http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 73, Number 2, Pages 554–568 (2024)
DOI:10.31801/cfsuasmas.1296991
ISSN 1303-5991 E-ISSN 2618-6470

Research Article; Received: May 14, 2023; Accepted: February 9, 2024

MINIMAL TRANSLATION SURFACES IN A STRICT WALKER 3-MANIFOLD

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Abstract. In this paper, we study minimal translations surfaces in a strict Walker 3-manifold. Based on the existence of two isometries, we classify minimal translation surfaces on this class of manifold. Some drawings have been added to illustrate the shape of certain surfaces.

1. Introduction

Minimal surfaces are the most natural objects in differential geometry, and have been studied during the last two and half centuries since J. L. Lagrange. In particular, minimal surfaces have encountered striking applications in other fields, like mathematical physics, conformal geometry, computer aided design, among others. In order to search for more minimal surfaces, some natural geometric assumptions arise. Translation surfaces were studied in the Euclidean 3-dimensional space and they are represented as graphs $z = \alpha(x)+\beta(y)$, where α and β are smooth functions. Scherk [11] proved in 1835 that, besides the planes, the only minimal translation surfaces are the surfaces given by

$$z = \frac{1}{a} \log \Big| \frac{\cos(ax)}{\cos(ay)} \Big|,$$

where a is a non-zero constant. Since then, minimal translation surfaces were generalized in several directions. For example, the Euclidean space $\mathbb{R}^3$ was replaced with other spaces of dimension 3-usually being 3-dimensional Lie groups and the

2020 *Mathematics Subject Classification.* 53A10, 53C42, 53C50.
Keywords. Minimal surfaces, translation surfaces, Walker manifolds.
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notion of translation was often replaced by using the group operation (see for example [6], [8], [14] and references therein). Another generalizations of Scherk surfaces are: affine translation surfaces in Euclidean 3-space [7], affine translation surfaces in affine 3-dimensional space [12] and translation surfaces in Galilean 3-space [14]. On the other hand, Scherk surfaces were generalized to minimal translation surfaces in Euclidean spaces of arbitrary dimensions(see [5], [9]). In [13], the authors introduce and define the notion of translation surfaces in the Heisenberg group $\mathbb{H}(1;1)$ as the formal analogue to those in the Euclidean 3-space.

In this paper, we define and classify minimal translation surfaces in a 3-dimensional strict Walker manifold. The strict Walker manifolds are described in terms of a suitable coordinates (x, y, z) of the manifolds $\mathbb{R}^3$ and their metric depends on an arbitrary function of two variables $f = f(y, z)$ and their metric tensor is given by

$$g_f^\epsilon = \epsilon dy^2 + 2dxdz + fdz^2 \qquad (1)$$

where $\epsilon = \pm 1$. These manifolds are denoted by (M, g_f^ϵ). In [4], the authors study a class of minimal surfaces in the three-dimensional Lorentzian Walker manifolds. Their proved the existence of minimal flat and non totally geodesic graphs on three dimensional Lorentizain Walker manifolds. In [2], the authors have found that the strict Walker manifold (M, g_f^ϵ) where f depends only on the variable y are very important. Here we will work with the manifold (M, g_f^ϵ) where f depends only on y and f is not locally a constant.

Three dimensional geometry plays a central role in the investigation of many problem in Riemannian and Lorentzian geometry. The fact that Ricci operator completly determines the curvature tensor is crucial to these investigations, (for detail see [1]).

The paper is organised as follow: in section 2, we recall some preliminaries results for three-dimensional Walker manifold (M, g_f^ϵ) and we give some basic formula for immersed surface in (M, g_f^ϵ). We consider two families of translation surfaces in (M, g_f^ϵ) which are used in the main result. In the last section we classify those which are minimal.

2. Preliminaries

A Walker n-manifold is a pseudo-Riemannian manifold, which admits a field of null parallel r-planes, with $r \leq \frac{n}{2}$. The canonical forms of the metrics were investigated by A. G. Walker [15]. Walker has derived adapted coordinates to a parallel plan field. Hence, the metric of a three-dimensional Walker manifold (M, g_f^ϵ) with coordinates (x, y, z) is expressed as

$$g_f^\epsilon = dx \circ dz + \epsilon dy^2 + f(x, y, z)dz^2 \qquad (2)$$

and its matrix form as

$$g_f^\epsilon = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \epsilon & 0 \\ 1 & 0 & f \end{pmatrix} \quad \text{with inverse} \quad (g_f^\epsilon)^{-1} = \begin{pmatrix} -f & 0 & 1 \\ 0 & \epsilon & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

for some function $f(x, y, z)$, where $\epsilon = \pm 1$ and thus $\mathcal{D} = \mathrm{Span}\{\partial_x\}$ as the parallel degenerate line field. Notice that, when $\epsilon = 1$ and $\epsilon = -1$ the Walker manifold has signature $(2, 1)$ and $(1, 2)$ respectively, and therefore is Lorentzian in both cases. Hence, if (M, g_f^ϵ) is a strict Walker manifolds i.e., $f(x, y, z) = f(y, z)$, then the associated Levi-Civita connection satisfies

$$\nabla_{\partial_y}\partial z = \frac{1}{2} f_y \partial_x, \quad \nabla_{\partial_z}\partial z = \frac{1}{2} f_z \partial_x - \frac{\epsilon}{2} f_y \partial_y. \tag{3}$$

Let now u and v be two vectors in M. Denoted by (e_1, e_2, e_3) the canonical frame in $\mathbb{R}^3$. The vector product of u and v in (M, g_f^ϵ) with respect to the metric g_f^ϵ is the vector denoted by $u \times v$ in M defined by

$$g_f^\epsilon(u \times v, w) = \det(u, v, w) \tag{4}$$

for all vector w in M, where $\det(u, v, w)$ is the determinant function associated to the canonical basis of $\mathbb{R}^3$. If $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ then by using (4), we have:

$$u \times v = \left(\begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} - f \begin{vmatrix} u_2 & v_2 \\ u_3 & v_3 \end{vmatrix} \right) e_1 - \epsilon \begin{vmatrix} u_1 & v_1 \\ u_3 & v_3 \end{vmatrix} e_2 + \begin{vmatrix} u_2 & v_2 \\ u_3 & v_3 \end{vmatrix} e_3$$

Let D be an open subset of the plane $\mathbb{R}^2$ satisfying this interval condition: horizontal or vertical lines intersect D in intervals (if at all). A *two-parameter map* is a smooth map $\varphi : D \to M$. Thus φ is composed of two interwoven families of *parameter curves*:

(1) the u-parameter curves $v = v_0$ of φ is $u \mapsto \varphi(u, v_0)$.
(2) the v-parameter curves $u = u_0$ of φ is $v \mapsto \varphi(u_0, v)$.

The partial velocities $\varphi_u = d\varphi(\partial_u)$ and $\varphi_v = d\varphi(\partial_v)$ are vector fields on φ. Evidently $\varphi_u(u_0, v_0)$ is the velocity vector at u_0 of the u-parameter curve $v = v_0$, and symmetrically for $\varphi_v(u_0, v_0)$. If φ lies in the domain of a coordinate system $x^1, \ldots, x^n$, then its coordinate functions $x_i \circ \varphi$ $(1 \leq i \leq n)$ are real-valued functions on D and

$$\varphi_u = \sum \frac{\partial x^i}{\partial u} \partial_i, \quad \varphi_v = \sum \frac{\partial x^i}{\partial v} \partial_i.$$

So far M could be a smooth manifold: now suppose it is pseudo-Riemannian. If Z is a smooth vector field on φ, its partial covariant derivatives are: $Z_u = \frac{\nabla Z}{\partial u}$, the covariant derivative of Z along u-parameter curves, and $Z_v = \frac{\nabla Z}{\partial v}$, the covariant derivative of Z along v-parameter curves. Explicitly, $Z_u(u_0, v_0)$ is the covariant derivative at u_0 of the vector field $u \mapsto Z(u, v_0)$ on the curve $u \mapsto \varphi(u, v_0)$. In terms of coordinates, $Z = \sum Z^i \partial_i$, where each $Z^i = Z(x^i)$ is a real valued function on D. Then

$$Z_u = \sum_k \Big\{ \frac{\partial Z^k}{\partial u} + \sum_{i,j} \Gamma_{ij}^k Z^i \frac{\partial x^j}{\partial u} \Big\} \partial_k. \tag{5}$$

In the special case $Z = \varphi_u$, the derivative $Z_u = \varphi_{uu}$ gives the accelerations of u-parameter curves, while φ_{vv} gives v-parameter accelerations. With coordinate notation as above, we have:

$$\varphi_{uv} = \sum_k \Big\{ \frac{\partial^2 x^k}{\partial v \partial u} + \sum_{i,j} \Gamma_{ij}^k \frac{\partial x^i}{\partial u} \frac{\partial x^j}{\partial v} \Big\} \partial_k. \tag{6}$$

Now we will assume that φ is an isometric immersion. The first fondamental form of the immersion φ is given by

$$\begin{cases} E = g_f\left(\varphi_*(\partial_u), \varphi_*(\partial_u)\right) \\ F = g_f\left(\varphi_*(\partial_u), \varphi_*(\partial_v)\right) \\ G = g_f\left(\varphi_*(\partial_v), \varphi_*(\partial_v)\right). \end{cases} \tag{7}$$

The coefficients of the second fundamental form of φ are

$$\begin{cases} L = \varepsilon_1 g_f\left(\varphi_{uu}, \xi\right) \\ M = \varepsilon_1 g_f\left(\varphi_{uv}, \xi\right) \\ N = \varepsilon_1 g_f\left(\varphi_{vv}, \xi\right) \end{cases} \tag{8}$$

where $\varepsilon_1 = g_f^\epsilon(\xi, \xi)$ the sign of the unit normal ξ along φ.
The mean curvature of φ is given by

$$H = \varepsilon_1 \frac{1}{2} \Big(\frac{LG - 2MF + NE}{EG - F^2} \Big). \tag{9}$$

The idea of translation surface have its origine in the classical text of G. Darboux [3] where the so-called "*surfaces définies par des propriétés cinématiques*" is introduced. A Darboux surface of translation is defined kinematically as the movement of a curve by a uniparameter family of rigid motion of $\mathbb{R}^3$. Hence, such a surface in locally described by $\varphi(s,t) = A(t).\alpha(s) + \beta(t)$ where α and β are parametrized curves in $\mathbb{R}^3$ and $A(t)$ is an orthogonal transformation. $A(t)$ being identity is the case which is most investigated. So a surface S in $\mathbb{R}^3$ is called a translation surface if S can be locally discribed as a sum

$$\varphi(s,t) = \alpha(s) + \beta(t).$$

Next, we consider a three-dimensional strict Walker manifold (M, g_f^ε), where f is not locally a constant and depends only on the variable y. For any real number a, the following two maps:

$$\begin{array}{rcl} \mathbb{R}^3 & \to & \mathbb{R}^3 \\ (x,y,z) & \mapsto & (x,y,z+a) \end{array}$$

and

$$\begin{array}{rcl} \mathbb{R}^3 & \to & \mathbb{R}^3 \\ (x,y,z) & \mapsto & (x+a,y,z) \end{array}$$

are isometries of (M, g_f^{ε}). Based in these isometries, we will define two types of translation surfaces.

Definition 1. *A non-degenerate surface S of sign ε_1 in (M, g_f^{ε}) is a translation surface if it can be described locally by an isometric immersion $\varphi : U \subset \mathbb{R}^2 \to (M, g_f^{\epsilon})$ of the form*

$$\varphi(u,v) = (u, v, \alpha(u) + \beta(v)), \qquad \textit{Type I} \tag{10}$$

or

$$\varphi(u,v) = (\alpha(u) + \beta(v), u, v), \qquad \textit{Type II} \tag{11}$$

where α and β are smooth functions on opens of $\mathbb{R}$.

The aim of this work is to classify the minimal translation surfaces in (M, g_f^{ε}) of the Type I and type II as above.

3. Main Results

3.1. **Minimal translation surfaces of Type I.** Let us consider a translation surface of Type I in (M, g_f^{ϵ}) parametrized by $\varphi(u,v) = (u, v, \alpha(u) + \beta(v))$. In this case we have $x = u$, $y = v$ and $z = \alpha(u) + \beta(v)$. For a function g of one variable u (respectively v) we denote $\frac{dg}{du}$ by $\dot{g}$ (respectively $\frac{dg}{dv}$ by g'). The tangent plane of S is spanned by

$$\varphi_u = \partial_x + \dot{\alpha}\partial_z \quad \text{and} \quad \varphi_v = \partial_y + \beta'\partial_z. \tag{12}$$

The unit normal ξ (up to orientation) is given by

$$\xi = \frac{1}{\Delta}\left[(1+\dot{\alpha}f)\partial_x - \varepsilon\beta'\partial_y - \dot{\alpha}\partial_z\right]. \tag{13}$$

where $\Delta = \|\varphi_u \times \varphi_v\|$. We obtain the coefficients of the first fundamental form of φ as

$$E = 2\dot{\alpha} + \dot{\alpha}^2 f, \quad F = \beta' + \dot{\alpha}\beta' f, \quad G = \varepsilon + \beta'^2 f. \tag{14}$$

And using (6) we have

$$\varphi_{uu} = \begin{pmatrix} 0 \\ -\frac{\varepsilon}{2}\dot{\alpha}^2 f_y \\ \ddot{\alpha} \end{pmatrix}, \quad \varphi_{uv} = \begin{pmatrix} \frac{1}{2}\dot{\alpha} f_y \\ -\frac{\varepsilon}{2}\dot{\alpha}\beta' f_y \\ 0 \end{pmatrix}, \quad \varphi_{vv} = \begin{pmatrix} \beta' f_y \\ -\frac{\varepsilon}{2}\beta'^2 f_y \\ \beta'' \end{pmatrix}. \tag{15}$$

Then the coefficients of the second fundamental form of φ

$$L = \frac{\varepsilon_1}{\Delta}\left[\frac{\varepsilon}{2}\beta'\dot{\alpha}^2 f_y + \ddot{\alpha}\right],$$

$$\begin{aligned} M &= \frac{\varepsilon_1}{\Delta}\left[-\frac{1}{2}\dot{\alpha}^2 f_y + \frac{\varepsilon}{2}\dot{\alpha}\beta'^2 f_y\right], \\ N &= \frac{\varepsilon_1}{\Delta}\left[-\dot{\alpha}\beta' f_y + \frac{\varepsilon}{2}\beta'^3 f_y + \beta''\right]. \end{aligned} \tag{16}$$

Consequently, the minimality condition (9) may be expressed as follows:

$$\ddot{\alpha}(\varepsilon + \beta'^2 f) + \dot{\alpha}^2(-\frac{1}{2}\beta' f_y + f\beta'') + 2\dot{\alpha}\beta'' = 0. \tag{17}$$

Since $y = v$, we can rewrite the minimal condition for Type I in the form

$$\ddot{\alpha}(\varepsilon + \beta'^2 f) + \dot{\alpha}^2(-\frac{1}{2}\beta' f' + f\beta'') + 2\dot{\alpha}\beta'' = 0. \tag{18}$$

We have the following solutions:

(1) **Case 1:** Assume that $\dot{\alpha} = 0$ that is $\alpha = \alpha_0$ (constant). We get the following surface:

$$(s_1): \quad \varphi(u, v) = (u, v, \alpha_0 + \beta(v))$$

for any smooth functions β.

(2) **Case 2:** Assume that $\dot{\alpha} \neq 0$ and $\ddot{\alpha} = 0$. Equation (18) becomes

$$\frac{\ddot{\alpha}}{\dot{\alpha}}(\varepsilon + \beta'^2 f) + \dot{\alpha}(-\frac{1}{2}\beta' f' + f\beta'') + 2\beta'' = 0. \tag{19}$$

Since $\ddot{\alpha} = 0$, from (19) we have:

$$\begin{cases} \alpha(u) = au + b \ \text{ with } \ a \in \mathbb{R}^*,\, b \in \mathbb{R} \\ (af + 2)\beta'' = \frac{1}{2}af'\beta'. \end{cases} \tag{20}$$

(a) If $\beta' = 0$, then $\beta = \beta_0$ is a constant with $\alpha(u) = au + b$, $a \in \mathbb{R}^*$ satisfy (19) as (18). Thus we have the plan:

$$(s_2): \quad \varphi(u, v) = (u, v, au + \tilde{b}), \quad a \in \mathbb{R}^*,\, \tilde{b} \in \mathbb{R}$$

(b) Now assume $\beta' \neq 0$. An easy integration of the second equation in (20) gives

$$\beta(v) = \tilde{c}\int_{v_*}^{v} \sqrt{|2 + af|}dv,$$

where $\tilde{c} \in \mathbb{R}^*$, v_* is a real number such that v and v_* belong to interval on which $(2 + af > 0)$ or $(2 + af < 0)$. So we get the solution

$$(s_3): \quad \varphi(u, v) = \Big(u, v, au + b + \tilde{c}\int_{v_*}^{v} \sqrt{|2 + af|}dv\Big), \quad a, \tilde{c} \in \mathbb{R}^*,\, b \in \mathbb{R}.$$

(3) **Case 3:** Assume that $\dot{\alpha} \neq 0$ and $\ddot{\alpha} \neq 0$. Then equation (18) can be written as (19) anywhere where $\dot{\alpha} \neq 0$. By differentiating the equation (19) with respect to u and v, we get:

$$\frac{d}{du}\left(\frac{\ddot{\alpha}}{\dot{\alpha}}\right)(\varepsilon + \beta'^2 f)' + \ddot{\alpha}(-\frac{1}{2}\beta' f' + f\beta'')' = 0. \tag{21}$$

(a) **Case 3-1:** $(\varepsilon + \beta'^2 f)' = 0$. Since $\ddot{\alpha} \neq 0$, the equation (21) gives $(-\frac{1}{2}\beta' f' + f\beta'')' = 0$. So we get

$$\begin{cases} \varepsilon + \beta'^2 f &= c_1 \\ -\frac{1}{2}f'\beta' + f\beta'' &= c_2, \end{cases} \tag{22}$$

where $c_1, c_2 \in \mathbb{R}$. Then the equation(19) becomes

$$\left(\frac{\ddot{\alpha}}{\dot{\alpha}}\right) c_1 + \dot{\alpha} c_2 = -2\beta''. \tag{23}$$

Since the left member depends only on u and the right member depends only on v, then there exist a constant c_3 and we have:

$$\begin{cases} \beta' &= -\frac{1}{2}c_3 v + c_4 \\ \left(\frac{\ddot{\alpha}}{\dot{\alpha}}\right) c_1 + \dot{\alpha} c_2 &= c_3, \end{cases} \tag{24}$$

where $c_3, c_4 \in \mathbb{R}$. If $c_3 = 0$, then $\beta'' = 0$ and $\beta' = c_4$. From (22), one gets $\varepsilon + c_4^2 f = c_1$. Then $c_4^2 f' = 0$ and $c_4 = 0$ by the hypothesis on f. So $\beta' = 0$ implies $c_2 = 0$ and $c_1 = \varepsilon$. Using this with (24) we get $\ddot{\alpha} = 0$ (contradiction with the hypothesis). So $c_3 \neq 0$. And then $\beta' \neq 0$ and $\beta'' = -\frac{1}{2}c_3 \neq 0$. Then (22) becomes

$$\begin{cases} f &= \frac{c_1 - \varepsilon}{(-\frac{1}{2}c_3 v + c_4)^2} \\ -\frac{1}{2}f'\beta' + f\beta'' &= c_2. \end{cases} \tag{25}$$

So we get $f' = \frac{c_3(c_1-\varepsilon)}{(-\frac{1}{2}c_3 v + c_4)^3}$. Thus (25) gives $\frac{c_3(c_1-\varepsilon)}{(-\frac{1}{2}c_3 v + c_4)^2} = c_2$, and then we must have $c_2 = 0$ and $c_1 = \varepsilon$. Then we get $f = 0$ (a contradiction). So the sub-case $(\varepsilon + \beta'^2 f)' = 0$ is not possible.

(b) **Case 3-2:** $(\varepsilon + \beta'^2 f)' \neq 0$. The equation (21) becomes

$$\frac{\frac{d}{du}\left(\frac{\ddot{\alpha}}{\dot{\alpha}}\right)}{\ddot{\alpha}} = -\frac{(-\frac{1}{2}\beta' f' + f\beta'')'}{(\varepsilon + \beta'^2 f)'}. \tag{26}$$

Since the left member depends only on u and the right member depends only on v, its must be constant c. So we get $\frac{d}{du}\left(\frac{\ddot{\alpha}}{\dot{\alpha}}\right) = c\ddot{\alpha}$ and $(-\frac{1}{2}\beta' f' + f\beta'')' = -c(\varepsilon + \beta'^2 f)'$. Then, there exist constants $c_1, c_2 \in \mathbb{R}$ such that

$$\frac{\ddot{\alpha}}{\dot{\alpha}} = c\dot{\alpha} + c_1 \quad \text{and} \quad (-\frac{1}{2}\beta' f' + f\beta'') = -c(\varepsilon + \beta'^2 f) + c_2. \tag{27}$$

If we put the equations (27) in (19), we get

$$c_1(\varepsilon + \beta'^2 f) + \dot{\alpha} c_2 + 2\beta'' = 0.$$

If we differentiate with respect to u, we obtain $\ddot{\alpha} c_2 = 0$ i.e., $c_2 = 0$. So we get:

$$\begin{cases} c_1(\varepsilon + \beta'^2 f) + 2\beta'' &= 0 \\ -c(\varepsilon + \beta'^2 f) &= -\frac{1}{2}\beta' f' + f\beta'' \\ \frac{\ddot{\alpha}}{\dot{\alpha}} &= c\dot{\alpha} + c_1 \end{cases} \tag{28}$$

And now we have two possibilities: $c_1 = 0$ or $c_1 \neq 0$.

- **Case 3-2-1:** $c_1 = 0$. We have $c \neq 0$ otherwise $\ddot{\alpha} = 0$. The first equation in (28) gives $\beta'' = 0$, so $\beta' = \beta'_0 \in \mathbb{R}$. And we get

$$-c(\varepsilon + \beta'^2 f) = -\frac{1}{2} f' \beta'_0. \tag{29}$$

If $\beta'_0 = 0$, then by using (29) we get $c\varepsilon = 0$, which is impossible. Therefore $\beta'_0 \neq 0$. An easy integration of (29) gives $f(v) = Ke^{2c\beta'_0 v} - \frac{\varepsilon}{(\beta'_0)^2}$ and $\beta = \beta'_0 v + \beta_0$. The equation $\frac{\ddot{\alpha}}{\dot{\alpha}} = c\dot{\alpha}$ gives $\alpha = -\frac{1}{c}\log|cu + c_1|$, $c \in \mathbb{R}^*$ and $c_1 \in \mathbb{R}$. Then we get solution of the form

$$(s_4) : \begin{cases} \varphi(u,v) &= (u, v, -\frac{1}{c}\log|cu + c_1| + \beta'_0 v + \beta_0) \\ f(v) &= Ke^{2c\beta'_0 v} - \frac{\varepsilon}{(\beta'_0)^2} \end{cases}$$

where $K, c, \beta'_0 \in \mathbb{R}^*$ and $c_1, \beta_0 \in \mathbb{R}$.

- **Case 3-2-2:** $c_1 \neq 0$. The first and the second equations in (28) give:

$$\begin{cases} (f - \frac{2c}{c_1})\beta'' &= \frac{1}{2} f' \beta' \\ \beta'^2 f &= -(2\beta'' + \varepsilon c_1). \end{cases}$$

If $\beta' = 0$ then $\beta'' = 0$ and $\varepsilon c_1 = 0$, which is impossible since $c_1 \neq 0$. Therefore we have $\beta' \neq 0$. So we get

$$\begin{cases} f &= -\frac{2\beta'' + \varepsilon c_1}{\beta'^2} \\ \frac{\beta''}{\beta'} &= \frac{1}{2}\frac{f'}{f - \frac{2c}{c_1}}. \end{cases} \tag{30}$$

The second equation of (30) gives

$$\beta' = \pm c_* \sqrt{\left| f - \frac{2c}{c_1} \right|} \quad \text{with} \quad c_* \in \mathbb{R}^*_+.$$

Denoted by $\mu = \operatorname{sign}\left(f - \frac{2c}{c_1}\right)$ and we get:

$$\begin{cases} \beta'^2 &= \mu c_*^2 \left(f - \frac{2c}{c_1}\right) \\ \beta'' &= \pm c_* \frac{\mu f'}{2\sqrt{\mu\left(f - \frac{2c}{c_1}\right)}} \end{cases} \tag{31}$$

The first equation of (31) gives: $\beta = \pm \int_{v_*}^{v} \sqrt{\left| f - \frac{2c}{c_1} \right|} d\tau$ where v_* and v belong to an intervall on which $\left(f - \frac{2c}{c_1}\right) > 0$ or $\left(f - \frac{2c}{c_1}\right) <$

0.
The first equation of (30) gives

$$f = -\frac{\pm c_* \frac{\mu f'}{2\sqrt{\mu\left(f-\frac{2c}{c_1}\right)}} + \varepsilon c_1}{\mu c_*^2\left(f - \frac{2c}{c_1}\right)}.$$

If we put $t = \sqrt{\mu\left(f - \frac{2c}{c_1}\right)}$ then $t^2 = \mu\left(f - \frac{2c}{c_1}\right)$, we have $f = \mu t^2 + \frac{2c}{c_1}$ and t satisfy $-\mu c_*^2(t^2 + \frac{2c}{c_1})t^2 \pm c_* t' = \varepsilon c_1$. We get the solution:

$$(s_5): \quad \varphi(u,v) = (u, v, \alpha(u) + \beta(v))$$

where α and β are given by:

(i) $\alpha(u) = Ae^{c_1 u} + B$ and $\beta(v) = \pm c_* \int_{v_*}^{v} \sqrt{|f|}d\tau$ with $f = \mu t^2$ ($\mu = \pm 1$) where t is solution of differential equation $-\mu c_*^2 t^4 \pm c_* t' = \varepsilon c_1$;

(ii) $\alpha(u) = \int_{u_*}^{u} \frac{d\tau}{Ke^{-c_1 u} - \frac{c}{c_1}}$ and $\beta(v) = \pm c_* \int_{v_*}^{v} \sqrt{|f - \frac{2c}{c_1}|}d\tau$, where $K, c, c_1 \in \mathbb{R}^*, c_* > 0$ with $f = \mu(t^2 + \frac{2c}{c_1})$ where t is solution of $-\mu c_*^2(t^2 + \frac{2c}{c_1})t^2 \pm t' = \varepsilon c_1$.

We conclude with the following:

Theorem 1. *A translation surface S of Type I in (M, g_f^ϵ) where f depends only on y and not locally a constant, is minimal if and only if there is an interval I ($u \in I$) and an interval J ($v \in J$) such that on $I \times J$ the surface take one of the following form*

1) $\varphi(u,v) = (u, v, \alpha_0 + \beta(v))$ *for any smooth functions β where $\alpha_0 \in \mathbb{R}$.*
2) $\varphi(u,v) = (u, v, au + \tilde{b})$, *where $a \in \mathbb{R}^*, \tilde{b} \in \mathbb{R}$.*
3) $\varphi(u,v) = \left(u, v, au + b + \tilde{c}\int^{v} \sqrt{|2 + af|}d\tau\right)$, *where $a, \tilde{c} \in \mathbb{R}^*$, $b \in \mathbb{R}$.*
4) $\varphi(u,v) = (u, v, -\frac{1}{c}\log|cu + c_1| + \beta_0' v + \beta_0)$ *where the function $f(v) = Ke^{2c\beta_0' v} - \frac{\varepsilon}{(\beta_0')^2}$ and $K, c, \beta_0' \in \mathbb{R}^*$ and $c_1, \beta_0 \in \mathbb{R}$.*
5) $\varphi(u,v) = (u, v, \alpha(u) + \beta(v))$ *where α and β are given by*
   (i) $\alpha(u) = Ae^{c_1 u} + B,\ A \in \mathbb{R}^*, B \in \mathbb{R}$ *and* $\beta(v) = \pm c_* \int^{v} \sqrt{|f|}d\tau$, *with $f = \mu t^2$ where $t = t(v)$ is solution of differential equation $\pm c_* t' = \mu c_*^2 t^4 + \varepsilon c_1$;*
   (ii) $\alpha(u) = \int^{u} \frac{d\tau}{Ke^{-c_1 u} - \frac{c}{c_1}}$ *and* $\beta(v) = \pm c_* \int^{v} \sqrt{|f - \frac{2c}{c_1}|}d\tau;\ K, c, c_1 \in \mathbb{R}^*$, *$c_* > 0$ with $f = \mu(t^2 + \frac{2c}{c_1})$ where $t = t(v)$ is solution of $\pm c_* t' = \mu c_*^2(t^2 + \frac{2c}{c_1})t^2 \varepsilon c_1$.*

Example 1. *Let (M, g_f^ϵ) be a Walker manifold where the function $f(y) = y^2$. Let S be a translation surface in M satisfying the condition of the theorem 1. In 3) of the above theorem, if we take $a = 2, b = 0, \tilde{c} = 1$ then the surface S is given by (see figure (A)):*

$$\varphi(u, v) = \left(u, v, 2u + \frac{1}{\sqrt{2}} \ln(v + \sqrt{1 + v^2}) + \frac{1}{\sqrt{2}} v\sqrt{1 + v^2}\right). \tag{32}$$

In 5)i), if we take $A = 1, B = -3, c_* = 1, c_1 = 1$ then the surface S is given by (see figure (B)):

$$\varphi(u, v) = \left(u, v, e^u + \frac{1}{2}v^2 - 3\right). \tag{33}$$

3.2. **Minimal translation surfaces of Type II.** Let us consider a translation surface S of Type II in (M, g_f^ϵ) parametrized by $\varphi(u, v) = (\alpha(u) + \beta(v), u, v)$. In this case we have $x = \alpha(u) + \beta(v)$, $y = u$ and $z = v$. For a function g of one variable u (respectively v) we denote $\frac{dg}{du}$ by $\dot{g}$ (respectively $\frac{dg}{dv}$ by g'). The tangent plane of S is spanned by

$$\varphi_u = \dot{\alpha}\partial_x + \partial_y \quad \text{and} \quad \varphi_v = \beta'\partial_x + \partial_z, \tag{34}$$

while the unit normal ξ (up to orientation) is given by

$$\xi = \frac{1}{\Delta}\left[(-\beta' - f)\partial_x - \varepsilon\dot{\alpha}\partial_y + \partial_z\right] \tag{35}$$

where $\Delta = \|\varphi_u \times \varphi_v\|$. We obtain the coefficients of the first fundamental form of φ as

$$E = \varepsilon, \quad F = \dot{\alpha}, \quad G = 2\beta' + f. \tag{36}$$

And we have by using (6)

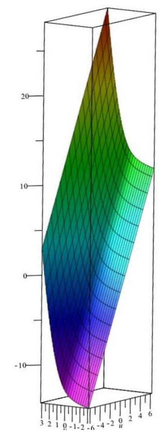
$$\varphi_{uu} = \begin{pmatrix} \ddot{\alpha} \\ 0 \\ 0 \end{pmatrix}, \quad \varphi_{uv} = \begin{pmatrix} \frac{1}{2}f_y \\ 0 \\ 0 \end{pmatrix}, \quad \varphi_{vv} = \begin{pmatrix} \beta'' \\ -\frac{\varepsilon}{2}f_y \\ 0 \end{pmatrix}. \tag{37}$$

Then the coefficients of the second fundamental form of φ

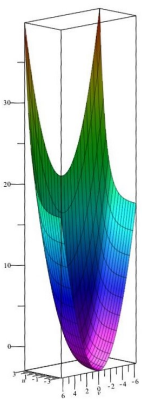
$$\begin{aligned} L &= \frac{\varepsilon_1}{\Delta}\big(\ddot{\alpha}\big), \\ M &= \frac{\varepsilon_1}{\Delta}\Big(\frac{1}{2}f_y\Big), \\ N &= \frac{\varepsilon_1}{\Delta}\Big(\beta'' + \frac{\varepsilon}{2}\dot{\alpha}f_y\Big). \end{aligned} \tag{38}$$

Consequently, the minimality condition (9) may be expressed as follows:

$$\ddot{\alpha}(2\beta' + f) - \frac{1}{2}\dot{\alpha}\dot{f} + \varepsilon\beta'' = 0 \tag{39}$$



(A) $\varphi(u,v) = \left(u, v, 2u + \frac{1}{\sqrt{2}} \ln(v + \sqrt{1+v^2}) + \frac{1}{\sqrt{2}} v\sqrt{1+v^2}\right)$



(B) $\varphi(u,v) = \left(u, v, e^u + \frac{1}{2}v^2 - 3\right)$

FIGURE 1. Figures of the Example 1

Taking the derivatives with respect to v, we get

$$2\ddot{\alpha}\beta'' + \varepsilon\beta''' = 0. \tag{40}$$

We will consider the following cases:

(1) **Case 1:** Assume that $\ddot{\alpha} = 0$. Since (40), we get $\beta'' = \beta_0'' \in \mathbb{R}$ and $\dot{\alpha} = \dot{\alpha}_0 \in \mathbb{R}$. And the equation (39) becomes $-\frac{1}{2}\dot{\alpha}_0\dot{f} + \varepsilon\beta_0'' = 0$. We have the following two subcases:

(a) **Case 1-1:** $\dot{\alpha}_0 = 0$. If $\dot{\alpha}_0 = 0$ then $\beta_0'' = 0$. Thus we get $\alpha = \alpha_0$ and $\beta(v) = av + b$. We get the plane

$$(s_1') : \quad \varphi(u,v) = (a_1 v + a_2, u, v); \quad a_1, a_2 \in \mathbb{R}.$$

(b) **Case 1-2:** $\dot{\alpha} \neq 0$. If $\dot{\alpha} \neq 0$ then $\beta_0'' \neq 0$ and we get $\dot{f} = \frac{2\varepsilon\beta_0''}{\dot{\alpha}}$. We get the solution

$$(s_2') : \begin{cases} \varphi(u,v) & = & (a_1 u + a_2 v + a_3, u, v) \\ f(u) & = & 2\varepsilon \frac{a_2}{a_1} u + a_4 \end{cases}$$

where $a_1, a_2 \in \mathbb{R}^*$, $a_3, a_4 \in \mathbb{R}$.

(2) **Case 2:** Assume that $\ddot{\alpha} \neq 0$. We will consider the following two sub-cases.

(a) **Case 2-1:** $\beta'' = 0$. If $\beta'' = 0$ then $\beta' = \beta_0' \in \mathbb{R}$. And the equation in (39) becomes

$$\frac{\ddot{\alpha}}{\dot{\alpha}} = \frac{1}{2}\left(\frac{\dot{f}}{2\beta_0' + f}\right),$$

which gives

$$\begin{cases} \alpha(u) & = & \tilde{c}\int_{u^*}^{u} \sqrt{|f+2a|}d\tau, \quad a \in \mathbb{R} \\ \beta(v) & = & av + d \end{cases}$$

where u^* and u belong to an interval on which $(f+2a > 0)$ or $(f+2a < 0)$. We get the solution

$$(s_3') : \begin{cases} \varphi(u,v) = \left(\tilde{c}\int_{u^*}^{u} \sqrt{|f+2a|}d\tau + av + d, u, v\right) \\ \tilde{c} \in \mathbb{R}^*, \ a, d \in \mathbb{R} \end{cases}$$

(b) **Case 2-2:** $\beta'' \neq 0$. If $\beta'' \neq 0$ then there exist $c \in \mathbb{R}^*$ such that

$$\begin{cases} 2\ddot{\alpha} & = & c \\ \frac{\beta'''}{\beta''} & = & -c\varepsilon. \end{cases}$$

Thus we have

$$\begin{cases} 2\dot{\alpha} & = & cu + c_1 \\ \beta'' & = & -c\varepsilon\beta' + c_2 \end{cases}$$

where $c_1, c_2 \in \mathbb{R}$. And the equation in (39) becomes

$$\frac{c}{2}(2\beta' + f) - \frac{1}{4}(cu + c_1)\dot{f} + \varepsilon(-\varepsilon c\beta' + c_2) = 0,$$

that is

$$\frac{c}{2}f = \frac{1}{4}(cu + c_1)\dot{f} + \varepsilon c_2.$$

And then we have the solution

$$(s_4) \quad \varphi(u,v) = \Big(\frac{1}{4}cu^2 + \frac{1}{2}c_1 u + \tilde{c}_1 + \frac{\varepsilon c_2}{c}v + K_1 e^{-\varepsilon c v}, u, v\Big)$$

with $f(u) = K_2(cu+c_1)^2 + \frac{2c_2\varepsilon}{c}$, where $c_1, \tilde{c}_1, c_2, \tilde{c}_2 \in \mathbb{R}$ and $K_1, K_2 \in \mathbb{R}^*$.

We have the following result:

Theorem 2. *A translation surface S of Type II in (M, g_f^ϵ) where f depends only on y and not locally a constant, is minimal if and only if there is an interval I ($u \in I$) and an interval J ($v \in J$) such that on $I \times J$ the surface take one of the following form*

(1) $\varphi(u,v) = (a_1 v + a_2, u, v);\quad a_1, a_2 \in \mathbb{R}$.
(2) $\varphi(u,v) = (a_1 u + a_2 v + a_3, u, v);\quad a_1, a_2 \in \mathbb{R}^*, a_3, a_4 \in \mathbb{R}$ *with* $f(u) = 2\varepsilon \frac{a_2}{a_1} u + a_4$.
(3) $\varphi(u,v) = \left(\tilde{c}\int_{u^*}^{u} \sqrt{|f+2a|}d\tau + av + d, u, v\right);\quad a, d \in \mathbb{R}$.
(4) $\varphi(u,v) = \left(\frac{1}{4}cu^2 + \frac{1}{2}c_1 u + \tilde{c}_1 + \frac{\varepsilon c_2}{c} v + K_1 e^{-\varepsilon c v}, u, v\right);\quad c_1, \tilde{c}_1, c_2, \tilde{c}_2 \in \mathbb{R}$, $c, K_1, K_2 \in \mathbb{R}^*$ *with* $f(u) = K_2(cu+c_1)^2 + \frac{2c_2\varepsilon}{c}$.

Example 2. *Let (M, g_f^ϵ) be a Walker manifold where the function $f(y) = 2y^2$. Let S be a translation surface in M satisfying the condition of the theorem 2. In 3) of the above theorem 2, if we take $a = 1, \tilde{c} = 1, d = 0$ then the surface S is given by (see figure 2a):*

$$\varphi(u,v) = \left(\frac{1}{\sqrt{2}}\ln(u+\sqrt{1+u^2}) + \frac{1}{\sqrt{2}}u\sqrt{1+u^2} + v, u, v\right). \qquad (41)$$

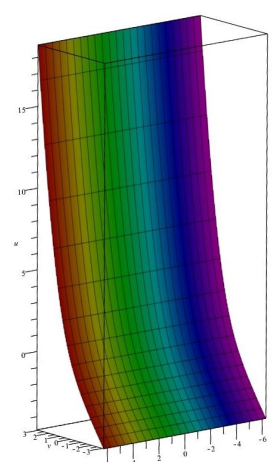


FIGURE 2. $\varphi(u,v) = \left(\frac{1}{\sqrt{2}}\ln(u+\sqrt{1+u^2}) + \frac{1}{\sqrt{2}}u\sqrt{1+u^2} + v, u, v\right)$, Figure of the Example 2.

4. Conclusion

In this paper we have defined two types of translation surfaces using two kind of isometries in a strict Walker manifold (M, g_f^ϵ). First we have studied and classified the minimality of the translation surface of type I and we draw some examples of these family of surfaces. Secondely, we considered the family of translation surfaces of type II and we studied their minimality. We classify these surfaces and draw some example.

Author Contribution Statements All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Declaration of Competing Interests On behalf of all authors, the corresponding author states that there is no conflict of interest.

Acknowledgements The authors would like to thank the reviewers for their valuable suggestions that lead to a better presentation of the article, as well as the journal editors who took care of the article

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