\sigma)-k)}}\alpha}{N(0)}\Big[\frac{\beta(m-1)+1}{m+1}\Big]\Big\}.$$

Hence, by the above estimates we can get the following for any $t\in[0,T]$

$$G^{'}(t)\geq DG^{\frac{1}{1-\sigma}}(t). \tag{3.36}$$

By integrating (3.36) on interval $(0,t)$, we have

$$G^{\frac{\sigma}{1-\sigma}}(t)\geq \frac{1}{G^{\frac{-\sigma}{1-\sigma}}(0)-D\frac{\sigma t}{1-\sigma}},\qquad \forall\ t\in[0,T]. \tag{3.37}$$

Therefore, relation (3.37) shows that $G(t)$ blows up in finite time

$$T^*\leq \frac{1-\sigma}{G^{\frac{\sigma}{1-\sigma}}(0)D\sigma}. \tag{3.38}$$

Because of the arbitrariness of time T, one can choose T such that $T \geq \frac{1-\sigma}{G^{\frac{\sigma}{1-\sigma}}(0)D\sigma}$. Hence, there exists a time $T^* \in (0, T]$ for which

$$\lim_{t \to T^{*-}} \left(\left\| \partial_t u \right\|^{k+1}_{\frac{N-1}{k+1}, \frac{N}{k+1}} + \left\| \nabla_{\mathbb{M}} u \right\|^{m+1}_{\frac{N-1}{2}, \frac{N}{2}} \right) = +\infty, \tag{3.39}$$

which contradicts with $T_{max} = +\infty$. Therefore, the solution of the problem 1.1 blows up in finite time. □

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