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IDEALS IN THE CENTER OF SYMMETRIC ALGEBRAS

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ABSTRACT. We study symmetric algebras A over a field F in which the Jacobson radical of the center of A, the socle of the center of A or the Reynolds ideal of A are ideals.

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1. Introduction

Let F be a field and let A be a finite-dimensional F-algebra. As customary, we denote its *center* by $Z(A)$, its *Jacobson radical* by $J(A)$ and its (left) *socle*, the sum of all simple left ideals of A, by $\operatorname{soc}(A)$. In this paper, we are interested in the Jacobson radical $J(Z(A))$ and the socle $\operatorname{soc}(Z(A))$ of $Z(A)$ as well as the *Reynolds ideal* $R(A) = \operatorname{soc}(A) \cap Z(A)$ of A. All three subspaces are ideals in $Z(A)$. We study the following properties:

Main Problem. *For which finite-dimensional F-algebras A is*

(P1) *the Jacobson radical $J(Z(A))$ of $Z(A)$, or*
(P2) *the socle $\operatorname{soc}(Z(A))$ of $Z(A)$, or*
(P3) *the Reynolds ideal $R(A)$ of A*

an ideal in A?

In this paper, an ideal I of A is always meant to be a two-sided ideal of A and we denote it by $I \trianglelefteq A$. Note that the property (P1) is equivalent to $A \cdot J(Z(A)) \subseteq J(Z(A))$, and the corresponding statement holds for (P2) and (P3). The properties (P1) – (P3) are trivially satisfied whenever A is commutative. Thus we can view these conditions as weak commutativity properties.

The question (P1) has already been answered for group algebras and their p-blocks by Clarke [5], Koshitani [8] and Külshammer [12]. The latter paper additionally contains some results on arbitrary symmetric algebras. Moreover, Landrock [13] has proven that $J(Z(A))$ is an ideal of A if A is a split symmetric local algebra of dimension at most 10.

In Section 2, we collect results that hold for arbitrary finite-dimensional algebras over fields. In particular, we prove that (P2) implies (P3), and that, for split local algebras, property (P1) implies (P2). Suppose that A_1 and A_2 are nonzero algebras such that $A_1/J(A_1)$ or $A_2/J(A_2)$ is separable. We show that $\mathrm{soc}(Z(A_1 \otimes A_2))$ is an ideal in $A_1 \otimes A_2$ if and only if $\mathrm{soc}(Z(A_i))$ is an ideal in A_i for $i = 1, 2$ (see Proposition 2.10). Section 3 is then devoted to the study of symmetric algebras. We show that for any symmetric algebra A such that $J(Z(A))$ is an ideal of A, and any ideal I of A such that A/I is symmetric, also $J(Z(A/I))$ is an ideal of A/I (see Proposition 3.10) and we prove an analogous result where the radical of the center is replaced by the socle of the center. In Section 4, we study the properties (P1) – (P3) for trivial extension algebras. In Section 5, we prove that $J(Z(A))$ is an ideal in any split symmetric local algebra A of dimension at most 11, and we present an example of a split symmetric local algebra A of dimension 12 in which $J(Z(A))$ is not an ideal (see Theorem 5.11). In the same spirit, we prove that $\mathrm{soc}(Z(A))$ is an ideal in any split symmetric local algebra A of dimension at most 16, and we give an example of a split symmetric local algebra A of dimension 20 in which $\mathrm{soc}(Z(A))$ is not an ideal (see Theorem 5.13). The dimensions 17, 18 and 19 remain open. These results will be used in a sequel to this paper dealing with group algebras.

2. Finite-dimensional algebras

We first investigate the properties (P1) – (P3) for arbitrary finite-dimensional algebras. We discuss some preliminary results as well as the relation between the three conditions. In the second part, we study the tensor product of two algebras.

2.1. Preliminaries. Let F be a field. All occurring F-algebras are supposed to be associative and unitary. We write $F\{a_1, \ldots, a_n\}$ for the subspace of an F-algebra A spanned by the elements $a_1, \ldots, a_n \in A$. For $a, b \in A$, we set $[a, b] := ab - ba$. For subspaces A_1, A_2 of A, we set $[A_1, A_2] = F\{[a_1, a_2] \colon a_1 \in A_1,\ a_2 \in A_2\}$. The subspace $K(A) = [A, A]$ is called the *commutator space* of A.

Lemma 2.1. ([11, Equation (3)] and [12, Remark 2.2]) *Let A be a finite-dimensional algebra over a field.*

(i) *We have $A \cdot K(A) = K(A) \cdot A$, and this is the smallest ideal I of A such that A/I is commutative.*
(ii) *For any ideal I of A, we have $K(A/I) = (K(A) + I)/I$.*

In our investigation, we mainly use the following criterion, which is stated in [12, Lemma 2.1]:

Lemma 2.2. *Let A be a finite-dimensional algebra over a field.*

(i) *For an element $z \in Z(A)$, we have $Az \subseteq Z(A)$ if and only if $K(A) \cdot z = 0$ holds.*
(ii) *$J(Z(A))$ is an ideal of A if and only if $J(Z(A)) \cdot K(A) = 0$ holds.*
(iii) *$\mathrm{soc}(Z(A))$ is an ideal of A if and only if $\mathrm{soc}(Z(A)) \cdot K(A) = 0$ holds.*
(iv) *$R(A)$ is an ideal of A if and only if $R(A) \cdot K(A) = 0$ holds.*

Proof. The statements (ii) – (iv) are subject of [12, Lemma 2.1], so it remains to prove (i). First assume $Az \subseteq Z(A)$. For $a, b \in A$, we then have $z[a, b] = [za, b] = 0$ since $za \in Z(A)$ holds, so z annihilates $K(A)$. Conversely, if $K(A) \cdot z = 0$ holds, we have $[za, b] = z[a, b] = 0$ for all $a, b \in A$ and hence $za \in Z(A)$ follows. $\square$

We now study the relations between the properties (P1) – (P3). Let A be a finite-dimensional algebra over a field F. Recall that the Jacobson radical of $Z(A)$ is given by $J(Z(A)) = J(A) \cap Z(A)$. For a subset $S \subseteq A$, we set $\mathrm{lAnn}_A(S)$ and $\mathrm{rAnn}_A(S)$ to be the left and the right annihilator of S in A, respectively, and write $\mathrm{Ann}_A(S)$ if both sets coincide. By [16, Theorem 1.8.18], we have $\mathrm{soc}(A) = \mathrm{rAnn}_A(J(A))$ and $\mathrm{soc}(Z(A)) = \mathrm{Ann}_{Z(A)}(J(Z(A)))$.

Lemma 2.3. *Let A be a finite-dimensional algebra over a field. If $\mathrm{soc}(Z(A))$ is an ideal in A, then $R(A)$ is an ideal in A.*

Proof. Let $\mathrm{soc}(Z(A))$ be an ideal of A. Lemma 2.2 then yields $\mathrm{soc}(Z(A)) \cdot K(A) = 0$. Clearly, $R(A)$ is contained in $\mathrm{soc}(Z(A))$, so we have $R(A) \cdot K(A) \subseteq \mathrm{soc}(Z(A)) \cdot K(A) = 0$ and hence $R(A)$ is an ideal of A by Lemma 2.2. $\square$

The converse of Lemma 2.3 does not hold. In fact, we will see below (Lemma 3.2) that $R(A)$ is an ideal in every symmetric split basic algebra A, but Example 5.15 presents a symmetric split local algebra A in which $\mathrm{soc}(Z(A))$ is not an ideal. Also, there is no immediate relation between the properties (P1) and (P2) or (P3).

Example 2.4.

(i) Let F be a field of odd characteristic. Consider the free algebra $F\langle X_1, X_2, X_3\rangle$ in variables X_1, X_2, X_3 and its quotient algebra

$$A \coloneqq F\langle X_1, X_2, X_3\rangle/(X_i^3, X_iX_j + X_jX_i)_{i,j=1,2,3,\ i\neq j}.$$

An F-basis of A is given by the set $\{x_1^{r_1}x_2^{r_2}x_3^{r_3} : r_1, r_2, r_3 \in \{0, 1, 2\}\}$, where x_i denotes the image of X_i in A for $i \in \{1, 2, 3\}$. It is easily verified that x_1^2 is contained in $J(Z(A))$, whereas $x_1^2x_2$ is not. Hence $J(Z(A))$ is not an ideal of A.

A short direct computation shows

$$\mathrm{soc}(Z(A)) = F\{x_1x_2^2x_3^2, x_1^2x_2x_3^2, x_1^2x_2^2x_3, x_1^2x_2^2x_3^2\} \trianglelefteq A.$$

By Lemma 2.3, we also have $R(A) \trianglelefteq A$. In fact, we have $R(A) = \mathrm{soc}(A)$ in this case.

(ii) Now let F be an arbitrary field and consider a non-commutative semisimple F-algebra A (e.g. the matrix algebra $\mathrm{Mat}_n(F)$ for $n \geq 2$). We have $J(Z(A)) = 0 \trianglelefteq A$ and $\mathrm{soc}(Z(A)) = R(A) = Z(A) \ntrianglelefteq A$.

However, in the important special case that A is a split local F-algebra, we now show that condition (P1) implies (P2). This is mainly due to the following observation:

Lemma 2.5. *Let A be a finite-dimensional split local algebra of dimension at least 2 over a field F. Then $\mathrm{soc}(Z(A)) \subseteq J(Z(A))$ holds.*

Proof. Since $\dim A \geq 2$ holds, we have $J(A) \neq 0$. Let $\ell \in \mathbb{N}$ be the least positive integer with $J(A)^\ell = 0$. Since A is of the form $A = F \cdot 1 \oplus J(A)$, we have $J(A)^{\ell-1} \subseteq Z(A)$. This implies $J(Z(A)) = J(A) \cap Z(A) \neq 0$. It follows that $\mathrm{soc}(Z(A))$ is a proper (nilpotent) ideal of $Z(A)$, which implies $\mathrm{soc}(Z(A)) \subseteq J(Z(A))$. □

We therefore obtain the following implication:

Corollary 2.6. *Let A be a finite-dimensional split local algebra over a field F. If $J(Z(A))$ is an ideal of A, then $\mathrm{soc}(Z(A))$ is an ideal of A.*

Proof. If $A \cong F$ holds, then A is commutative and hence $\mathrm{soc}(Z(A)) = \mathrm{soc}(A)$ is an ideal in A. Now assume $\dim A \geq 2$. By Lemmas 2.5 and 2.2, we have $\mathrm{soc}(Z(A)) \cdot K(A) \subseteq J(Z(A)) \cdot K(A) = 0$. Hence $\mathrm{soc}(Z(A))$ is an ideal of A again by Lemma 2.2. □

2.2. Tensor products. Throughout, let F be a field. We now study the properties (P1) – (P3) for the tensor product $A_1 \otimes A_2 := A_1 \otimes_F A_2$ of two nonzero finite-dimensional F-algebras A_1 and A_2.

Remark 2.7. It is well-known that for arbitrary finite-dimensional F-algebras A_1 and A_2, we have $Z(A_1 \otimes A_2) = Z(A_1) \otimes Z(A_2)$ and $K(A_1 \otimes A_2) = K(A_1) \otimes A_2 + A_1 \otimes K(A_2)$. Moreover, we have $J(A_1) \otimes A_2 + A_1 \otimes J(A_2) \subseteq J(A_1 \otimes A_2)$. If additionally $A_1/J(A_1)$ or $A_2/J(A_2)$ is separable, we even obtain $J(A_1 \otimes A_2) = J(A_1) \otimes A_2 + A_1 \otimes J(A_2)$ (see [15, Corollary 1.16.5]).

We now prove corresponding formulas for the socle and the Reynolds ideal of $A_1 \otimes A_2$, which are probably known as well:

Lemma 2.8. *Let A_1 and A_2 be finite-dimensional F-algebras such that $A_1/J(A_1)$ or $A_2/J(A_2)$ is separable.*

(i) *We have* $\operatorname{soc}(A_1 \otimes A_2) = \operatorname{soc}(A_1) \otimes \operatorname{soc}(A_2)$.

(ii) *We have $R(A_1 \otimes A_2) = R(A_1) \otimes R(A_2)$.*

Proof. Let $n_1, n_2 \in \mathbb{N}_0$ denote the dimensions of A_1 and A_2, respectively.

(i) We first show $\operatorname{soc}(A_1) \otimes A_2 = \operatorname{rAnn}_{A_1 \otimes A_2}(J(A_1) \otimes A_2)$. For $s \in \operatorname{soc}(A_1)$, $a, b \in A_2$ and $j \in J(A_1)$, we have $(j \otimes a) \cdot (s \otimes b) = js \otimes ab = 0 \otimes ab = 0$, so $\operatorname{soc}(A_1) \otimes A_2$ is contained in the right annihilator of $J(A_1) \otimes A_2$. For the converse inclusion, we choose F-bases $\{v_1, \ldots, v_{n_1}\}$ of A_1 and $\{w_1, \ldots, w_{n_2}\}$ of A_2. The set $\{v_i \otimes w_k \colon i = 1, \ldots, n_1,\ k = 1, \ldots, n_2\}$ is an F-basis of $A_1 \otimes A_2$. Consider an element $x \in \operatorname{rAnn}(J(A_1) \otimes A_2)$ and write $x \coloneqq \sum_{i=1}^{n_1} \sum_{k=1}^{n_2} \lambda_{ik}(v_i \otimes w_k)$ with $\lambda_{ik} \in F$ $(i = 1, \ldots, n_1,\ k = 1, \ldots, n_2)$. For any $j \in J(A_1)$, we obtain

$$0 = (j \otimes 1) \cdot x = \sum_{i=1}^{n_1} \sum_{k=1}^{n_2} \lambda_{ik}(jv_i \otimes w_k) = \sum_{k=1}^{n_2} \left(\sum_{i=1}^{n_1} \lambda_{ik} j v_i \right) \otimes w_k.$$

For $k = 1, \ldots, n_2$, this yields $0 = \sum_{i=1}^{n_1} \lambda_{ik} j v_i = j \cdot \sum_{i=1}^{n_1} \lambda_{ik} v_i$. We obtain $\sum_{i=1}^{n_1} \lambda_{ik} v_i \in \operatorname{soc}(A_1)$, which implies $x \in \operatorname{soc}(A_1) \otimes A_2$. Analogously, we show $A_1 \otimes \operatorname{soc}(A_2) = \operatorname{rAnn}_{A_1 \otimes A_2}(A_1 \otimes J(A_2))$. This yields

$$\begin{aligned} \operatorname{soc}(A_1 \otimes A_2) &= \operatorname{rAnn}_{A_1 \otimes A_2}(J(A_1) \otimes A_2 + A_1 \otimes J(A_2)) \\ &= \operatorname{rAnn}_{A_1 \otimes A_2}(J(A_1) \otimes A_2) \cap \operatorname{rAnn}_{A_1 \otimes A_2}(A_1 \otimes J(A_2)) \\ &= (\operatorname{soc}(A_1) \otimes A_2) \cap (A_1 \otimes \operatorname{soc}(A_2)) \\ &= \operatorname{soc}(A_1) \otimes \operatorname{soc}(A_2). \end{aligned}$$

(ii) With (i), we obtain

$$\begin{aligned} R(A_1 \otimes A_2) &= \operatorname{soc}(A_1 \otimes A_2) \cap Z(A_1 \otimes A_2) \\ &= (\operatorname{soc}(A_1) \otimes \operatorname{soc}(A_2)) \cap (Z(A_1) \otimes Z(A_2)) \\ &= (\operatorname{soc}(A_1) \cap Z(A_1)) \otimes (\operatorname{soc}(A_2) \cap Z(A_2)) \\ &= R(A_1) \otimes R(A_2). \end{aligned}$$

□

Lemma 2.9. *Let A_1 and A_2 be finite-dimensional F-algebras and let U_1 and U_2 be nonzero subspaces of A_1 and A_2, respectively. Then $U_1 \otimes U_2$ is an ideal in $A_1 \otimes A_2$ if and only if U_i is an ideal in A_i for $i = 1, 2$.*

Proof. Assume that U_i is an ideal of A_i for $i = 1, 2$. For $a_1 \in A_1$ and $a_2 \in A_2$, we then have $(a_1 \otimes a_2) \cdot (U_1 \otimes U_2) \subseteq U_1 \otimes U_2$ as well as $(U_1 \otimes U_2) \cdot (a_1 \otimes a_2) \subseteq U_1 \otimes U_2$, which shows that $U_1 \otimes U_2$ is an ideal of $A_1 \otimes A_2$.

Now assume conversely that $U_1 \otimes U_2$ is an ideal of $A_1 \otimes A_2$. We choose F-bases $\{v_1, \dots, v_{n_1}\}$ of A_1 and $\{w_1, \dots, w_{n_2}\}$ of A_2 such that $\{v_1, \dots, v_{k_1}\}$ and $\{w_1, \dots, w_{k_2}\}$ are bases of U_1 and U_2 for some $k_i \in \{1, \dots, n_i\}$ $(i = 1, 2)$, respectively. Then the set $\{v_{j_1} \otimes w_{j_2} \colon 1 \leq j_i \leq k_i \text{ for } i = 1, 2\}$ is an F-basis for $U_1 \otimes U_2$.

We show that $a_1 v_i \in U_1$ holds for all $a_1 \in A_1$ and $i = 1, \dots, k_1$. To this end, we set $a \coloneqq a_1 \otimes 1 \in A_1 \otimes A_2$ and $v \coloneqq v_i \otimes w_1 \in U_1 \otimes U_2$. Since $av \in U_1 \otimes U_2$ holds by assumption, there exist coefficients $\lambda_{rt} \in F$ for $1 \leq r \leq k_1$, $1 \leq t \leq k_2$ with $av = \sum_{r,t} \lambda_{rt} v_r \otimes w_t$. Expressing $a_1 v_i = \sum_{d=1}^{n_1} \mu_d v_d$ in terms of the basis of A_1 (with $\mu_1, \dots, \mu_{n_1} \in F$) yields

$$av = (a_1 \otimes 1) \cdot (v_i \otimes w_1) = a_1 v_i \otimes w_1 = \left(\sum_{d=1}^{n_1} \mu_d v_d \right) \otimes w_1 = \sum_{d=1}^{n_1} \mu_d (v_d \otimes w_1).$$

By comparing the coefficients in the two expressions for av, we obtain $\mu_d = 0$ for $d > k_1$. This shows $a_1 v_i \in U_1$. In a similar way, one shows $v_i a_1 \in U_1$, which proves that U_1 is an ideal of A_1. For U_2, we proceed analogously. □

Proposition 2.10. *Let A_1 and A_2 be nonzero finite-dimensional F-algebras such that $A_1/J(A_1)$ or $A_2/J(A_2)$ is separable.*

(i) $\operatorname{soc}(Z(A_1 \otimes A_2))$ *is an ideal of $A_1 \otimes A_2$ if and only if* $\operatorname{soc}(Z(A_i))$ *is an ideal of A_i for $i = 1, 2$.*
(ii) *$R(A_1 \otimes A_2)$ is an ideal of $A_1 \otimes A_2$ if and only if $R(A_i)$ is an ideal of A_i for $i = 1, 2$.*

Proof. Note that one of $Z(A_1)/J(Z(A_1))$ and $Z(A_2)/J(Z(A_2))$ is separable. Since $\operatorname{soc}(Z(A_1 \otimes A_2)) = \operatorname{soc}(Z(A_1) \otimes Z(A_2)) = \operatorname{soc}(Z(A_1)) \otimes \operatorname{soc}(Z(A_2))$ and $R(A_1 \otimes A_2) = R(A_1) \otimes R(A_2)$ hold by Lemma 2.8, the claim follows by Lemma 2.9. □

In contrast, the corresponding statement for the Jacobson radical of $Z(A_1 \otimes A_2)$ does not hold:

Example 2.11. Consider the quotient algebra $A = F[X]/(X^2)$ of the polynomial ring $F[X]$ and let $M \coloneqq \operatorname{Mat}_2(A)$ be the algebra of 2×2-matrices with entries in A. Note that $M \cong \operatorname{Mat}_2(F) \otimes_F A$ holds. We have $Z(M) = A \cdot \mathbb{1} \cong A$ and hence $J(Z(M)) = J(A) \cdot \mathbb{1}$. It is easily seen that $J(Z(M))$ is not closed under multiplication with arbitrary elements of M, so $J(Z(M))$ is not an ideal in M. On the other hand, we have $J(Z(A)) \trianglelefteq A$ since A is commutative and $J(Z(\operatorname{Mat}_2(F)))$ is an ideal in $\operatorname{Mat}_2(F)$ since $\operatorname{Mat}_2(F)$ is semisimple (see Example 2.4).

3. Symmetric algebras

Let F be a field. In this section, we investigate the main problem for symmetric algebras. In particular, we study the transition to certain quotient algebras.

A finite-dimensional F-algebra A is called *symmetric* if it admits a non-degenerate associative symmetric bilinear form $\beta\colon A \times A \to F$. The kernel of the associated linear form $\lambda\colon A \to F,\ a \mapsto \beta(1, a)$ then contains the commutator space $K(A)$, but no nonzero one-sided ideal of A (see [18, Theorem IV.2.2]). For a subspace X of A, we consider its orthogonal space $X^\perp = \{a \in A\colon \beta(a, x) = 0 \text{ for all } x \in X\}$ with respect to β.

Lemma 3.1. ([11, Equations (28) – (32), (35)]) *Let A be a symmetric F-algebra and consider subspaces X and Y of A. Then the following hold:*

(i) $\dim X + \dim X^\perp = \dim A$.
(ii) $(X^\perp)^\perp = X$.
(iii) $Y \subseteq X$ *implies* $X^\perp \subseteq Y^\perp$.
(iv) *We have* $(X + Y)^\perp = X^\perp \cap Y^\perp$ *and* $(X \cap Y)^\perp = X^\perp + Y^\perp$.
(v) *For an ideal I of A, we have $I^\perp = \mathrm{lAnn}_A(I) = \mathrm{rAnn}_A(I)$, and $I^\perp$ is an ideal of A as well. In particular, we obtain $J(A) = \mathrm{soc}(A)^\perp$.*
(vi) $K(A)^\perp = Z(A)$.

The symmetric algebras in which the Reynolds ideal is an ideal can be characterized as follows:

Lemma 3.2. *Let A be a split symmetric algebra. Then $R(A)$ is an ideal of A if and only if A is basic. In this case, we have $R(A) = \mathrm{soc}(A)$.*

Proof. Suppose first that $R(A)$ is an ideal of A. Since $A/J(A)$ is a semisimple and hence symmetric F-algebra (see [17, p. 630]), there is an element $s \in Z(A)$ such that $\mathrm{soc}(A) = J(A)^\perp = As$ (cf. Section 3.1 below). Then $s \in Z(A) \cap \mathrm{soc}(A) = R(A)$, and $\mathrm{soc}(A) = AR(A) = R(A) \subseteq Z(A)$. Hence $K(A) = Z(A)^\perp \subseteq \mathrm{soc}(A)^\perp = J(A)$, i.e. $A/J(A)$ is commutative, and A is basic. Conversely, suppose that A is basic. Since A is split, this implies that $A/J(A)$ is commutative. Thus $K(A) \subseteq J(A)$, and $\mathrm{soc}(A) = J(A)^\perp \subseteq K(A)^\perp = Z(A)$. Thus $R(A) = Z(A) \cap \mathrm{soc}(A) = \mathrm{soc}(A)$ is an ideal of A. □

In the following, we therefore focus on the study of $J(Z(A))$ and $\mathrm{soc}(Z(A))$.

3.1. Transition to quotient algebras.

Let F be a field. In this part, we consider various quotient algebras of A.

Lemma 3.3. *Let A be a symmetric F-algebra.*

(i) *$J(Z(A))$ is an ideal in A if and only if $K(B)$ is an ideal of $B \coloneqq A/\operatorname{soc}(A)$.*
(ii) *$\operatorname{soc}(Z(A))$ is an ideal in A if and only if $K(C)$ is an ideal of $C \coloneqq A/A \cdot J(Z(A))$.*

Proof. By Lemma 3.1, $J(Z(A))$ is an ideal of A if and only if $J(Z(A))^{\perp}$ is. Moreover, we have

$$J(Z(A))^{\perp} = (J(A) \cap Z(A))^{\perp} = J(A)^{\perp} + Z(A)^{\perp} = \operatorname{soc}(A) + K(A).$$

Since $\operatorname{soc}(A)$ is an ideal of A, $J(Z(A))^{\perp}$ is an ideal of A if and only if

$$A \cdot K(A) \cdot A \subseteq K(A) + \operatorname{soc}(A)$$

holds. Setting $B \coloneqq A/\operatorname{soc}(A)$, this is equivalent to $K(B) = (K(A)+\operatorname{soc}(A))/\operatorname{soc}(A)$ being an ideal of B (see Lemma 2.1). Similarly, $\operatorname{soc}(Z(A))$ is an ideal of A if and only if $\operatorname{soc}(Z(A))^{\perp}$ is. Note that $\operatorname{Ann}_A(J(Z(A))) = \operatorname{Ann}_A(A \cdot J(Z(A))) = (A \cdot J(Z(A)))^{\perp}$ follows by Lemma 3.1 (v), which yields

$$\operatorname{soc}(Z(A)) = Z(A) \cap \operatorname{Ann}_A(J(Z(A)) = Z(A) \cap (A \cdot J(Z(A)))^{\perp}.$$

By Lemma 3.1, this implies

$$\operatorname{soc}(Z(A))^{\perp} = Z(A)^{\perp} + A \cdot J(Z(A)) = K(A) + A \cdot J(Z(A)).$$

Hence $\operatorname{soc}(Z(A))$ is an ideal of A if and only if $K(A) + A \cdot J(Z(A))$ is, which is equivalent to $K(C)$ being an ideal of $C \coloneqq A/A \cdot J(Z(A))$. □

Remark 3.4. By Lemma 3.1 (v), $K(A)$ is an ideal of A if and only if $K(A)^{\perp} = Z(A)$ is, i.e., if and only if A is commutative. Hence if $J(Z(A))$ is an ideal of A and the algebra B defined in Lemma 3.3 is symmetric, then B is commutative. The analogous statement holds for $\operatorname{soc}(Z(A))$ and the algebra C defined in Lemma 3.3.

We now study quotient algebras of A which are again symmetric. Note that this is an additional condition which is not satisfied for arbitrary quotients of A:

Lemma 3.5. *Let A be a symmetric F-algebra with symmetrizing linear form λ. Moreover, let I be an ideal of A such that $\bar{A} := A/I$ is symmetric, and let $\bar{\lambda}$ be a symmetrizing linear form on $\bar{A}$. Then there exists a unique $z \in Z(A)$ such that $\bar{\lambda}(a + I) = \lambda(az)$ for all $a \in A$. In particular, we have $I = (Az)^{\perp}$. Conversely, for $z \in Z(A)$, the algebra $A/(Az)^{\perp}$ is symmetric with symmetrizing linear form $\bar{\lambda}$ defined by $\bar{\lambda}(a + I) = \lambda(az)$ for $a \in A$.*

Proof. The proof is given in [11, pages 429 – 430]. □

In particular, by applying Lemma 3.5 with $I = 0$, we see that two symmetrizing forms of a fixed symmetric algebra A only differ by a central unit. With the characterization given in Lemma 3.5, we can simplify the criterion for $J(Z(A)) \trianglelefteq A$ in case that A is local.

Lemma 3.6. *Let A be a symmetric local F-algebra. Then $J(Z(A)) \trianglelefteq A$ holds if and only if for all ideals $0 \neq I \trianglelefteq A$ such that A/I is symmetric, it follows that A/I is commutative.*

Proof. First assume $J(Z(A)) \trianglelefteq A$ and let $I \trianglelefteq A$ be a nonzero ideal such that A/I is symmetric. Since A is local, the ideal $I^\perp$ is contained in $J(A)$. This yields $\operatorname{soc}(A) \cdot I^\perp = 0$ and hence $\operatorname{soc}(A) \subseteq (I^\perp)^\perp = I$. The algebra A/I is therefore isomorphic to a quotient of $\bar{A} \coloneqq A/\operatorname{soc}(A)$. By Lemma 3.3, we have $K(\bar{A}) \trianglelefteq \bar{A}$ and hence $K(A/I) \trianglelefteq A/I$ follows by Lemma 2.1. Since A/I is symmetric, this implies $K(A/I) = 0$ (see Remark 3.4), so A/I is commutative. Conversely, assume that $K(A/I) = 0$ holds for every ideal $0 \neq I \trianglelefteq A$ for which the quotient A/I is symmetric. By Lemma 3.5, we have $I = (Az)^\perp$ for some $z \in J(Z(A))$. Lemma 2.1 then yields

$$K(A) \subseteq \bigcap_{z \in J(Z(A))} (Az)^\perp = \left(\sum_{z \in J(Z(A))} Az \right)^\perp = (A \cdot J(Z(A)))^\perp$$

and hence $J(Z(A)) \cdot K(A) \subseteq J(Z(A)) \cdot (A \cdot J(Z(A)))^\perp = 0$. By Lemma 2.2, $J(Z(A))$ is an ideal of A. □

3.2. Symmetric quotient algebras. Let A be a symmetric algebra over a field F, with symmetrizing linear form λ and corresponding bilinear form β. The aim of this section is to prove that the properties $J(Z(A)) \trianglelefteq A$ and $\operatorname{soc}(Z(A)) \trianglelefteq A$ are inherited by symmetric quotient algebras of A.

Thus let I be an ideal of A such that $\bar{A} := A/I$ is symmetric, with symmetrizing linear form $\bar{\lambda}$ and corresponding bilinear form $\bar{\beta}$. We denote the canonical map $A \to \bar{A}$, $a \mapsto \bar{a} := a + I$, by ν and consider its adjoint map $\nu^* : \bar{A} \to A$ defined by requiring

$$\beta(\nu^*(\bar{x}), y) = \bar{\beta}(\bar{x}, \nu(y)) \tag{3.1}$$

for all $x, y \in A$. By Lemma 3.5, there exists a unique $z \in Z(A)$ with $\bar{\lambda}(\bar{a}) = \beta(a, z) = \lambda(az)$ for all $a \in A$, and $I = (Az)^\perp$ holds.

Lemma 3.7. *The map ν^* has the following properties:*

(i) *It is explicitly given by $\nu^*(\bar{x}) = xz$ for all $x \in A$.*
(ii) *For all $x, y \in A$, we have $\nu^*(\bar{x}) \cdot y = \nu^*(\bar{x} \cdot \bar{y})$ and $x \cdot \nu^*(\bar{y}) = \nu^*(\bar{x} \cdot \bar{y})$.*

(iii) *The map ν^* is injective. Thus it induces an isomorphism of A-bimodules between $\bar{A}$ and Az.*

Proof. (i) The defining property (3.1) of ν^* is equivalent to $\lambda\big(\nu^*(\bar{x})\cdot y\big) = \bar{\lambda}(\bar{x}\cdot\bar{y}) = \lambda(xyz) = \lambda(xzy)$ for all $x, y \in A$. Hence the right ideal $(\nu^*(\bar{x}) - xz)A$ is contained in the kernel of λ. Since $\mathrm{Ker}(\lambda)$ contains no nontrivial one-sided ideals, this implies $\nu^*(\bar{x}) = xz$.

(ii) This directly follows from (i).

(iii) This follows from the fact that the map ν is surjective. □

Lemma 3.8. *We obtain the following relations:*

(i) $\nu^*\big(Z(\bar{A})\big) = Z(A) \cap \mathrm{Im}(\nu^*) = Z(A) \cap Az.$

(ii) $\nu^*(J(Z(\bar{A}))) = Z(A) \cap \nu^{-1}(\mathrm{soc}(\bar{A}))^\perp \subseteq J(Z(A)) \cap Az.$

(iii) *For $x \in Z(\bar{A})$, we have $\nu^*(x) \in \mathrm{soc}(Z(A))$ if and only if $x \cdot \nu(J(Z(A))) = 0$ holds. In particular, we obtain $\nu^*\big(\mathrm{soc}(Z(\bar{A}))\big) \subseteq \mathrm{soc}(Z(A))$.*

Proof. Let U be any subspace of $\bar{A}$, and let $x \in A$. Then $x \in \nu^*(U)^\perp$ is equivalent to $0 = \beta(\nu^*(U), x) = \bar{\beta}(U, \bar{x})$, which implies $\bar{x} \in U^\perp$. This shows $\nu^*(U)^\perp = \nu^{-1}(U^\perp)$ and $\nu^*(U) = \nu^{-1}(U^\perp)^\perp$.

(i) For $U = Z(\bar{A})$, we have $U^\perp = K(\bar{A}) = (K(A)+I)/I$ and $\nu^{-1}(U^\perp) = K(A)+I$. Thus $\nu^*(U) = (K(A)+I)^\perp = Z(A) \cap I^\perp = Z(A) \cap Az$ follows.

(ii) Let $U = J(Z(\bar{A})) = Z(\bar{A}) \cap J(\bar{A})$. Then we have $U^\perp = K(\bar{A}) + \mathrm{soc}(\bar{A})$ and $\nu^{-1}(U^\perp) = K(A) + \nu^{-1}(\mathrm{soc}(\bar{A}))$, so that

$$\nu^*(U) = (K(A) + \nu^{-1}(\mathrm{soc}(\bar{A})))^\perp = Z(A) \cap \nu^{-1}(\mathrm{soc}(\bar{A}))^\perp.$$

Moreover, since $\nu(\mathrm{soc}(A) + I) = \nu(\mathrm{soc}(A)) \subseteq \mathrm{soc}(\bar{A})$ holds, we have $\mathrm{soc}(A) + I \subseteq \nu^{-1}(\mathrm{soc}(\bar{A}))$. This implies $\nu^{-1}(\mathrm{soc}(\bar{A}))^\perp \subseteq (\mathrm{soc}(A) + I)^\perp = J(A) \cap I^\perp$ and

$$Z(A) \cap \nu^{-1}(\mathrm{soc}(\bar{A}))^\perp \subseteq Z(A) \cap J(A) \cap I^\perp = J(Z(A)) \cap Az.$$

(iii) For $x \in Z(\bar{A})$, we have $\nu^*(x) \in Z(A)$. By using Lemma 3.7 and the injectivity of ν^*, we obtain the equivalence

$$\begin{aligned}\nu^*(x) \in \mathrm{soc}(Z(A)) &\Leftrightarrow \nu^*(x) \cdot J(Z(A)) = 0 \Leftrightarrow \nu^*\big(x \cdot \nu(J(Z(A)))\big) = 0 \\ &\Leftrightarrow x \cdot \nu\big(J(Z(A))\big) = 0.\end{aligned}$$

Now let $x \in \mathrm{soc}(Z(\bar{A}))$. We then have $\nu\big(J(Z(A))\big) \subseteq J(\bar{A}) \cap Z(\bar{A}) = J(Z(\bar{A}))$, which implies $x \cdot \nu\big(J(Z(A))\big) = 0$. The above equivalence then yields $\nu^*(x) \in \mathrm{soc}(Z(A))$. □

Remark 3.9. Lemma 3.8 (iii) shows that $\operatorname{soc}(Z(A)) \cap \operatorname{Im} \nu^*$ is precisely the image under ν^* of the annihilator of $\nu(J(Z(A)))$ in $Z(\bar{A})$.

We now prove that the properties $J(Z(A)) \trianglelefteq A$ and $\operatorname{soc}(Z(A)) \trianglelefteq A$ are inherited by symmetric quotient algebras of A:

Proposition 3.10. *Let A be a symmetric F-algebra and consider an ideal $I \trianglelefteq A$ for which $\bar{A} := A/I$ is symmetric.*

(i) *If $J(Z(A)) \trianglelefteq A$ holds, then $J(Z(\bar{A})) \trianglelefteq \bar{A}$ follows.*
(ii) *If $\operatorname{soc}(Z(A))$ is an ideal of A, then $\operatorname{Ann}_{Z(\bar{A})}(\nu(J(Z(A))))$ and $\operatorname{soc}(Z(\bar{A}))$ are ideals of $\bar{A}$.*

Proof.

(i) Suppose that $J(Z(A))$ is an ideal in A. By Lemma 2.2, we have $J(Z(A)) \cdot K(A) = 0$. In particular, this yields $0 = \nu^*(J(Z(\bar{A}))) \cdot K(A) = \nu^*(J(Z(\bar{A})) \cdot K(\bar{A}))$ (see Lemmas 3.8 and 3.7). Since ν^* is injective, this implies $J(Z(\bar{A})) \cdot K(\bar{A}) = 0$. Hence $J(Z(\bar{A})) \trianglelefteq \bar{A}$ follows again by Lemma 2.2.
(ii) Suppose that $\operatorname{soc}(Z(A))$ is an ideal in A. Consider an element $a \in A$ with $\bar{a} \in \operatorname{Ann}_{Z(\bar{A})}(\nu(J(Z(A))))$. We then have $\bar{a} \cdot \nu(J(Z(A))) = 0$, so $\nu^*(\bar{a}) \in \operatorname{soc}(Z(A))$ follows by Lemma 3.8 (iii). Since $\operatorname{soc}(Z(A))$ is an ideal of A, this implies
$$0 = \nu^*(\bar{a}) \cdot K(A) = \nu^*\big(\bar{a} \cdot K(\bar{A})\big).$$
Thus $\bar{a} \cdot K(\bar{A}) = 0$ follows. By Lemma 2.2 (i), this yields $\bar{A}\bar{a} \subseteq Z(\bar{A})$ and we conclude that $\bar{A}\bar{a} \subseteq \operatorname{Ann}_{Z(\bar{A})}(\nu(J(Z(A))))$ holds. The statement concerning $\operatorname{soc}(Z(\bar{A}))$ can be proven similarly to (i). ☐

The following example demonstrates that in general, the properties (P1) and (P2) are not transferred to quotient algebras, that is, the prerequisites of Proposition 3.10 are necessary.

Example 3.11. Let F be a field of characteristic $\operatorname{char}(F) = 5$ and let $q \in F^\times$ be an element of order 24. We consider the (non-symmetric) algebra

$$A := F\langle X_1, X_2, X_3\rangle/(X_1^5, X_2^5, X_3^2, X_1X_2 + X_2X_1, X_3X_1 - qX_1X_3, X_3X_2 - qX_2X_3).$$

Here, $F\langle X_1, X_2, X_3\rangle$ denotes the free F-algebra in variables X_1, X_2, X_3. An F-basis of A is given by

$$\left\{x_1^{\ell_1} x_2^{\ell_2} x_3^{\ell_3} : \ell_1, \ell_2 \in \{0, \ldots, 4\},\ \ell_3 \in \{0, 1\}\right\},$$

where x_i denotes the image of X_i in A for $i = 1, 2, 3$. One can verify directly that $Z(A) = F\{1, x_1^4 x_2^4 x_3\}$ is 2-dimensional and that $J(Z(A)) = \operatorname{soc}(Z(A)) = \operatorname{soc}(A)$ is an ideal of A. Now consider the algebra

$$B \coloneqq F\langle X_1, X_2\rangle/(X_1^2, X_2^4, X_1X_2 + X_2X_1),$$

which can be viewed as a quotient algebra of A. We write y_1 and y_2 for the images of X_1 and X_2 in B, respectively. An F-basis of B is given by

$$\left\{y_1^{\ell_1} y_2^{\ell_2} : \ell_1 \in \{0,1\},\ \ell_2 \in \{0,\ldots,3\}\right\}.$$

A short computation shows that $J(Z(B)) = \operatorname{soc}(Z(B)) = F\{y_2^2, y_1y_2^3\}$ is 2-dimensional. However,

$$B \cdot J(Z(B)) = B \cdot \operatorname{soc}(Z(B)) = F\left\{y_2^2, y_2^3, y_1y_2^2, y_1y_2^3\right\}$$

is of dimension 4, so $J(Z(B)) = \operatorname{soc}(Z(B))$ is not an ideal in B.

4. Trivial extension algebras

Let F be a field. In the following, we consider trivial extension algebras, which arise in various contexts in the representation theory of finite-dimensional algebras.

For an F-vector space V, we set $V^* \coloneqq \operatorname{Hom}_F(V,F)$ to be the space of F-linear forms on V. Let A be an F-algebra. Recall that A^* becomes an A-A-bimodule by setting $(af)(x) \coloneqq f(xa)$ and $(fa)(x) \coloneqq f(ax)$ for $x, a \in A$ and $f \in A^*$. The trivial extension algebra $T \coloneqq T(A)$ of A is the vector space $A \oplus A^*$, endowed with the multiplication law

$$(a,f)\cdot(b,g) \coloneqq (ab, ag + fb) \text{ for all } a,b \in A \text{ and } f,g \in A^*.$$

We denote the elements of T by tuples (a,f) with $a \in A$ and $f \in A^*$. By [1, Proposition 3.1], T is a symmetric algebra and a symmetrizing linear form is given by $\lambda\colon T \to F,\ (a,f) \mapsto f(1)$.

In the following, we identify subspaces $V \subseteq A$ and $W \subseteq A^*$ with the subspaces $V \oplus 0$ and $0 \oplus W$ of T, respectively. In this way, A can be viewed as a subalgebra of T. Similarly, we identify A^* with the ideal $0 \oplus A^*$ of T, which squares to zero. For a subspace U of A, we view $(A/U)^*$ as a subset of A^* by identifying the map $f\colon A/U \to F$ with $f^\wedge\colon A \to F$ defined by setting $f^\wedge(x) = f(x+U)$ for all $x \in A$.

We first determine the substructures of T investigated in this paper, thereby extending some results of [1] and [3].

Lemma 4.1. *Let A be an F-algebra and let $T \coloneqq T(A)$ be the trivial extension algebra of A. Then the following identities hold:*

(i) $Z(T) = Z(A) \oplus (A/K(A))^*$

(ii) $K(T) = K(A) \oplus [A, A^*]$, *where* $[A, A^*]$ *is the* F*-subspace of* A^* *spanned by the elements* $af - fa$ *with* $a \in A$ *and* $f \in A^*$*.*

(iii) $J(T) = J(A) \oplus A^*$

(iv) $J(Z(T)) = J(Z(A)) \oplus (A/K(A))^*$

(v) $\operatorname{soc}(T) = 0 \oplus (A/J(A))^*$

(vi) $\operatorname{soc}(Z(T)) = \{b \in \operatorname{soc}(Z(A))\colon Ab \subseteq K(A)\} \oplus (A/(K(A) + A \cdot J(Z(A))))^*$

(vii) $R(T) = 0 \oplus (A/(K(A) + J(A)))^*$.

Proof. The statements of (i) and (ii) are proven in [1] as well as [3]. Since $J(A)\oplus A^*$ is a nilpotent ideal of T, we have $J(A) \oplus A^* \subseteq J(T)$. On the other hand, we have $J(T) = (J(T) \cap A) \oplus A^*$ and $J(T) \cap A$ is a nilpotent ideal of A, which yields $J(T) \cap A \subseteq J(A)$. This shows the identity in (iii). Combined with (i), this yields the formula for $J(Z(T))$ given in (iv). Since $A/J(A)$ is a semisimple right T-module, its dual $(A/J(A))^*$ is a semisimple left T-module. Hence $0 \oplus (A/J(A))^*$ is contained in $\operatorname{soc}(T)$. Since $\dim(0 \oplus (A/J(A))^*) = \dim T - \dim J(T)$ follows by (iii), we obtain the equality given in (v).

Now we show (vi). To this end, set $I \coloneqq K(A) + A \cdot J(Z(A))$. Consider $t \in \operatorname{soc}(Z(T))$, $a \in J(Z(A))$ and $f \in (A/K(A))^*$. By (i), we have $t = (b, g)$ with $b \in Z(A)$ and $g \in (A/K(A))^*$. Moreover, (iv) implies $(a, f) \in J(Z(T))$. Thus $0 = (b, g)(a, f) = (ba, bf + ga)$, i.e., we have $ba = 0$ and $0 = (bf + ga)(x) = f(xb) + g(ax)$ for $x \in A$. Since a is arbitrary, this forces $b \in \operatorname{soc}(Z(A))$ and $f(xb) = 0$ for $x \in A$, i.e. $f(Ab) = 0$. Since f is arbitrary, we obtain $Ab \subseteq K(A)$ and $g(ax) = 0$ for $x \in A$, so that $g(aA) = 0$. Since a is arbitrary, this also implies $g \in (A/I)^*$.

Now let $b \in \operatorname{soc}(Z(A))$ with $Ab \subseteq K(A)$ and $g \in (A/I)^*$. Note that (b, g) is contained in $Z(T)$. Consider an arbitrary element $(a, f) \in J(Z(T))$. Because of $a \in J(Z(A))$, we have $ab = 0$. Moreover, for any $x \in A$, we obtain $(ag + fb)(x) = g(xa) + f(bx) = 0$ since we have $bx = xb \in Ab \subseteq K(A) \subseteq \operatorname{Ker}(f)$ and $xa \in A \cdot J(Z(A)) \subseteq \operatorname{Ker}(g)$. This shows $(a, f) \cdot (b, g) = 0$ and hence $(b, g) \in \operatorname{soc}(Z(T))$.

Finally, using (i) and (v), we obtain

$$R(T) = \operatorname{soc}(T) \cap Z(T) = 0 \oplus (A/(K(A) + J(A)))^*,$$

which proves (vii). We remark that this statement, for fields of positive characteristic, is already proven in [1]. $\square$

Remark 4.2. For $b \in A$, requiring $Ab \subseteq K(A)$ as in Lemma 4.1 (iv) forces $b \in J(A)$: For $A' \coloneqq A/J(A)$ and $b' \coloneqq b + J(A)$, we have $A'b' \subseteq K(A')$. Since A' is semisimple and hence symmetric, we have $A'b' = 0$ since $K(A')$ does not contain any nontrivial left ideal. This implies $b \in J(A)$.

Theorem 4.3. *Let A be an F-algebra with trivial extension algebra $T := T(A)$.*

(i) *$J(Z(T))$ is an ideal in T if and only if $J(Z(A))$ and $K(A)$ are ideals in A.*

(ii) *$\mathrm{soc}(Z(T))$ is an ideal in T if and only if $I := K(A) + A \cdot J(Z(A))$ and $S := \{b \in \mathrm{soc}(Z(A))\colon Ab \subseteq K(A)\}$ are ideals of A.*

Proof. (i) By Lemmas 2.2 and 4.1, $J(Z(T))$ is an ideal of T if and only if we have

$$\begin{aligned} 0 &= J(Z(T)) \cdot K(T) \\ &= J(Z(A)) \cdot K(A) \oplus (J(Z(A)) \cdot [A, A^*] + (A/K(A))^* \cdot K(A)) . \end{aligned}$$

The condition $J(Z(A)) \cdot K(A) = 0$ is equivalent to $J(Z(A))$ being an ideal of A (see Lemma 2.2). If $J(Z(A))$ is an ideal of A, we have

$$J(Z(A)) \cdot [A, A^*] = [J(Z(A)) \cdot A, A^*] = [J(Z(A)), A^*] = 0.$$

Thus the second component of $J(Z(T)) \cdot K(T)$ is zero if and only if $(A/K(A))^* \cdot K(A)$ is. This is equivalent to $A \cdot K(A) = K(A) \cdot A$ being contained in $K(A)$ (see Lemma 2.1), that is, to $K(A)$ being an ideal in A.

(ii) Again, $\mathrm{soc}(Z(T))$ is an ideal in T if and only if we have

$$0 = \mathrm{soc}(Z(T)) \cdot K(T) = S \cdot K(A) \oplus (S \cdot [A, A^*] + (A/I)^* \cdot K(A)) .$$

If $S \cdot K(A) = 0$ holds, then we have $Ab \subseteq Z(A)$ for all $b \in S$ (see Lemma 2.2 (i)). Since Ab annihilates $J(Z(A))$, we even obtain $Ab \subseteq \mathrm{soc}(Z(A))$. Moreover, we have $A(Ab) \subseteq Ab \subseteq K(A)$ and hence $Ab \subseteq S$. This shows that S is an ideal of A. Conversely, if S is an ideal of A, then $S \cdot K(A) = 0$ follows by [12, Lemma 2.1]. If S is an ideal of A, we have

$$S \cdot [A, A^*] = [S, A^*] = 0.$$

Hence the second component of $\mathrm{soc}(Z(T)) \cdot K(T)$ is zero if and only if $(A/I)^* \cdot K(A) = 0$ holds. As before, this is equivalent to $A \cdot K(A) = K(A) \cdot A \subseteq I$, that is, to I being an ideal of A. $\square$

Remark 4.4. In the special case that the algebra A itself is symmetric, $J(Z(T))$ is an ideal of T if and only if A is commutative (see Remark 3.4).

5. Symmetric local algebras of small dimension

Let F be an algebraically closed field. Landrock showed in [13] that $J(Z(A))$ is an ideal in A for every symmetric local F-algebra A of dimension at most 10. In this section, we extend his result by proving that $J(Z(A))$ is an ideal in every symmetric local F-algebra A of dimension at most 11. We will also show that this bound is sharp. Moreover, we study the analogous problem for the socle of the center. In the following, we write J^i for the powers $J^i(A) := J(A)^i$ of the Jacobson radical of A.

Lemma 5.1. ([9, Lemma E]) *Let I be an ideal of an F-algebra A and let n be a positive integer. Suppose that*

$$I^n = F\{x_{i1}\cdots x_{in} \colon i = 1,\dots,d\} + I^{n+1}$$

holds for elements $x_{ij} \in I$. Then we have

$$I^{n+1} = F\{x_{j1}x_{i1}\cdots x_{in} \colon i,j = 1,\dots,d\} + I^{n+2}$$

and

$$I^{n+1} = F\{x_{i1}\cdots x_{in}x_{jn} \colon i,j = 1,\dots,d\} + I^{n+2}.$$

Lemma 5.2. ([3, Lemma 2.7]) *Let A be a local algebra with $\dim A/K(A) = 4$ and $\dim J^2/J^3 \geq 2$. Then either $\dim A \leq 8$ holds or there exist elements $a, b \in J$ such that $a^2 + J^3$ and $ab + J^3$ or $a^2 + J^3$ and $ba + J^3$ are linearly independent in J^2/J^3.*

Lemma 5.3. ([3, Lemma 0.3] and [9, Lemma G]) *Suppose that A is a local algebra. If $\dim J^i/J^{i+1} = 1$ holds for some positive integer i, then we have $J^i \subseteq Z(A)$. If A is additionally symmetric, then we even have $J^{i-1} \subseteq Z(A)$.*

The next statement can be found in the proof of [13, Theorem 3.2].

Lemma 5.4. *Let A be a non-commutative symmetric algebra. Then $\dim A \geq \dim Z(A) + 3$ holds.*

Proof. Since A is not commutative, we have $\dim Z(A) < \dim A$. If $\dim A = \dim Z(A) + 1$ holds, then we have $A = Fx \oplus Z(A)$ for some $x \in A$ and hence A is commutative, a contradiction. If $\dim A = \dim Z(A) + 2$ holds, we may write $A = Fx \oplus Fy \oplus Z(A)$ with $x, y \in A$. This yields $K(A) \subseteq F[x, y]$ and hence $1 \geq \dim K(A) = \dim A - \dim Z(A) = 2$, which is a contradiction. □

We now collect some properties of symmetric local algebras.

Lemma 5.5. ([14, Lemma 3.1]) *Let A be a symmetric local algebra. Then:*

(i) $\dim \operatorname{soc}(A) = 1$ *and* $\operatorname{soc}(A) \subseteq \operatorname{soc}(Z(A))$.
(ii) $K(A) \cap \operatorname{soc}(A) = 0$ *and $Z(A)$ is local.*
(iii) *We have $J^{\ell-1} = \operatorname{soc}(A)$, where ℓ denotes the minimal positive integer with $J^\ell = 0$.*

Theorem 5.6. *Let A be a symmetric local algebra. If $\dim Z(A) \leq 4$ holds, then A is commutative. For $\dim Z(A) = 5$, one of the following cases occurs:*

(i) $\dim A = 5$ *and A is commutative.*

(ii) $\dim A = 8$ *and there are two possibilities for the Loewy structure of A:*

  (a) $\dim J/J^2 = \dim J^2/J^3 = 3$ *and* $\dim J^3 = 1$*, or*

  (b) $\dim J/J^2 = \dim J^2/J^3 = \dim J^3/J^4 = 2$ *and* $\dim J^4 = 1$*.*

Proof. The result for $\dim Z(A) \leq 4$ is the statement of [9, Theorem B]. Now let $\dim Z(A) = 5$. The fact that the algebra A is of dimension 5 or 8 is the main result of [4]. If A is of dimension 5, then A is commutative, so let $\dim A = 8$. If $J^3 = 0$ holds, then Lemma 5.5 yields $\dim J^2/J^3 \leq 1$ and by Lemma 5.3, we obtain $J \subseteq Z(A)$. This implies that A is commutative, which is a contradiction. Hence we have $J^3 \neq 0$, which yields $\dim J^3/J^4 \geq 1$ by Nakayama's lemma. If $\dim J^3/J^4 \geq 2$ holds, then Lemma 5.1 yields $\dim J/J^2 \geq 2$ and $\dim J^2/J^3 \geq 2$. Furthermore, we have $\dim J^4/J^5 \geq 1$ by Lemma 5.5. As $\dim A = 8$ and $\dim A/J = 1$ hold, this implies that A has the Loewy structure given in (b). It remains to consider the case $\dim J^3/J^4 = 1$. Here, we obtain $J^2 \subseteq J(Z(A))$ by Lemma 5.3 and hence $\dim J^2 \leq 4$ follows. On the other hand, we have $K(A) \subseteq J^2$ and hence

$$(J^2)^\perp \subseteq K(A)^\perp \cap J = Z(A) \cap J = J(Z(A)),$$

which yields $\dim (J^2)^\perp \leq \dim J(Z(A)) = 4$. Lemma 3.1 (i) then implies $\dim J^2 = \dim (J^2)^\perp = 4$ and hence $J^2 = J(Z(A)) = (J^2)^\perp$. By Lemma 3.1 (v), this implies $J^4 = J^2 \cdot J^2 = J^2 \cdot (J^2)^\perp = 0$ and hence we obtain $\dim J^3 = 1$, so A has the Loewy structure given in (a). □

Both Loewy structures given in Theorem 5.6 (ii) occur as the following example demonstrates.

Example 5.7.

(i) For a field F of odd characteristic, we consider the algebra

$$A = F\langle X, Y, Z\rangle / \langle X^4, Y^2, Z^2, YX + XY, ZX + XZ, YZ - X^2, ZY + X^2\rangle.$$

Here, $F\langle X, Y, Z\rangle$ denotes the free algebra in variables X, Y, Z. One can show that $\dim A = 8$ and $\dim Z(A) = 5$ hold, that A is a symmetric local algebra and that the Loewy structure of A is of type (a) in Theorem 5.6 (ii).

(ii) Let F be a field of characteristic $p = 2$. Then the group algebra FD_8 of the dihedral group of order 8 over F has dimension 8, a 5-dimensional center and a Loewy structure of the second type described in Theorem 5.6 (ii).

We conclude this part with the following results on symmetric local F-algebras with a 6-dimensional center, which are proven in [6].

Lemma 5.8. ([6, Main theorem]) *Let A be a symmetric local F-algebra with $\dim Z(A) = 6$ and $\dim J/J^2 = 2$. Then $\dim A \leq 12$ holds.*

Lemma 5.9. ([6, Lemma 2.6]) *Let A be a symmetric local F-algebra of dimension at least 11 with $\dim Z(A) = 6$ and $\dim J/J^2 = \dim J^2/J^3 = 3$. By possibly replacing A by its opposite algebra, we find elements $a, b, c \in J$ with $J = F\{a, b, c\} + J^2$ such that either $J^2 = F\{a^2, ab, ac\} + J^3$ or $J^2 = F\{a^2, ab, ba\} + J^3$ holds.*

Remark 5.10. We take the opportunity to point out that Gerhard proved in her diploma thesis [7] that any symmetric local algebra A satisfying $\dim Z(A) = 6$ and $\dim J^2/J^3 = 3$ has dimension at most 21.

5.1. Jacobson radical. As before, let F be an algebraically closed field. In this section, we investigate the Jacobson radical of $Z(A)$. We prove that $J(Z(A))$ is an ideal in A if A is a symmetric local algebra of dimension at most 11 and we provide an example of a symmetric local F-algebra A of dimension 12 in which $J(Z(A))$ is not an ideal. This extends [13, Theorem 3.2], in which it is shown that $J(Z(A))$ is an ideal in A if A is a symmetric local algebra of dimension at most 10.

Theorem 5.11. *Let A be a symmetric local F-algebra with dimension $\dim A \leq 11$ over an algebraically closed field F. Then $J(Z(A))$ is an ideal in A.*

Proof. Let A be a symmetric local algebra of minimal dimension in which $J(Z(A))$ is not an ideal, and assume $\dim A \leq 11$. By Lemma 3.6, there exists a nonzero ideal I of A such that $A' := A/I$ is symmetric and non-commutative. By Theorem 5.6 and Lemma 5.4, this implies $\dim Z(A') \geq 5$ and $\dim A' \geq 8$. By Lemma 3.5, there exists an element $z \in J(Z(A))$ with $I = (Az)^{\perp}$. Note that we have $\dim Az = \dim A'$ and $Az \subseteq J$. If $\dim Az > 8$ holds, then there exists $x \in J$ with $J = Fx + Az = Fx + Fz + J^2$ since we have $\dim J \leq 10$. Since A is then generated by x and z (see [10, Proposition 5.2]), the algebra is commutative, a contradiction. Hence we have $\dim Az = 8$ and $\dim Z(A') = 5$. Moreover, $z' := z + I$ is contained in $J(Z(A'))$ and we have $\dim A'/A'z' \leq \dim A/Az \leq 3$, which implies $\dim A'z' > 4$. Since $\dim A' < \dim A$ holds, $J(Z(A'))$ is an ideal in A' by assumption. In particular, $A'z'$ is contained in $J(Z(A'))$ and hence we have $\dim A'z' \leq \dim J(Z(A')) = 4$, a contradiction. $\square$

Example 5.12. Let F be a field of odd characteristic. We consider the unitary subalgebra A of $\mathrm{Mat}_{12}(F)$ generated by the matrices

$$M = \begin{pmatrix}
. & . & . & . & . & . & . & . & . & . & . & . \\
1 & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . \\
. & 1 & . & . & . & . & . & . & . & . & . & . \\
. & . & 1 & . & . & . & . & . & . & . & . & . \\
. & . & . & 1 & . & . & . & . & . & . & . & . \\
. & . & . & . & 1 & . & . & . & . & . & . & . \\
. & . & . & . & . & 1 & . & . & . & . & . & . \\
. & . & . & . & . & . & 1 & . & . & . & . & . \\
. & . & . & . & . & . & . & 1 & . & . & . & . \\
. & . & . & . & . & . & . & . & 1 & . & . & . \\
. & . & . & . & . & . & . & . & . & 1 & . & .
\end{pmatrix}$$

and

$$N = \begin{pmatrix}
. & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . \\
1 & . & . & . & . & . & . & . & . & . & . & . \\
. & . & 1 & . & . & . & . & . & . & . & . & . \\
. & -1 & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & -1 & . & . & . & . & . & . & . \\
. & . & . & 1 & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & 1 & . & . & . & . & . \\
. & . & . & . & . & -1 & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & -1 & . & . & . \\
. & . & . & . & . & . & . & 1 & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & 1 & .
\end{pmatrix}.$$

Here, zero entries are represented by dots. A short computation shows $M^7 = M^5N = NM + MN = N^2 - M^2 = 0$. Actually, one can show that these are the defining relations of A as a quotient of a free algebra. The set

$$B := \left\{\mathbb{1}, M, N, M^2, MN, M^3, M^2N, M^4, M^3N, M^5, M^4N, M^6\right\}$$

generates A as an F-vector space. One easily verifies that these elements are linearly independent, so B is an F-basis of A. In particular, we obtain $\dim A = 12$. Since all nontrivial basis elements are nilpotent, the algebra A is local.

Let $s \in \mathrm{soc}(A)$ and write $s = \sum_{v \in B} c_v v$ with coefficients $c_v \in F$. The condition $sM^6 = 0$ translates to $c_{\mathbb{1}} = 0$ since all other products vanish. Due to $M^5N = M^7 = 0$, we obtain $0 = s \cdot M^5 = c_M \cdot M^6$, so $c_M = 0$, and $0 = s \cdot M^4N = c_N \cdot M^6$, which yields $c_N = 0$. Continuing this way yields $s = c_{M^6} \cdot M^6$, so $\mathrm{soc}(A) = FM^6$ is one-dimensional.

By directly calculating the commutators of the elements in B, we see that an F-basis of $K(A)$ is given by $\{MN, M^2N, M^3N, M^4N, M^3, M^5\}$ and hence we obtain $\dim K(A) = 6$. Note that $K(A) \cap \mathrm{soc}(A) = 0$ holds. Thus we can define a linear form $\lambda\colon A \to F$ by setting $\lambda(M^6) = 1$ and $\lambda(b) = 0$ for $b \in B \backslash \{M^6\}$, and extending this F-linearly to A. Then the kernel of λ does not contain any nonzero left ideals of A. Moreover, λ vanishes on $K(A)$. In particular, $\lambda(ab) - \lambda(ba) = \lambda(ab - ba) = \lambda([a, b]) = 0$ holds and hence λ is symmetric. By [18, Theorem IV.2.2], this shows that (A, λ) is a symmetric local algebra.

It is easily verified that the set $\{\mathbb{1}, M^2, M^4, M^5, M^4N, M^6\}$ is contained in $Z(A)$. This is even an F-basis of $Z(A)$ since we have $\dim Z(A) = \dim A - \dim K(A) = 6$. As M^2 is nilpotent, we have $M^2 \in J(Z(A))$. However, $M \cdot M^2 = M^3$ is not contained in $Z(A)$, so $J(Z(A))$ is not an ideal of A.

5.2. Socle. We now investigate the corresponding problem for the socle. First we show that $\mathrm{soc}(Z(A))$ is an ideal of A if A is a symmetric local algebra of dimension at most 16. In the second part of this section, we prove that there exists a local trivial extension algebra T of dimension 20 with $\mathrm{soc}(Z(T)) \ntrianglelefteq T$. The dimensions 17, 18 and 19 remain open.

Theorem 5.13. *Let A be a symmetric local algebra over an algebraically closed field F. If $\dim A \leq 16$ holds, then $\mathrm{soc}(Z(A))$ is an ideal in A.*

Proof. Assume that A is a symmetric local algebra of dimension at most 16 in which $\mathrm{soc}(Z(A))$ is not an ideal. By Lemma 2.2, there exists an element $z \in \mathrm{soc}(Z(A))$ with $z \cdot K(A) \neq 0$. By Lemma 3.5, $A' \coloneqq A/(Az)^{\perp}$ is a symmetric local algebra. Since $K(A)$ is not contained in $(Az)^{\perp}$, the algebra A' is not commutative, which yields $\dim Az = \dim A' \geq \dim Z(A') + 3 \geq 8$ by Theorem 5.6 and Lemma 5.4. Since $z \in \mathrm{soc}(Z(A)) \subseteq J(Z(A))$ holds (see Lemma 2.5), we obtain $z^2 = 0$. In particular, we have $Az \subseteq (Az)^{\perp}$ and hence Lemma 3.1 yields $\dim A = \dim Az + \dim (Az)^{\perp} \geq 2 \cdot \dim Az \geq 16$. By assumption, we then obtain $\dim A = 16$, which yields $\dim A' = \dim Az = \dim (Az)^{\perp} = 8$ and hence $Az = (Az)^{\perp}$.

We also see $\dim Z(A') = 5$ and conclude that $3 = \dim Z(A')^{\perp} = \dim K(A') = \dim(K(A) + Az)/Az$ holds. This implies

$$\dim K(A) + Az = 11 \text{ and } 5 = \dim(K(A) + Az)^{\perp} = \dim Z(A) \cap Az.$$

Moreover, $z \in \mathrm{soc}(Z(A))$ forces

$$J(Z(A)) \subseteq Z(A) \cap (Az)^{\perp} = Z(A) \cap Az \subseteq J(Z(A)),$$

so that $J(Z(A)) = Z(A) \cap Az$ and $\dim Z(A) = 6$. Hence we have $\dim K(A) = 10$ and $\dim K(A) \cap Az = \dim K(A) + \dim Az - \dim(K(A)+Az) = 10+8-11 = 7$. Since $\mathrm{soc}(A) \subseteq Az$ and $\mathrm{soc}(A) \cap K(A) = 0$ we obtain $Az = (K(A) \cap Az) \oplus \mathrm{soc}(A) \subseteq J^2$, and this yields $\dim J/J^2 = \dim J(A')/J(A')^2$.

The two possible Loewy structures of A' are given in Theorem 5.6. If A' has a Loewy structure of type (b), we have $\dim J/J^2 = \dim J(A')/J^2(A') = 2$. By Lemma 5.8, this yields the contradiction $\dim A \leq 12$. Hence the Loewy structure of A' is of type (a) in Theorem 5.6. We have $\dim J/J^2 = \dim J(A')/J^2(A') = 3$ and $\dim J^2/J^3 \geq \dim J^2(A')/J^3(A') = 3$.

Writing $J = F\{a, b, c\} + J^2$ for elements $a, b, c \in J$, we obtain

$$K(A) \subseteq F\{[a, b], [b, c], [a, c]\} + J^3.$$

Thus $\dim A/J^3 \leq 3 + \dim A/(K(A) + J^3)$ follows. On the other hand, we have $Az = (K(A) \cap Az) + \mathrm{soc}(A) \subseteq K(A) + J^3$, which implies $\dim A/(K(A) + J^3) = \dim A'/(K(A') + J(A')^3) < \dim A'/K(A') = 5$. Hence

$$\dim A/J^3 \leq 7 = \dim A'/J(A')^3 \leq \dim A/J^3,$$

so that $\dim A/J^3 = 7$ and $\dim J^3 = 9$. We also conclude that $Az \subseteq J^3$ and $Jz \subseteq J^4$ hold.

On the other hand, we have $J(A')^4 = 0$ and hence $J^4 \subseteq Az$. Since J^3 is not contained in $Z(A)$, Lemma 5.3 yields $\dim J^3/J^4 \geq 2$ and hence J^4 is a proper subset of Az. As in Lemma 3.7, the map

$$\varphi\colon A' \to Az,\ a + Az \mapsto az$$

is an A-bimodule isomorphism. With this, we see that Jz is the unique maximal submodule of Az and hence $J^4 = Jz$ follows. Furthermore, we obtain $\dim J^5 = \dim J^2z = 4$ and $\dim J^6 = \dim J^3z = 1$.

If we have $J^3 = xJ^2 + J^4$ for some $x \in J$, then $J^4 = x^2J^2 + J^5$ follows and we obtain the contradiction $\dim J^4/J^5 \leq \dim J^3/J^4 = 2$. Hence

$$\dim(xJ^2 + J^4)/J^4 < \dim J^3/J^4 = 2$$

holds for every $x \in J$. Similarly, we obtain $\dim(J^2x + J^4)/J^4 < 2$ for every $x \in J$. By Lemma 5.9, there exist elements $a, b, c \in J$ with $J = F\{a, b, c\} + J^2$ and $J^2 = F\{a^2, ab, ac\} + J^3$ or $J^2 = F\{a^2, ab, ba\} + J^3$. In the first case, we have $J^3 = a \cdot J^2 + J^4$ (see Lemma 5.1), which yields a contradiction as before. Hence $J^2 = F\{a^2, ab, ba\} + J^3$ follows. By Lemma 5.1, we obtain $J^3 = F\{a^3, a^2b, aba, ba^2, bab, b^2a\} + J^4$. Assume that a^3 is not contained in J^4. Then aJ^2 and J^2a are contained in $Fa^3 + J^4$. Hence $J^3 = F\{a^3, bab\} + J^4$ follows, so bab is not contained in J^4. This yields the contradiction

$$J^4 = F\{a^4, abab, ba^3, b^2ab\} + J^5 = F\{a^4, a^3b\} + J^5.$$

Hence a^3 is contained in J^4. Now assume $a^2b \notin J^4$. Then $J^3 = F\{a^2b, ba^2, b^2a\}+J^4$ holds. First let $J^3 = F\{a^2b, ba^2\} + J^4$. Then ba^2 is not contained in J^4 and we obtain the contradiction

$$J^4 = F\{a^3b, aba^2, ba^2b, b^2a^2\} + J^5 = F\{ba^2b, b^2a^2\} + J^5,$$

since $a^3b \in J^5$ and $aba^2 \in Fba^3+J^5 = J^5$ hold. Analogously, let $J^3 = F\{a^2b, b^2a\}+ J^4$, that is, b^2a is not contained in J^4. This yields the contradiction

$$J^4 = F\{a^3b, ab^2a, ba^2b, b^3a\} + J^5 = F\{ba^2b, b^3a\} + J^5$$

due to $a^3b \in J^5$ and $ab^2a \in Fa^3b + J^5 = J^5$. Hence we may assume $a^2b \in J^4$. In case that aba is not contained in J^4, we obtain $J^3 = F\{aba, bab\} + J^4$ and bab is not contained in J^4. This yields the contradiction

$$J^4 = F\{a^2ba, abab, baba, b^2ab\} + J^5 = F\{abab, baba\} + J^5$$

as in the previous cases. Hence we may additionally assume $aba \in J^4$. This implies $J^3 = bJ^2 + J^4$, which is a contradiction. □

For local trivial extension algebras, we obtain the following result:

Lemma 5.14. *Let A be a local algebra of dimension $\dim A \leq 9$ over an algebraically closed field F. Then:*

(i) *$I \coloneqq K(A) + A \cdot J(Z(A))$ is an ideal of A.*
(ii) *$S \coloneqq \{b \in \operatorname{soc}(Z(A)) \colon Ab \subseteq K(A)\}$ is an ideal of A.*
(iii) *For the corresponding trivial extension algebra $T \coloneqq T(A)$, we have*

$$\operatorname{soc}(Z(T)) \trianglelefteq T.$$

Proof.

(i) Assume that A is a counterexample. Then $A \cdot K(A) = K(A) \cdot A$ is not contained in I. Since $A = F \cdot 1 \oplus J$ holds, we obtain $JK(A) \not\subseteq I$. Because of $K(A) = [A, A] = [J, J] \subseteq J^2$, we have $JK(A) \subseteq J^3$, so J^3 is not contained in I and, in particular, not in $J(Z(A))$. Then J/J^2, J^2/J^3 and J^3/J^4 are of dimension at least 2 (see Lemma 5.3), which implies $\dim J^4 \leq 2$. This yields $J^4 \subseteq Z(A)$ since for $J^5 = 0$, we obtain $[A, J^4] = [J, J^4] \subseteq J^5 = 0$ and for $\dim J^5 = 1$, we have $\dim J^4/J^5 \leq 1$ and the statement follows by Lemma 5.3. On the other hand, $J^3 \nsubseteq J(Z(A))$ implies $J^4 \neq 0$, so $\dim J/J^2 \leq 3$ holds. Furthermore, we have $\dim(K(A)+J^3)/J^3 \leq 2$ since otherwise $\dim J^2/J^3 \geq 3$ and hence $\dim J/J^2 \leq 2$ follows. Writing $J = F\{x, y\} + J^2$ for some $x, y \in J$ then yields the contradiction $K(A) = [J, J] \subseteq F[x, y] + J^3$.

Set $B := A/J^4$ and $N := J(B)$. Then $N^4 = 0$ holds and there exist $a, b, c \in N$ with $N = F\{a, b, c\} + N^2$. Furthermore, $\dim(K(B) + N^3)/N^3 \leq 2$ holds and $NK(B)$ is not contained in $K(B) \subseteq F\{[a, b], [a, c], [b, c]\}+N^3 \subseteq N^2$. Hence we have $NK(B) = N[a, b]+N[a, c]+N[b, c]$. Due to symmetry reasons, we may assume that $N[a, b]$ is not contained in $K(B)$. Since $K(B)$ contains the elements $a[a, b]$ and $b[a, b]$, it follows that $c[a, b]$ is not contained in $K(B)$. On the other hand, since $\dim(K(B) + N^3)/N^3 \leq 2$ holds, there exists a nonzero tuple $(\alpha, \beta, \gamma) \in F^3$ with

$$\alpha[a, b] + \beta[a, c] + \gamma[b, c] \in N^3.$$

If α is nonzero, then $[a, b]$ is contained in $F\{[a, c], [b, c]\} + N^3$. We then obtain the contradiction $c[a, b] \in F\{c[a, c], c[b, c]\} \subseteq K(B)$. Hence we have $\alpha = 0$ and then $[\beta a + \gamma b, c] \in N^3$. Due to symmetry reasons, we may assume $\gamma \neq 0$. By dividing by γ, we may even assume $\gamma = 1$. But then we have $[\beta a + b, c]a \in N^4 = 0$ and we obtain the contradiction

$$\begin{aligned} c[a, b] &= c[a, \beta a + b] = ca(\beta a + b) - c(\beta a + b)a = ca(\beta a + b) - (\beta a + b)ca \\ &= [ca, \beta a + b] \in K(B). \end{aligned}$$

(ii) Assume that there exists an element $z \in \operatorname{soc}(Z(A))$ with $Az \subseteq K(A)$ such that $Az \nsubseteq Z(A)$ holds. Since we have $z \in K(A) \subseteq J^2$, this yields $J^3 \nsubseteq Z(A)$ and hence $J^4 \neq 0$. As in the proof of (i), it follows that J/J^2, J^2/J^3 and J^3/J^4 are at least 2-dimensional and we have $\dim J/J^2 \leq 3$ as well as $J^4 \subseteq Z(A)$. In particular, we obtain $z \notin J^3$, but $z^2 = 0$ holds due to $z \in \operatorname{soc}(Z(A))$.

By Lemma 2.2, we have $K(A) \cdot z \neq 0$. Hence there exist elements $a, b \in J$ with $0 \neq [a, b]z = [az, b] = [a, bz]$. This yields $az \notin Z(A)$, so $az \notin J^4$ holds. Due to $[Fa, Faz + J^4] = [Fa, J^4] = 0$, we have $bz \notin Faz + J^4$, so $az + J^4$ and $bz + J^4$ are linearly independent in J^3/J^4. Hence $a + J^2$ and $b + J^2$ are linearly independent in J/J^2.

Assume that $[a, b] + J^3$ and $z + J^3$ are linearly dependent in J^2/J^3. Then we have $[a, b] = \beta z + y$ for some $\beta \in F$ and $y \in J^3$. Hence $0 \neq [a, b]z =$

$\beta z^2 + yz = yz$ follows. In particular, we have $J^5 \neq 0$. Hence the Loewy layers J^i/J^{i+1} $(i = 0, \ldots, 5)$ of A are of dimensions 1,2,2,2,1,1. This implies $J^3 = F\{az, bz\} + J^4$ and $J^3 z = 0$, a contradiction to $yz \neq 0$.

Hence $[a, b] + J^3$ and $z + J^3$ are linearly independent in $(K(A) + J^3)/J^3 \subseteq J^2/J^3$. In particular, we obtain $\dim(K(A) + J^3)/J^3 \geq 2$. As in (i), this implies $\dim J/J^2 = 3$. It follows that the Loewy layers J^i/J^{i+1} $(i = 0, \ldots, 4)$ of A are of dimensions 1,3,2,2,1. This yields

$$J^2 = K(A) + J^3 = K(A) + Az = K(A).$$

By Lemma 5.2, there exist elements $x, y, w \in J$ such that $J = F\{x, y, w\} + J^2$ holds and $x^2 + J^3$ and $xy + J^3$ form a basis of J^2/J^3 (by possibly going over to the opposite algebra of A). By Lemma 5.1, $x^3 + J^4$ and $x^2 y + J^4$ form an F-basis of J^3/J^4. Write $xw \equiv \alpha x^2 + \beta xy \pmod{J^3}$ for some $\alpha, \beta \in F$. By replacing w by $\bar{w} \coloneqq w - \alpha x - \beta y$, we may assume $xw \in J^3$. Furthermore, there exist coefficients $\alpha_i, \beta_i \in F$ $(i = 1, \ldots, 4)$ with

$$\begin{aligned}
yw &\equiv \alpha_1 x^2 + \beta_1 xy \pmod{J^3}\\
wx &\equiv \alpha_2 x^2 + \beta_2 xy \pmod{J^3}\\
wy &\equiv \alpha_3 x^2 + \beta_3 xy \pmod{J^3}\\
w^2 &\equiv \alpha_4 x^2 + \beta_4 xy \pmod{J^3}.
\end{aligned}$$

With this, we obtain

$$\begin{aligned}
0 \equiv (xw)x &\equiv x(wx) \equiv \alpha_2 x^3 + \beta_2 x^2 y \pmod{J^4}\\
0 \equiv (xw)y &\equiv x(wy) \equiv \alpha_3 x^3 + \beta_3 x^2 y \pmod{J^4}\\
0 \equiv (xw)w &\equiv xw^2 \quad\equiv \alpha_4 x^3 + \beta_4 x^2 y \pmod{J^4}.
\end{aligned}$$

Comparing the coefficients yields $\alpha_2 = \beta_2 = \alpha_3 = \beta_3 = \alpha_4 = \beta_4 = 0$. This implies

$$\begin{aligned}
0 \equiv yw^2 \equiv (yw)w \equiv \alpha_1 x^2 w + \beta_1 xyw &\equiv \beta_1 xyw\\
&\equiv \beta_1(\alpha_1 x^3 + \beta_1 x^2 y) \pmod{J^4},
\end{aligned}$$

which yields $\beta_1 = 0$. Furthermore, we obtain

$$0 \equiv y(wy) \equiv (yw)y \equiv \alpha_1 x^2 y \pmod{J^4}$$

and hence $\alpha_1 = 0$. This yields $[x, w], [y, w] \in J^3$ and hence

$$J^2 = K(A) \subseteq F\{[x, y], [x, w], [y, w]\} + J^3 \subseteq F[x, y] + J^3,$$

which is a contradiction to $\dim J^2/J^3 = 2$.

(iii) This follows by Theorem 4.3 together with (i) and (ii). □

Hence every local trivial extension algebra T of dimension at most 18 satisfies $\mathrm{soc}(Z(T)) \trianglelefteq T$. The following example demonstrates that this bound is tight:

Example 5.15. Let F be a field of characteristic 2. We consider the unitary subalgebra A of $\mathrm{Mat}_{10}(F)$ generated by the matrices

$$M = \begin{pmatrix} . & . & . & . & . & . & . & . & . & . \\ 1 & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & 1 & . & . & . & . & . & . & . & . \\ . & . & 1 & . & . & . & . & . & . & . \\ . & . & . & 1 & . & . & . & . & . & . \\ . & . & . & . & 1 & . & . & . & . & . \\ . & . & . & . & . & 1 & . & . & . & . \\ . & . & . & . & . & . & 1 & . & . & . \\ . & . & . & . & . & . & . & 1 & 1 & . \end{pmatrix}$$

and

$$N = \begin{pmatrix} . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ 1 & . & . & . & . & . & . & . & . & . \\ . & 1 & . & . & . & . & . & . & . & . \\ . & 1 & . & . & . & . & . & . & . & . \\ . & 1 & . & . & . & . & . & . & . & . \\ . & 1 & . & 1 & 1 & . & . & . & . & . \\ . & . & . & 1 & . & 1 & . & . & . & . \\ . & . & . & . & 1 & 1 & . & . & . & . \\ . & . & . & . & . & 1 & 1 & 1 & 1 & . \end{pmatrix}.$$

Again, zero entries are represented by dots. One can check that these matrices satisfy the relations $M^6 = N^2 = 0$ as well as $NM = M^2 + MN + M^3 + M^2N$. Moreover, $M^4N = M^5$ holds. It is easy to see that the set $B := \{M^{\ell_1}N^{\ell_2} : \ell_1 \in \{0, \dots, 4\},\ \ell_2 \in \{0, 1\}\}$ is an F-basis for A. As before, we set $J := J(A)$. The Loewy layers J^i/J^{i+1} $(i = 0, \dots, 5)$ of A have the dimensions $1, 2, 2, 2, 2, 1$, respectively. In particular, A is a local algebra. One computes that

$$J(Z(A)) = F\{M^4, M^5\} \subseteq F\left\{M^2, M^3 + M^3N, M^2N + M^3N, M^4, M^5\right\} = K(A)$$

holds. Thus $J(Z(A))$ is an ideal of A. Since $K(A)$ contains M^2, but not M^3, we obtain $K(A) + A \cdot J(Z(A)) = K(A) \neq A \cdot K(A)$. In particular, $K(A)$ is not an ideal of A. The trivial extension algebra $T \coloneqq T(A)$ has dimension 20 and by Theorem 4.3, $\mathrm{soc}(Z(T))$ is not an ideal of T.

Combined with Theorem 5.13, this yields $\dim A \in \{17, 18, 19, 20\}$ for a symmetric local F-algebra A of minimal dimension in which $\mathrm{soc}(Z(A))$ is not an ideal.

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