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Research Article

On Hawaiian homology groups

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Abstract

In this paper, we introduce a kind of homology which we call Hawaiian homology to study and classify pointed topological spaces. The Hawaiian homology group has advantages over Hawaiian groups. Moreover, the first Hawaiian homology group is isomorphic to the abelianization of the first Hawaiian group for path-connected and locally path-connected topological spaces. Since Hawaiian homology has concrete elements and abelian structure, its calculation is easier than that of the Hawaiian group. Thus we use Hawaiian homology groups to compare Hawaiian groups, and then we obtain some information about Hawaiian groups of some wild topological spaces.

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1. Introduction and motivation

Homology is a well-known useful tool of algebraic topology. Since homology has concrete interpretation, different fields of science apply it as an algebraic modelling of geometric properties to study their observations mathematically [13]. Homology is a general way of associating a sequence of algebraic objects, such as abelian groups or modules, with other mathematical objects such as topological spaces. Homology groups were originally defined in algebraic topology and then it is generalized in a wide variety of other contexts, such as group theory, theory of Lie algebra, Galois theory and algebraic geometry.

The original motivation for defining homology groups was to distinguish shapes by examining their holes. Homology group has a famous characteristic, the Betti number, which is known as the various counting of the number of holes of the spaces, and by this fact, the Betti number helps to classify spaces. The nth Betti number is defined as the rank of the nth homology group of the given topological space.

There are many different homology theories, each of which has its advantages, applications, and defects [1, 20]. A particular type of mathematical object, such as a topological space or a group, may have one or more associated homology theories. When the underlying object has a geometric interpretation as a topological space, the nth homology group

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represents the behavior in the dimension n. Most homology groups or modules may be formulated as derived functors on appropriate abelian categories, measuring the failure of a functor to be exact.

In Section 2, we define a Hawaiian homology similar to the singular homology theory associated with a chain complex of shrinking sequences of singular simplexes tending to the base point. Thus, it is natural that Hawaiian homology groups depend on the choice of the base point. More precisely, the Hawaiian homology is a covariant functor from the category of pointed topological spaces to the category of abelian groups. We see that Hawaiian homology groups depend on the local behavior of the spaces at their base points. This is a great difference between the Hawaiian homology and the singular homology, the latter depends only on the homotopy type of space, while the base point has an essential role in calculating the Hawaiian homology. In fact, the Hawaiian homology is invariant under base point preserving homotopies, called pointed homotopies. We present some examples to clarify the effects of choice of the base point and pointed homotopies.

Also, we investigate basic properties of the Hawaiian homology, including the Hawaiian homology of the singletons, disjoint union, homotopic maps, and so on. Then we present a close relation between the Hawaiian homology group and Hawaiian group of topological spaces. The first Hawaiian homology group is isomorphic to the abelianization of the first Hawaiian group for any path-connected and locally path-connected topological space. The nth Hawaiian group of a pointed topological space was defined by Karimov and Repovš [15] as the set of all pointed homotopy classes of continuous maps with a group operation coming from the operation of nth homotopy groups. The nth Hawaiian group, defined below, is a covariant functor from the category of pointed topological spaces to the category of groups, denoted by $\mathcal{H}_n(X, x_0)$, where (X, x_0) is a pointed topological space.

Definition 1.1 ([15])**.** Let (X, x_0) be a pointed space, let $[-]$ denote the class of pointed homotopy, and let $n = 1, 2, \ldots$. Then the nth Hawaiian group of (X, x_0) is defined by $\mathcal{H}_n(X, x_0) = \{[f] : f : (\mathbb{HE}^n, \theta) \to (X, x_0)\}$ with the multiplication induced by $(f{*}g)|_{\mathbb{S}^n_k} = f|_{\mathbb{S}^n_k} \cdot g|_{\mathbb{S}^n_k}$ $(k \in \mathbb{N})$ for any $[f], [g] \in \mathcal{H}_n(X, x_0)$, where $\cdot$ denotes the concatenation of n-loops.

The Hawaiian group functor has some advantages over other well-known covariant functors from the category of topological spaces to the category of groups. Unlike homotopy and homology group, Hawaiian group is not preserved by free homotopies. As an example, consider the cone space $C(\mathbb{HE}^1)$, where $\mathbb{HE}^1$ denotes the one-dimensional Hawaiian earring. Since cone spaces are contractible, their homotopy, homology, and cohomology groups are trivial [15], but the first Hawaiian group of $C(\mathbb{HE}^1)$ is uncountable [15]. Also, by using this functor, we can study some local behaviors of spaces. For instance, if X has a countable local basis at x_0 and the nth Hawaiian group $\mathcal{H}_n(X, x_0)$ is countable, then X is locally n-simply connected at x_0 (see [15, Theorem 2]). For a space X having a countable local basis at x_0, $\mathcal{H}_n(CX, x)$ is trivial if and only if X is locally n-simply connected at x_0 and it is uncountable otherwise [2, Corollaries 2.16 and 2.17]. Hence, unlike homotopy groups, Hawaiian groups depend on the behavior of the space at the base point. In this regard, there are path-connected spaces whose Hawaiian groups are not isomorphic at different points, such as the n-dimensional Hawaiian earring, where $n \geq 2$ (see [2, Corollary 2.11]). In this paper, we show that the Hawaiian homology groups have the advantages of Hawaiian groups, from the viewpoint of homology. Moreover, Hawaiian homology groups have more concrete elements, and also they can be computed by techniques of abelian groups (see theorem 6.2), while there rarely exist similar techniques to calculate Hawaiian groups. By using these facts, we use Hawaiian homology groups to study and classify pointed topological spaces; for instance, the first Hawaiian homology group of the Harmonic archipelago at the origin is not isomorphic to the first Hawaiian homology group at any other points. Also since the first Hawaiian homology group for any locally path-connected space is the abelianization of the first Hawaiian group, it can help us to investigate the structure of Hawaiian groups. Since the first Hawaiian homology groups are not isomorphic at different points, we conclude that the first Hawaiian groups at different points of Harmonic archipelago are not isomorphic.

In Section 3, by employing Hawaiian homology groups, we compare Hawaiian groups of some pointed spaces, for instance, the Harmonic archipelago, Griffiths space, Hawaiian earrings, and so on. Moreover, we calculate Hawaiian homology groups of the Harmonic archipelago and prove that the first Hawaiian homology and the first singular homology groups of the Harmonic archipelago are isomorphic. The Harmonic archipelago is an element of a class of spaces called archipelago spaces constructed by attaching loops to some weak join of spaces. Hawaiian earrings belong to the class of weak join spaces defined as follows.

Definition 1.2 ([6,10])**.** The weak join of a family of spaces $\{(X_i, x_i); i \in I\}$, denoted by $(X, x_*) = \widetilde{\bigvee}_{i\in I}(X_i, x_i)$, is the underlying space of wedge space $(\hat{X}, \hat{x}_*) = \bigvee_{i\in I}(X_i, x_i)$ with the weak topology with respect to X_i's, except at the common point. Every open neighborhood in X at x_* is of the form $\bigcup_{i\in F} U_i \cup \bigcup_{i\in I\setminus F} X_i$, where U_i is an open neighborhood at x_i in $X_i \subset X$ and F is a finite subset of I (see [6, Section 2] and [10, p. 18]).

In [17, Definition 2.1], small n-Hawaiian loop was defined as a small map from $\mathbb{HE}^n$ to any space X.

In this paper, all spaces are assumed to be first countable. Moreover, by $\widehat{\ }^{\,p}$ we mean the p-adic completion of a group and P denotes the set of all primes.

2. Hawaiian homology groups

In this section, we introduce the Hawaiian homology as a new invariant from the category of pointed topological spaces to the category of abelian groups. First, we define the Hawaiian simplex in the natural way. That is, the disjoint union of a countably infinite family of Euclidean simplexes whose diameters tend to zero together with the origin as the cluster point.

Definition 2.1. Let $\Delta_k^n = \Delta^n/k = \{a \in \mathbb{R}^{n+1} |\ ka \in \Delta^n\}$, for $k = 1, 2, \ldots$, where Δ^n denotes the standard Euclidean n-simplex, and let $\Delta_0^n = \{\theta\}$, where θ denotes the origin. The n-Hawaiian simplex is defined as follows:

$$H\Delta^n = \bigcup_{k=0}^{\infty} \Delta_k^n.$$

Moreover, an n-Hawaiian simplex in the given space (X, x_0) is a continuous map $\sigma^n : (H\Delta^n, \theta) \to (X, x_0)$. The restriction of σ^n on the kth simplex Δ_k^n can be regarded as a singular n-chain in X, denoted by $\sigma_k^n = \sigma^n|_{\Delta_k^n}$. Note that the sequence $\{\sigma_k^n\}$ is null-convergent; That is for any open set U around x_0, all the images of σ_k^n's are contained in U except a finite number.

As in the definition of singular homology, the group $S_n(X, x_0)$ is defined as the free abelian group generated by all n-Hawaiian simplexes in (X, x_0). To define the boundary homomorphism $\partial_n : S_n(X, x_0) \to S_{n-1}(X, x_0)$, we need to introduce face maps. Let $k \in \mathbb{N}$. Consider $\varepsilon_{i,k}^n : \Delta_k^{n-1} \to \Delta_k^n$ as the map defined by the ith face-map on Euclidean $(n-1)$-simplex together with the appropriate coefficient associated with $k \in \mathbb{N}$. Then let $\varepsilon_i^n = \bigcup_{k=0}^{\infty} \varepsilon_{i,k}^n : \bigcup_{k=0}^{\infty} \Delta_k^{n-1} \to \bigcup_{k=0}^{\infty} \Delta_k^n$ be the map obtained by the union of all $\varepsilon_{i,k}^n$'s for $k \in \mathbb{N}$ and the constant map for $k = 0$. The mapping ε_i^n is continuous, because Δ_k^{n-1} is mapped into Δ_k^n by a continuous mapping for each $k \in \mathbb{N}$ and then it maps each convergent sequence to a convergent sequence. Then ε_i^n is called the ith face-map of the

n-Hawaiian simplex. Now the boundary map $\partial_n : S_n(X, x_0) \to S_{n-1}(x, x_0)$ is defined by $\partial_n(\sigma^n) = \sum_{i=0}^{n}(-1)^i \sigma \circ \varepsilon_i^n$. Then we have the following chain complex:

$$\cdots \longrightarrow S_n(X, x_0) \xrightarrow{\partial_n} S_{n-1}(X, x_0) \xrightarrow{\partial_{n-1}} \cdots \longrightarrow S_1(X, x_0) \xrightarrow{\partial_1} S_0(X, x_0) \xrightarrow{\partial_0} 0.$$

Remark 2.2. To define Hawaiian homology group, we must verify the equality $\partial_n\partial_{n+1} = 0$. It can be checked by the definition of boundary homomorphism as follows. Let σ be an $(n+1)$-Hawaiian simplex.

$$\begin{aligned}
\partial_n\partial_{n+1}(\sigma) &= \partial_n\big(\sum_{j=0}^{n+1}(-1)^j \sigma \circ \varepsilon_j^{n+1}\big) \\
&= \sum_{i,j}(-1)^{i+j}\sigma \circ \varepsilon_j^{n+1} \circ \varepsilon_i^n \\
&= \sum_{i,j}(-1)^{i+j} \bigcup_{k=0}^{\infty} (\sigma_k \circ \varepsilon_{j,k}^{n+1} \circ \varepsilon_{i,k}^n), \\
&= \sum_{j\le i}(-1)^{i+j} \bigcup_{k=0}^{\infty} (\sigma_k \circ \varepsilon_{j,k}^{n+1} \circ \varepsilon_{i,k}^n) + \sum_{i<j}(-1)^{i+j} \bigcup_{k=0}^{\infty} (\sigma_k \circ \varepsilon_{j,k}^{n+1} \circ \varepsilon_{i,k}^n).
\end{aligned}$$

By [20, Lemma 4.5], if $i < j$, then $\varepsilon_{j,k}^{n+1}\varepsilon_{i,k}^n = \varepsilon_{i,k}^{n+1}\varepsilon_{j-1,k}^n$. Thus by replacement in the second sum,

$$\partial_n\partial_{n+1}\sigma = \sum_{j\le i}(-1)^{i+j} \bigcup_{k=0}^{\infty} (\sigma_k\varepsilon_{j,k}^{n+1}\varepsilon_{i,k}^n) + \sum_{i<j}(-1)^{i+j} \bigcup_{k=0}^{\infty} (\sigma_k\varepsilon_{i,k}^{n+1}\varepsilon_{j-1,k}^n).$$

Change variables in the second sum as $p := i$ and $q := j - 1$. Then

$$\sum_{j<i}(-1)^{i+j} \bigcup_{k=0}^{\infty} (\sigma_k\varepsilon_{i,k}^{n+1}\varepsilon_{j-1,k}^n) = \sum_{p\le q}(-1)^{p+q+1} \bigcup_{k=0}^{\infty} (\sigma_k\varepsilon_{p,k}^{n+1}\varepsilon_{q,k}^n).$$

Again replace indexes with $j := p$ and $i := q$, and then

$$\begin{aligned}
\partial_n\partial_{n+1}(\sigma) &= \sum_{j\le i}(-1)^{i+j} \bigcup_{k=0}^{\infty} (\sigma_k\varepsilon_{j,k}^{n+1}\varepsilon_{i,k}^n) + \sum_{j\le i}(-1)^{j+i+1} \bigcup_{k=0}^{\infty} (\sigma_k\varepsilon_{j,k}^{n+1}\varepsilon_{i,k}^n) \\
&= \sum_{j\le i} \big((-1)^{i+j} + (-1)^{j+i+1}\big)\Big(\bigcup_{k=0}^{\infty} (\sigma_k\varepsilon_{j,k}^{n+1}\varepsilon_{i,k}^n)\Big) \\
&= \sum_{j\le i} (0)\Big(\bigcup_{k=0}^{\infty} (\sigma_k\varepsilon_{j,k}^{n+1}\varepsilon_{i,k}^n)\Big) \\
&= 0.
\end{aligned}$$

Now, we are able to define the Hawaiian homology group as follows.

Definition 2.3. Let (X, x_0) be a pointed topological space. Then the nth Hawaiian homology group of (X, x_0), denoted by $\mathbb{H}_n(X, x_0)$, equals the quotient group $\frac{Z_n(X,x_0)}{B_n(X,x_0)}$, where $Z_n(X, x_0) = \ker \partial_n$ and $B_n(X, x_0) = \operatorname{im} \partial_{n+1}$. By $\operatorname{cls} \alpha = \alpha + B_n(X, x_0)$ we mean the homology class of α.

As a simple example, we compute the first Hawaiian homology group of the unit circle.

Example 2.4. Let $\mathbb{S}^1$ denote the circle with radius 1 in the Euclidean space. Then $\mathbb{H}_1(\mathbb{S}^1, a) = \frac{Z_n(\mathbb{S}^1,a)}{B_n(\mathbb{S}^1,a)}$. Define $\psi : Z_n(\mathbb{S}^1, a) \to \sum_{\aleph_0} H_1(\mathbb{S}^1)$ by the rule $\psi(\sigma^n) = \{\operatorname{cls} \sigma^n|_{\Delta_k^n}\}_{k\in\mathbb{N}}$. Since simplexes $\{\sigma^n|_{\Delta_k^n}\}_{k\in\mathbb{N}}$ tend to be small, they are contained in a contractible subset, and then they may be replaced with constant simplexes. Hence just a finite number of non-null-homotopic simplexes remain. Therefore, $\mathbb{H}_1(\mathbb{S}^1, a) \cong \sum_{\aleph_0} H_1(\mathbb{S}^1) \cong \sum_{\aleph_0} \mathbb{Z}$.

By Definition 2.3, for each pointed topological space, there corresponds a sequence of abelian groups. To define a functor $\mathbb{H}_n : Top_* \to Ab$, $n \geq 0$, from the category of pointed topological spaces to the category of abelian groups, we need to define morphisms; that is, for each pointed map $f : (X, x) \to (Y, y)$, a homomorphism $\mathbb{H}_n(f) : \mathbb{H}_n(X, x) \to \mathbb{H}_n(Y, y)$ exists that satisfies

(i) $\mathbb{H}_n(\mathrm{id}_X) = \mathrm{id}_{\mathbb{H}_n(X,x)}$,
(ii) $\mathbb{H}_n(f \circ g) = \mathbb{H}_n(f) \circ \mathbb{H}_n(g)$.

As in the definition of the functor of singular homology group [20, p. 66], we define $\mathbb{H}_n(f) : \mathbb{H}_n(X, x) \to \mathbb{H}_n(Y, y)$ by $\mathrm{cls}\, z_n \mapsto \mathrm{cls}\, f_\#(z_n)$, where $f_\# : S_n(X, x) \to S_n(Y, y)$ is defined by $f_\#(\sum m_\sigma \sigma) = \sum m_\sigma (f \circ \sigma)$. The same argument as used in [20, Theorem 4.10] implies that $\mathbb{H}_n(f)$ satisfies conditions (i) and (ii), and then $\mathbb{H}_n$ is a functor. Then it is natural to ask if the Hawaiian homology functor preserves pointed homotopy mappings. There are some differences between the singular homology group, the most familiar homology theory, and the Hawaiian homology group. The first one is the preservation of homotopy mappings; for the singular homology group, every free homotopy equivalence induces an isomorphism, but for the Hawaiian homology group, if the homotopy mapping does not preserve the base point, it does not necessarily induce an isomorphism (see Example 3.6).

Now it is natural to know if the Hawaiian homology functors satisfy the homology axioms.

3. Axioms of homology

In homology theory, there are five properties called axioms of homology or sometimes called Eilenberg-Steenrood axioms, which consists of dimension axiom, additivity, homotopy, exactness and excision. These axioms classify different theories of homology and also help to calculate the homology group of some spaces such as n-spheres. We see that the Hawaiian homology groups do not satisfy all of these axioms. In the following we investigate homology axioms for Hawaiian homology. Three axioms of dimension, excision and exactness hold for arbitrary spaces, but two axioms of homotopy and additivity hold with some conditions and modifications.

3.1. Dimension, excision and exactness axioms

Dimension axiom. For the Hawaiian homology group, similar to the singular homology group, the axiom of dimension can be verified directly by calculating boundary homomorphism as follows; see also [20, Theorem 4.12].

Proposition 3.1. *Let $X = \{x\}$ be the one-point space. Then $\mathbb{H}_n(X, x) = 0$ for $n \geq 1$.*

Proof. Let $n \geq 1$. Since X has only one point, there is just one chain and all the groups $S_n(X, x)$ are isomorphic to the infinite cyclic group $\mathbb{Z}$. Also for the boundary homomorphism

$$\partial_n(\sigma) = \sum_{i=0}^{n} \sigma \circ \varepsilon_i^n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ -1 & \text{if } n \text{ is even} \end{cases}, \tag{3.1}$$

because $\sigma \circ \varepsilon_i^n$ is an $(n-1)$-simplex, and there is just one $(n-1)$-simplex being the generator 1. Then by equality (3.1),

$$Z_n(X, x) = \ker \partial_n = \begin{cases} S_n(X, x) & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases},$$

and

$$B_n(X,x) = \operatorname{im}\partial_{n+1} = \begin{cases} 0 & \text{if } n+1 \text{ is odd} \\ S_n(X,x) & \text{if } n+1 \text{ is even} \end{cases} = \begin{cases} S_n(X,x) & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}.$$

Hence if n is odd, $\mathbb{H}_n(X,x) = \frac{\ker\partial_n}{\operatorname{im}\partial_{n+1}} = \frac{S_n(X,x)}{S_n(X,x)} = 0$, and if n is even, $\mathbb{H}_n(X,x) = \frac{\ker\partial_n}{\operatorname{im}\partial_{n+1}} = \frac{0}{0} = 0$. Then we are done. □

Exactness axiom. For the rest of section, we address two axioms of exactness and excision. The axiom of exactness in homology states that each pair of spaces (X,A) induces a long exact sequence in homology groups, via the inclusions $i : A \to X$ and $j : X \to (X,A)$;

$$\cdots \to H_n(A) \xrightarrow{H_n(i)} H_n(X) \xrightarrow{H_n(j)} H_n(X,A) \xrightarrow{d^s} H_{n-1}(A) \to \cdots .$$

The homomorphism d^s is called the standard connecting homomorphism, which is obtained by the Zig-zag Lemma [19, Lemma 24.1]. We first, need to define relative Hawaiian homology. Let $x_0 \in A \subseteq X$. One can consider $S_n(A,x_0) \subseteq S_n(X,x_0)$, and homomorphism δ_n is induced by the boundary homomorphism ∂_n. Then one achieve chain complex

$$\cdots \longrightarrow \frac{S_{n+1}(X,x_0)}{S_{n+1}(A,x_0)} \xrightarrow{\delta_{n+1}} \frac{S_n(X,x_0)}{S_n(A,x_0)} \xrightarrow{\delta_n} \frac{S_{n-1}(X,x_0)}{S_{n-1}(A,x_0)} \longrightarrow \cdots .$$

Then the relative n-Hawaiian cycles and the relative n-Hawaiian boundaries can be defined as follows:

$$Z_n(X,A,x_0) = \{\gamma \in S_n(X,x_0);\ \partial_n\gamma \in S_{n-1}(A,x_0)\},$$
$$B_n(X,A,x_0) = B_n(X,x_0) + S_n(A,x_0).$$

Relative homology groups are defined as $\mathbb{H}_n(X,A,x_0) = \frac{Z_n(X,A,x_0)}{B_n(X,A,x_0)}$ for $n \geq 0$.

We intend to rewrite and prove the axiom of exactness for Hawaiian homology groups, by using Zig-zag Lemma in Proposition 3.2. The Zig-zag Lemma states that for chain complexes $\mathcal{C} = \{C_p, \partial_C\}$, $\mathcal{D} = \{D_p, \partial_D\}$ and $\mathcal{E} = \{E_p, \partial_E\}$ and chain maps ϕ and ψ, the following short exact sequence of chain complexes $0 \longrightarrow \mathcal{C} \xrightarrow{\phi} \mathcal{D} \xrightarrow{\psi} \mathcal{E} \longrightarrow 0$ leads to a long exact homology sequence:

$$\cdots \longrightarrow H_p(\mathcal{C}) \xrightarrow{\phi_*} H_p(\mathcal{D}) \xrightarrow{\psi_*} H_p(\mathcal{E}) \xrightarrow{\partial_*} H_{p-1}(\mathcal{D}) \longrightarrow \cdots ,$$

where ∂_* is induced by the boundary operator in $\mathcal{D}$ [19, Lemma 24.1].

Proposition 3.2. *Let $x_0 \in A \subseteq X$. Then the following sequence is exact*

$$\cdots \to \mathbb{H}_n(A,x_0) \xrightarrow{i_*} \mathbb{H}_n(X,x_0) \xrightarrow{j_*} \mathbb{H}_n(X,A,x_0) \xrightarrow{d} \mathbb{H}_{n-1}(A,x_0) \to \cdots . \tag{3.2}$$

Proof. By Zig-zag lemma we need to verify that the following sequence of chain maps is exact

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S_{n-1}(A,x_0) & \xrightarrow{i_*} & S_{n-1}(X,x_0) & \xrightarrow{q} & \frac{S_{n-1}(X,x_0)}{S_{n-1}(A,x_0)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S_{n}(A,x_0) & \xrightarrow{i_*} & S_{n}(X,x_0) & \xrightarrow{q} & \frac{S_{n}(X,x_0)}{S_{n}(A,x_0)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S_{n+1}(A,x_0) & \xrightarrow{i_*} & S_{n+1}(X,x_0) & \xrightarrow{q} & \frac{S_{n+1}(X,x_0)}{S_{n+1}(A,x_0)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

Since i_* is induced by the inclusion, it is injective. Also, q is the quotient homomorphism on the image of i_*, and then $\ker q = \operatorname{im} i_*$. Moreover, q is a quotient homomorphism, and then it is onto. Therefore, the sequence of chain complexes $0 \longrightarrow S(A, x_0) \xrightarrow{i_*} S(X, x_0) \xrightarrow{q} S(X, x_0)/S(A, x_0) \longrightarrow 0$ is exact. Thus by Zig-zag Lemma, the sequence (3.2) is exact. □

Excision axiom. The axiom of excision in homology theory states that for two subspaces X_1 and X_2 of X, if $X = X_1^\circ \cup X_1^\circ$, then the inclusion $i : (X_1, X_1 \cap X_2) \hookrightarrow (X, X_2)$ induces isomorphism on homology groups.

Proposition 3.3. *Let $X = X_1^\circ \cup X_1^\circ$ and $x_0 \in X_1^\circ \cap X_1^\circ$. Then*

$$i_* : \mathbb{H}_n(X_1, X_1 \cap X_2) \to \mathbb{H}_n(X, X_2),$$

is an isomorphism for all n.

Proof. By [20, Lemma 6.11], it suffices to show that the inclusion $S_*(X_1) + S_*(X_2) \to S_*(X)$ induces isomorphism in homology. We need a straightforward adaption of the proof of [20, Theorem 6.17]. First define $\theta : [\gamma_1 + \gamma_2] \mapsto \operatorname{cls}(\gamma_1 + \gamma_2)$. As the rule is defined by addition, θ is homomorphism. Then θ is surjective and injective by a similar argument as used in [20, Theorem 6.17], when [20, Lemma 6.12 and Lemma 6.16] are rewritten for Hawaiian cases. □

3.2. Homotopy and additivity axioms

Homotopy axiom. For the Hawaiian homology groups, homotopy axiom holds if the homotopy mapping preserves the base point (Proposition 3.4). Moreover, consider the cone over the 1-dimensional Hawaiian earring. Its 1st Hawaiian homology group at some point is not trivial (See Example 3.6), while it is contractible.

Proposition 3.4. *Let $f, g : (X, x_0) \to (Y, y_0)$ be pointed maps and let $F : f \simeq g$ rel $\{x_0\}$. Then $\mathbb{H}_n(f) = \mathbb{H}_n(g)$ for $n \geq 0$.*

Proof. To prove the equality $\mathbb{H}_n(f) = \mathbb{H}_n(g)$, for every n-Hawaiian simplex α, we show that $\mathbb{H}_n(f)(\operatorname{cls}\alpha) = \mathbb{H}_n(g)(\operatorname{cls}\alpha)$. It is equivalent to $f_\#(\alpha) - g_\#(\alpha)$ being the boundary of an $(n+1)$-Hawaiian simplex. By [20, Theorem 4.23], we have $F : f \simeq g$, and then $f^s_\#(\alpha|_{\Delta^n_k}) - g^s_\#(\alpha|_{\Delta^n_k})$ is equal to the boundary of an $(n+1)$-simplex β_k in Y; the simplex β_k is constructed by $F(\alpha|_{\Delta^n_k}, -)$ for $k = 0, 1, 2, \ldots$. Since F is a pointed homotopy mapping, and $\mathbb{I}$ is compact, the images of β_k's are convergent to the point y_0 (for a detailed proof see [2, Theorem 2.13]). Then one can define the $(n+1)$-Hawaiian simplex $\beta = \cup_{k=0}^{\infty}\beta_k$, where β_0 is the constant map at the point y_0. Moreover, since $f^s_{\#}(\alpha|_{\Delta^n_k}) - g^s_{\#}(\alpha|_{\Delta^n_k})$ for each $k = 0, 1, \ldots,$ $f_{\#}(\alpha) - g_{\#}(\alpha)$ equals the boundary of β. □

A space X is called *semi-locally contractible at* x_0 if there exists an open neighborhood U of x_0 such that the inclusion map $i : U \to X$ is nullhomotopic. A space X is called *semi-locally strongly contractible at* the point x_0 if there exists an open neighborhood U of x_0 such that the inclusion map $i : U \to X$ is homotopic to the constant map relative to the point x_0. Obviously if X is semi-locally strongly contractible at x, it is semi-locally contractible at x too. If X is semi-locally stronlgy contractible, then since the image of any Hawaiian simplex is contained in U except a finite number of standard simplexes and U can be contracted to x_0, Hawaiian simplexes have no information more than finite standard simplexes. Example 2.4 calculates the Hawaiian homology group of the circle $\mathbb{S}^1$, and it is formally generalized as follows.

Corollary 3.5. *Let X be semi-locally strongly contractible at the point x_0. Then*

$$\mathbb{H}_n(X, x_0) \cong \sum_{\aleph_0} H_n(X).$$

Proof. Since $\sum_{\aleph_0} H_n(X)$ is a subgroup of $\prod_{\aleph_0} H_n(X)$, the elements are of the form $\{b_k\}_{k\in\mathbb{N}}$ where $b_k = 0$ for all but finitely many k's. For each $k \in \mathbb{N}$, we consider the kth factor $H_n(X)$ as a subgroup of $\mathbb{H}_n(X, x_0)$, where each simplex vanishes except the kth one corresponded to the elements of $H_n(X)$. The intersections vanish, because each factor is corresponded to a unique $k \in \mathbb{N}$. It remains to verify that $\mathbb{H}_n(X, x_0)$ is generated by $\sum_{\aleph_0} H_n(X)$. Let σ be an n-Hawaiian simplex in X at x_0. Since X is semi-locally strongly contractible at x_0, there exists a neighborhood U of x_0 that is contractible in X at x_0. Since each n-Hawaiian simplex is a union of null-convergent standard n-simplexes (see [2]), there is $K \in \mathbb{N}$ such that $im(\sigma_k) \subseteq U$ for $k \geq K$. Then $\bigcup_{k\geq K} \sigma_k$ is null-homotopic in X, and thus $\mathbb{H}_n(X, x_0)$ is generated by $\sum_{\aleph_0} H_n(X)$. □

We present a space in Example 3.6 to prove that a free homotopy equivalence does not necessarily induce the isomorphism on Hawaiian homology groups.

Example 3.6. Let $C(\mathbb{HE}^1) = \mathbb{HE}^1 \times [0, 1]/\mathbb{HE}^1 \times \{1\}$ be the cone over the one-dimensional Hawaiian earring and θ be the origin at the bottom $\mathbb{HE}^1$. Since the cones are contractible, the homotopy axiom implies that the homology group must vanish, but it does not hold for the first Hawaiian homology group. To prove this, consider the simple 1-Hawaiian simplex σ whose image is the Hawaiian earring at the bottom $\mathbb{HE}^1 \times \{0\}$. This Hawaiian simplex is contained in $Z_1(C\mathbb{HE}^1, \theta) = \ker \partial_1$, and also there is no 2-Hawaiian simplex whose boundary equals σ. Therefore $\mathbb{H}_1(C(\mathbb{HE}^1, \theta) \neq 0$.

Example 3.6 emphasises the difference between Hawaiian homology and singular homology groups. That is all the singular homology groups vanishes for contractible spaces, but it does not hold for Hawaiian homology groups as seen in Example 3.6. However, there are some relations between Hawaiian homology and singular homology groups, such as the isomorphism proven in Corollary 3.5.

Null convergent homology groups. Hawaiian homology group is defined by convergent sequences of simplexes. Now there is a question; may we define Hawaiian homology by convergent sequences of singular n-chains not using Hawaiian simplexes? If the group is defined by convergent sequences of n-chains, the abelian group structure does not seem to be the same as that of the Hawaiian homology group. First, we intend to explain what we mean by convergence. For a sequence (c_i) of n-chains $c_i = \sum_\sigma a_\sigma \sigma$, where each $\sigma : \Delta^n \to X$ is a singular n-simplex, let $\text{supp}\,(c_i) = \bigcup \sigma(\Delta^n)$. Let us say that the sequence

(c_i) converges to a point x_0 of X, written as $\lim_{n\to\infty} c_i = x_0$, if for each neighborhood U of x_0, there exists an integer n such that $\operatorname{supp}(c_m) \subseteq U$ for each $m \geq n$.

Definition 3.7. Let $HS_n(X, x_0)$ be the subgroup of the countably infinite product of standard n-chains consisting of convergent ones, that is

$$HS_n(X, x_0) = \{(c_i) | \lim_{i\to\infty} c_i = x_0\}.$$

Consider the boundary operator $\partial_n : HS_n(X, x_0) \to HS_{n-1}(X, x_0)$ as the restriction of the countable product of the standard boundary operators on standard n-chains. Then the group $HL_n(X, x_0)$ could be defined by $HL_n(X, x_0) = \ker \partial_n / \operatorname{im} \partial_{n+1}$.

One can straightforwardly check that $HS_n(X, x_0)$ and then $HL_n(X, x_0)$ have group structures that seems to be related to Hawaiian homology groups.

Problem 3.8. Are groups $HL_n(X, x_0)$ and $\mathbb{H}_n(X, x_0)$ isomorphic for $n \in \mathbb{N}$?

There is a natural relation between group $HL_n(X, x_0)$ and countably infinite product of nth singular homology group as follows.

Proposition 3.9. *Let X be a path-connected space. The natural corresponding ψ : $HL_n(X, x_0) \to \prod_{\aleph_0} H_n(X)$ is a homomorphism.*

Proof. Define $\psi : HL_n(X, x_0) \to \prod_{\aleph_0} H_n(X)$ by the rule Let $(c_i)\operatorname{im} \partial_{n+1}$ be an element of $HL_n(X, x_0)$. Put $\psi(c_i)\operatorname{im} \partial_{n+1} := (c_i \operatorname{im} \partial^s_{n+1})$, where ∂^s is the standard boundary homomorphism. To prove well-definedness, let $(c_i)\operatorname{im} \partial_{n+1} = (b_i)\operatorname{im} \partial_{n+1}$. Then $(c_i - b_i) \in \operatorname{im} \partial_{n+1}$. Thus there is $(a_i) \in HS_{n+1}(X, x_0)$ such that $\partial_{n+1}(a_i) = (c_i - b_i)$. That is $c_i - b_i = \partial^s_{n+1} a_i$, for each $i \in \mathbb{N}$, because the boundary homomorphism ∂_{n+1} is the product of standard boundary homomorphisms on the components. Then $c_i - b_i \in \operatorname{im} \partial^s_{n+1}$. Hence $c_i \operatorname{im} \partial^s_{n+1} = b_i \operatorname{im} \partial^s_{n+1}$ for each $i \in \mathbb{N}$. This means that ψ is well-defined. To prove ψ is a homomorphism, consider $(c_i)\operatorname{im} \partial_{n+1}$ and $(b_i)\operatorname{im} \partial_{n+1}$ as arbitrary elements of $HL_n(X, x_0)$.

$$\begin{aligned}
\psi((c_i)\operatorname{im} \partial_{n+1} + (b_i)\operatorname{im} \partial_{n+1}) &= \psi((c_i + b_i)\operatorname{im} \partial_{n+1}) \\
&= (c_i + b_i \operatorname{im} \partial^s_{n+1}) \\
&= (c_i \operatorname{im} \partial^s_{n+1} + b_i \operatorname{im} \partial^s_{n+1}) \\
&= (c_i \operatorname{im} \partial^s_{n+1}) + (b_i \operatorname{im} \partial^s_{n+1}) \\
&= \psi(c_i)\operatorname{im} \partial_{n+1} + \psi(b_i)\operatorname{im} \partial_{n+1},
\end{aligned}$$

and then ψ is a homomorphism. □

The homomorphic image of the homomorphism ψ introduced in Proposition 3.9, is an abelian group which we denote by $\mathbb{L}_n(X, x_0)$. This group consists of all sequences of classes of convergent n-chains to the base point x_0. Recall that homotopy contractions make Hawaiian homology groups vanish whenever they preserve the base points. However $\mathbb{L}_n$ group is a subgroup of $\prod_{\aleph_0} H_n(X)$, and then it is trivial for contractible spaces, distinguishing it from the Hawaiian homology group. Although, these two groups may be isomorphic for some locally trivial spaces as proven in the following for semi-locally contractible spaces; See Corollary 3.5.

Proposition 3.10. *Let X be a path-connected space, which is semi-locally contractible at $x_0 \in X$. Then*

$$\mathbb{L}_n(X, x_0) \cong \sum_{\aleph_0} H_n(X).$$

Proof. Since $\mathbb{L}_n(X, x_0)$ is the image of homomorphism defined in Proposition 3.9, ψ : $HL_n(X, x_0) \to \prod_{\aleph_0} H_n(X)$, it suffices to show that $\operatorname{im} \psi = \sum_{\aleph_0} H_n(X)$. Let $\psi(c_i)\operatorname{im} \partial_{n+1} = (c_i \operatorname{im} \partial^s_{n+1}) \in \operatorname{im} \psi$. There is an open neighborhood U of x_0 such that the inclusion $i : U \to X$ is nullhomotopic. Since $(c_i) \in HS_n(X, x_0)$, there is $m \in \mathbb{N}$ such that

$\mathrm{supp}\,(c_i) \subseteq U$ for $i \geq m$. Then c_i can be considered as an n-chain in U for $i \geq m$. Since i is nullhomotopoic, by the homotopy axiom for singular homology, $H_n(i)$ is the trivial homomorphism. Then $c_i \mathrm{im}\,\partial^s_{n+1} = H_n(i)(c_i \mathrm{im}\,\partial^s_{n+1}) = 0$ for $i \geq m$. That is $\psi(c_i) = (c_i \mathrm{im}\,\partial^s_{n+1}) \in \sum_{\aleph_0} H_n(X)$. Now let $(c_i \mathrm{im}\,\partial^s_{n+1}) \in \sum_{\aleph_0} H_n(X)$. Then for some $m \in \mathbb{N}$, we have $c_i \mathrm{im}\,\partial^s_{n+1} = 0$ if $i \geq m$. Put $b_i = c_i$ for $i < m$ and $b_i = 0$ for $i \geq m$. Thus $\psi(b_i)\mathrm{im}\,\partial_{n+1} = (b_i \mathrm{im}\,\partial^s_{n+1}) = (c_i \mathrm{im}\,\partial^s_{n+1})$. Also, since $b_i = 0$ for $i \geq m$, $\mathrm{supp}\,(b_i) \subseteq U$ if $i \geq m$ for any open neighborhood U of x_0. Hence $\lim b_i = x_0$, and then $(b_i) \in HS_n(X, x_0)$. Moreover, note that ∂_n is induced by countably infinite copies of the standard boundary homomorphism ∂^s_n; That is $\partial_n(b_i) = (\partial^s_n b_i) = (\partial^s_n c_1, \ldots, \partial^s_n c_{m-1}, \partial^s_n 0, \partial^s_n 0, \ldots) = (0, \ldots, 0)$, because $(c_i) \in \sum_{\aleph_0} H_n(X)$, and then $c_i \in \ker \partial^s_n$, and also $\partial^s_n 0 = 0$ for $i \geq m$. Therefore, $(b_i) \in \ker \partial_n$. □

Now by Corollary 3.5 and Proposition 3.10, if X is semi-locally strongly contractible at x_0, then

$$\mathbb{H}_n(X, x_0) \cong \sum_{\aleph_0} H_n(X) \cong \mathbb{L}_n(X, x_0).$$

Note that a similar isomorphism holds for the nth Hawaiian group and the L_n group defined in [2, Theorem 2.5]. The L_n groups were defined by null-convergent sequences of n-loops; For a pointed space (X, x), $L_n(X, x)$ is the subset of $\prod_{\aleph_0} \pi_n(X, x)$ consisting of all sequences of homotopy classes $\{[f_i]\}$, where $\{f_i\}$ is null-convergent [2, Definition 2.6].

Additivity axiom. A modified version of the additivity axiom holds for Hawaiian homology group as stated in Theorem 3.11. Moreover, Example 3.12 shows that classical additivity axiom does not hold for Hawaiian homology. Recall that X is called *semi-locally path-connected* at $x \in X$ if there is an open neighborhood U of x such that U is path-connected in X; That is any two points x_1 and x_2 in U can be connected by a path in X.

Theorem 3.11. *Let X be semi-locally path-connected at point x_0, let $\{X_\lambda\}$ be the family of path components with $x_0 \in X_{\lambda_0}$, and let $n \geq 0$. Then*

$$\mathbb{H}_n(X, x_0) \cong \Big(\sum_{\lambda \neq \lambda_0} \sum_{\aleph_0} H_n(X_\lambda)\Big) \oplus \mathbb{H}_n(X_{\lambda_0}, x_0). \tag{3.3}$$

Proof. First, note that for each finite sequence of standard simplexes, one may construct a Hawaiian simplex by vanishing all terms of the sequence except the finite number of given simplexes. Therefore, $\sum_{\aleph_0} H_n(X_\lambda)$ can be considered as a natural subgroup of $\mathbb{H}_n(X, x_0)$. Moreover, consider two elements $\mathrm{cls}\,\sigma, \mathrm{cls}\,\beta \in \mathbb{H}_n(X_{\lambda_0}, x_0)$. If $\mathrm{cls}\,\sigma = \mathrm{cls}\,\beta$, then $\sigma - \beta$ equals the boundary of some $(n+1)$-Hawaiian simplexes in X_{λ_0}. Thus it holds for $\sigma - \beta$ as an n-Hawaiian simplex in X. Then $\mathrm{cls}\,\sigma = \mathrm{cls}\,\beta$ in $\mathbb{H}_n(X, x_0)$. Hence, $\mathbb{H}_n(X_{\lambda_0}, x_0)$ is a subgroup of $\mathcal{H}_n(X, x_0)$. Also, since X is semi-locally path-connected, there exists a path-connected open neighborhood, namely U of X at x_0. We show that the abelian group $\mathbb{H}_n(X, x_0)$ is generated by the family of subgroups $\{\sum_{\aleph_0} H_n(X_\lambda)\}_{\lambda \neq \lambda_0} \cup \{\mathbb{H}_n(X_{\lambda_0}, x_0)\}$. To prove this, consider a Hawaiian simplex σ. Then there is a natural number K such that if $k \geq K$, then $im(\sigma(\Delta^n_k)) \subseteq U$. Note that since U is path-connected, $U \subseteq X_{\lambda_0}$, and then $im(\sigma|_{\bigcup_{k \geq K} \Delta^n_k}) \subseteq X_{\lambda_0}$. Therefore, $\sigma|_{\bigcup_{k \geq K} \Delta^n_k}$ can be considered as an element of $\mathbb{H}_n(X_{\lambda_0}, x_0)$ and the other standard simplexes Δ^n_k, $k < K$, are generated by $\sum_{\lambda \neq \lambda_0} \sum_{\aleph_0} H_n(X_\lambda)$, as desired. Since these subgroups have trivial intersections, because of the path-connectedness of simplexes, the isomorphism holds. □

Now, we present an example to show that the isomorphism (3.3) differs from the corresponding statement for the singular homology,

$$H_n(X) \cong \sum_\lambda H_n(X_\lambda).$$

Example 3.12. Consider the space $X = \mathbb{HE}^2 \dot{\cup} \mathbb{S}^2$ and the point $a \in \mathbb{S}^2$. Then $\mathbb{H}_2(X, a) \cong \sum_{\aleph_0} H_2(\mathbb{HE}^2) \oplus \mathbb{H}_2(\mathbb{S}^2, a) = \sum_{\aleph_0} \prod_{\aleph_0} \mathbb{Z} \oplus \sum_{\aleph_0} \mathbb{Z}$ [10]. On the other hand, $\mathbb{H}_2(\mathbb{HE}^2, \theta) \oplus \mathbb{H}_2(\mathbb{S}^2, a) \cong \prod_{\aleph_0} \sum_{\aleph_0} \mathbb{Z} \oplus \sum_{\aleph_0} \mathbb{Z}$ and these two groups are not isomorphic [11].

4. Hurewicz theorem for Hawaiian homology

By a well-known fact called Hurewicz theorem, for path-connected spaces, two functors of homotopy and homology are connected. In fact, for a path-connected space X, there exists a surjective homomorphism $\phi_s : \pi_1(X, x_0) \to H_1(X)$ whose kernel equals the commutator subgroup of $\pi_1(X, x_0)$. A similar relation exists between Hawaiian homology and Hawaiian groups of pointed spaces. For each pointed space (X, x_0), there exists a homomorphism $\phi : \mathcal{H}_1(X, x_0) \to \mathbb{H}_1(X, x_0)$. The homomorphism ϕ is surjective and its kernel equals the commutator subgroup if X is path-connected and locally path-connected at x_0.

Theorem 4.1. *Let X be a path-connected space that is locally path-connected at x_0. Then the homomorphism $\phi : \mathcal{H}_1(X, x_0) \to \mathbb{H}_1(X, x_0)$ is surjective whose kernel equals the commutator subgroup of $\mathcal{H}_1(X, x_0)$.*

Proof. First, we correspond an element of $\mathbb{H}_1(X, x_0)$ for each map $f : (\mathbb{HE}^1, \theta) \to (X, x_0)$. Let $\eta : \bigcup_{k=0}^{\infty} \Delta_k^1 \to \mathbb{HE}^1$ maps each Δ_k^1 onto $\mathbb{S}_k^1$ homeomorphically except at the vertices mapped to the base point θ. Then $f\eta$ is a 1-Hawaiian simplex in X at x_0. Also $\partial(f\eta) = \bigcup_{k=0}^{\infty} f_k\eta_k(e_1^k) - \bigcup_{k=0}^{\infty} f_k\eta_k(e_0^k) = \bigcup_{k=0}^{\infty} (f_k\eta_k(e_1^k) - f_k\eta_k(e_0^k)) = \bigcup_{k=0}^{\infty} 0 = 0$, where $\eta_k : \Delta_k \to \mathbb{S}_k^1$ is the restriction of η to the kth standard 1-simplex. Thus $f\eta \in im(\partial_1)$. Hence one can define $\phi : \mathcal{H}_1(X, x_0) \to \mathbb{H}_1(X, x_0)$ by $\phi([f]) = \text{cls}\, f\eta$.

To prove well-definedness, let $f \simeq g\ rel\{\theta\}$. Then $f_k \simeq g_k\ rel\{\theta\}$ for each $k = 0, 1, \ldots$, where f_k and g_k are the restrictions of f and g, respectively, to the kth circle $\mathbb{S}_k^1$. By the standard Hurewicz theorem, $\text{cls}\, f_k\eta_k = \text{cls}\, g_k\eta_k$ for $k = 0, 1, \ldots$. Thus $\text{cls}\,(f_k\eta_k - g_k\eta_k) = 0$, and then $f_k\eta_k - g_k\eta_k$ belongs to the image of ∂, the boundary of some linear composition of standard 2-simplex $\sum_{j=1}^{n} m_j\sigma_k^j$. By the proof of the standard Hurewicz theorem [20, Lemma 4.26], this standard 2-simplex is constructed by the homotopy mapping $H_k : f_k \simeq g_k$ for $k = 1, 2, \ldots$. Since H_k is the restriction of the homotopy $H : f \simeq g$ to the kth circle, the image of H_k is null-convergent. Hence for any open neighborhood U of x_0, $\text{im}\,(H_k) \subseteq U$ if $k \geq K$ for some $K \in \mathbb{N}$ (see [2, Definition 2.1]). Therefore, $im(\cup_{j=0}^{n}\sigma_k^j)$ is null-convergent and then $\cup_{k=0}^{\infty} \sum_{j=1}^{n} m_n\sigma_k^j$, where the 0-simplex is the constant standard simplex, is a linear composition of 2-Hawaiian simplexes, whose boundary equals $f\eta - g\eta$. Hence $\text{cls}\ f - g = 0$, and then $\text{cls}\ f = \text{cls}\ g$. It implies that ϕ is well-defined.

Moreover, ϕ is a homomorphism. To prove this, consider two elements $[f]$ and $[g]$ in the group $\mathcal{H}_1(X, x_0)$. Then the following sequence of equalities holds:

$$\phi([f][g]) = \phi([f * g]) = \text{cls}\ (f * g)\eta = \text{cls}\ \ \cup_{k=0}^{\infty} (f_k * g_k)\eta_k.$$

We must verify the equality

$$\text{cls}\ \ \cup_{k=0}^{\infty} (f_k * g_k)\eta_k = \text{cls}\ \ \cup_{k=0}^{\infty} f_k\eta_k + \text{cls}\ \ \cup_{k=0}^{\infty} g_k\eta_k.$$

Equivalently, we must find a composition of 1-Hawaiian simplexes whose boundary equals $\cup_{k=0}^{\infty}(f_k * g_k)\eta_k - \cup_{k=0}^{\infty} f_k\eta_k - \cup_{k=0}^{\infty} g_k\eta_k$. By the proof of the standard Hurewicz theorem [20, Theorem 4.27], there exists a sequence of composition of standard 2-simplexes, namely $\sum_{j=1}^{n} m_n\sigma_k^j$ ($k \in \mathbb{N}$) constructed by the maps $f_k * g_k$, f_k, and g_k such that $\partial(\sum_{j=1}^{n} m_n\sigma_k^j) = (f_k * g_k)\eta_k - f_k\eta_k - g_k\eta_k$. It remains to verify that the sequence $\{\sum_{j=1}^{n} m_n\sigma_k^j\}$ is null-convergent. Since the maps $f_k * g_k$, f_k, and g_k are null-convergent, so is $\sum_{j=1}^{n} m_n\sigma_k^j$ ($k \in \mathbb{N}$). Therefore, $\cup_{k=0}^{\infty} \sum_{j=1}^{n} m_n\sigma_k^j$ is a 2-Hawaiian simplex, and also its boundary equals $\cup_{k=0}^{\infty}(f_k * g_k)\eta_k - \cup_{k=0}^{\infty} f_k\eta_k - \cup_{k=0}^{\infty} g_k\eta_k = (f * g)\eta - f\eta - g\eta$. Thus $\text{cls}\ \ (f * g)\eta = \text{cls}\ \ f\eta + \text{cls}\ \ g\eta$, and then ϕ is a homomorphism.

To prove surjectivity of the homomorphism ϕ, we consider an element cls σ of $\mathbb{H}_1(x, x_0)$. By the restriction to the kth 1-simplex, denoted by σ_k, one can use the standard Hurewicz theorem [20, Theorem 4.29], to obtain a sequence of maps f_k defined from $\mathbb{S}^1$ to X such that $\phi_s([f_k]) = \text{cls } \sigma_k$. If $\sigma_k = \sum_{i=0}^m \sigma_k^i$, then by the proof of [20, Theorem 4.29], f_k is defined by the concatenation of a sequence of simplexes and some paths between the base point x_0 and the end points of simplexes. Since X is locally path-connected at x_0 having a countable local basis, one can consider a sequence of nested open neighborhoods $\{U_i\}_{i\in\mathbb{N}}$ of x_0 such that each element U_i is path-connected in U_{i+1}. Therefore, the sequence f_k can be considered to be null-convergent. Then define $f|_{\mathbb{S}^1_k} = f_k$. Since $\{f_k\}$ is null-convergent, f is continuous by [2, Lemma 2.2], and also $\phi([f]) = \sigma$.

To compute the kernel of ϕ, we use some parts of proof of the original Hurewicz theorem. Recall that $(\mathcal{H}_1(X, x_0))' \subseteq \ker\phi$, because the group $\mathbb{H}_1(X, x_0)$ is abelian. Now let $[f] \in \ker\phi$ be an arbitrary element. Since $[f] \in \ker\phi$, $\phi([f]) = 0$; That is cls $f\eta = 0$. Thus $f\eta \in \text{im } \partial_2$. Equivalently there exists an 2-Hawaiian chain $\sigma = \sum_i n_i\sigma_i$ such that $\partial_2(\sigma) = f\eta$. We can write $n_i = \pm 1$, and then for each σ_i, $\partial\sigma_i = \tau_{i0} - \tau_{i1} + \tau_{i2}$ for three 1-Hawaiian simplexes τ_{ij}. Notice that

$$f\eta = \partial \sum_i n_i\sigma_i = \sum_i n_i \partial\sigma_i = \sum_i n_i(\tau_{i0} - \tau_{i1} + \tau_{i2}) = \sum_{i,j} (-1)^j n_i \tau_{ij}. \tag{4.1}$$

But $f\eta$ is a 1-Hawaiian simplex, and all 1-Hawaiian simplexes make a basis for the set of 1-chains. That is all the τ_{ij} must form cancelling pairs except for one being equal to $f\eta$. Now for each $k \in \mathbb{N}$, by the [20, proof of Theorem 4.29], one can consider $[f|_{\mathbb{S}^1_k}]$ as an element of $\pi_1(X, x_0)'$ by a rearrangement $\gamma_k = a_{k,1}b_{k,1}a_{k,1}^{-1}b_{k,1}^{-1}\ldots a_{k,m}, b_{k,m}a_{k,m}^{-1}b_{k.m}^{-1}$ formed by $\sum_{i,j}(-1)^j n_i \tau_{ij}|_{\Delta^1_k}\eta^{-1}|_{\mathbb{S}^1_k}$ together with some paths joining the base point x_0 to the standard simpexes $\tau_{ij}|_{\Delta^1_k}$. Since X is locally path-connected at x_0, we can consider those joining paths tend to x_0. Also $\tau_{ij}|_{\Delta^1_k}$ tend to x_0, because τ_{ij} is a 1-Hawaiian simplex. Hence γ_k tend to x_0. Define $\gamma : \mathbb{HE}^1 \to X$ by $\gamma|_{\mathbb{S}^1_k} = \gamma_k$. Then $[\gamma]$ belongs to $\mathcal{H}_1(X, x_0)$, because γ_k's are null-convergent. It remains to prove that $[\gamma] \in \mathcal{H}_1(X, x_0)'$. We know that $[\gamma_k]$'s belong to $\pi_1(X, x_0)'$ and they have some form $\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\ldots\alpha_j\beta_j\alpha_j^{-1}\beta_j^{-1}$. But by [20, proof of Theorem 4.29], γ_k's are obtained by a fixed pattern using restrictions of Hawaiian simplexes τ_{ij}'s introduced in Equation (4.1). Therefore the length of sequence, m does not depend on k and it is fixed for all γ_k's. Also all of the sequences of loops $\{a_{k,1}\}, \ldots, \{a_{k,m}\}, \{b_{k,1}\}, \ldots, \{b_{k,m}\}$ are null-convergent. Thus one can define $a, b : (\mathbb{HE}^1, \theta) \to (X, x_0)$ by $a_i|_{\mathbb{S}_k} = a_{k,i}$ and $b_i|_{\mathbb{S}_k} = b_{k,i}$. Then $\gamma = a_1b_1a_1^{-1}b_1^{-1}\ldots a_mb_ma_m^{-1}b_m^{-1}$ and therefore $[\gamma]$ belongs to $\mathcal{H}_1(X, x_0)'$ by [2, Lemma 2.2 (i)]. Moreover, since $f|_{\mathbb{S}^1_k}$'s and γ_k's are pairwise homotopic and only different in some null-convergent paths, the sequence of homotopies is null-convergent. Then by [2, Lemma 2.2 (ii)], $[f] = [\gamma]$ in $\mathcal{H}_1(X, x_0)$. □

By Theorem 4.1 and the first isomorphism theorem, it holds that

$$\mathbb{H}_1(X, x_0) \cong \frac{\mathcal{H}_1(X, x_0)}{(\mathcal{H}_1(x, x_0))'},$$

where $(\mathcal{H}_1(x, x_0))'$ denotes the commutator subgroup of the given group $(\mathcal{H}_1(x, x_0))$. Then the first Hawaiian homology group is isomorphic to the abelianization of the first Hawaiian group for path-connected and locally path-connected pointed topological spaces. This fact helps us to study Hawaiian groups of topological spaces by Hawaiian homology.

5. Higmann-complete Hawaiian groups

Herfort and Hojka proved that the fundamental group of archipelago spaces is a Higman-complete group, a group introduced as a non-abelian form of cotorsion groups; for more details, see [14, Definition 1]. In the rest of the paper, all the spaces are assumed to be path-connected.

Definition 5.1 ([14])**.** A group G is Higman-complete if for any sequence $f_1, f_2, \dots \in G$ and for a given sequence of words with two alphabets $w_1, w_2, \dots$, there exists a sequence $h_1, h_2, \dots \in G$ such that all equations $h_i = w_i(f_i, h_{i+1})$ hold simultaneously.

The first example of non-abelian Higman-complete groups are the fundamental groups of the spaces having small loops [14]. Virk [22] defined small loop as a loop α based at a point x such that for every open neighbourhood U of x, there exists a loop homotopic to α contained in U. The following lemma states that the first Hawaiian group of any space whose loops at the base point are small, belongs to the class of Higman-complete groups. This fact also holds for the fundamental group of archipelago spaces [14]. The well-known archipelago space was defined and studied by Virk [22], which is an example of spaces containing small loops.

Lemma 5.2. *If all* 1*-loops at the point* $x \in X$ *are small, then* $\mathcal{H}_1(X, x)$ *is Higmann-complete. If the same is true for* n*-loops,* $n \geq 2$*, then* $\mathcal{H}_n(X, x)$ *is cotorsion.*

Proof. Let all 1-loops be small at $x \in X$. Assume that $f : \mathbb{HE}^1 \to X$ is a 1-Hawaiian loop at x. First, we show that f is a small 1-Hawaiian loop. That is for each open neighborhood U of x, there is a homotopic representative $g \simeq f$ with image contained in U. Since f is continuous, there is $K \in \mathbb{N}$ such that $im(f|_{\mathbb{S}^1_k}) \subseteq U$ for $k \geq K$. Moreover, by the assumption, all $f|_{\mathbb{S}^1_k}$'s are small. For $k < K$, define g_k as the homotopic representative 1-loop of $f|_{\mathbb{S}^1_k}$ with image in U. Now put $g : \mathbb{HE}^1 \to X$ as $g|_{\mathbb{S}^1_k} = g_k$ for $k < K$, and $g|_{\mathbb{S}^1_k} = f|_{\mathbb{S}^1_k}$ for $k \geq K$, which is homotopic to f by [2, Lemma 2.2]. In [14, Theorem 4], it was shown that $\pi_1(X, x)$ is Higmann-complete if all 1-loops at x in X are small. Let $f_1, f_2, \dots$ be a sequence of elements in $G = \mathcal{H}_1(X, x)$ and $w_1, w_2, \dots$ a sequence of words. Also let $\{U_i\}_{i \in \mathbb{N}}$ be the local basis at x. Similar to [14, Proof of Theorem 4], we inductively construct maps η_i by subdivision $\mathbb{HE}^1$ into equal pieces such that for $h_i := [\eta_i] \in G$ all equations $h_i = w_i(f_i, h_{i+1})$ hold.

By the first part of the proof, each f_i is small. Then a representing γ_i of f_i enclosed in the neighbourhood U_i of x exists. We use $\overline{\gamma_i}$ as the reversed map corresponding to the group element f_i^{-1}. The word w_1 has finite word length, so subdivide each circle in $\mathbb{HE}^1$, as the domain of η_1, into accordingly many pieces of equal size. For each place in w_1 occupied by an f_1 (or f_1^{-1}) define η_1, restricted to the corresponding piece of $\mathbb{HE}^1$, equal to an appropriately scaled copy of γ_1 (or $\overline{\gamma_1}$), and leave the other parts undefined for now. Now proceed in this manner for w_2, again splitting $\mathbb{HE}^1$ into as many pieces as the length of w_2, and then setting η_2 equal to a scaled copy of γ_2 (or $\overline{\gamma_2}$) for each piece corresponding to f_2 (or f_2^{-1}). Then fill the partially defined η_2 into the pieces of the domain of η_1 that correspond to h_2 in the word w_1 (and the reversed $\overline{\eta_2}$ for h_2^{-1}). Going forward, each time the partial definition of η_i is extended, it is reinserted in the definition of η_i^{-1}, and recursively all the way to η_1. Note that each endpoint of an piece is always mapped to the basepoint by the γ_i, so bordering definitions do match up properly. After running through all infinitely many steps of this construction, each η_i is defined everywhere on $\mathbb{HE}^1$. but for a closed, totally disconnected set of limit points; set η_i constant to the basepoint on this set. Thus η_i is well-defined. Now clearly, η_i is continuous restricted to the interior of a piece corresponding to some f_n (or f_n^{-1}), with $n \geq i$. For $b \in \mathbb{HE}^1$ not within such a piece, b is mapped to the base point and the left (resp. right) continuity follows either from the continuity of the adjacent piece, or, in the absence of such, from the fact that the pieces converging to b eventually enclose their image in an arbitrary neighbourhood U of x; in this case, the (one-sided) neighbourhood around b can be chosen small enough that it does not intersect any of the finitely many pieces that are not mapped into U.

In summary, we have constructed a sequence of loops η_i, each concatenation according to the word w_i of the loops γ_i and η_{i+1}. Hence, the equations $h_i = w_i(f_i, h_{i+1})$ of the corresponding elements $f_i, h_i \in G$ all hold, as desired. Therefore, $\mathcal{H}_1(X, x)$ is Higmann-complete. Similarly, if all n-loops are small, $n \geq 2$, then $\pi_n(X, x)$ and $\mathcal{H}_n(X, x)$ are Higmann-complete. Since $\pi_n(X, x)$ and $\mathcal{H}_n(X, x)$ are abelian, by [14, Theorem 3], they are cotorsion. □

Theorem 4.1 presents a version of Hurewicz theorem for Hawaiian groups and Hawaiian homology groups. Now we use that theorem to obtain the following corollary.

Corollary 5.3. *Let X be locally path-connected at x, and let all 1-loops at x be small. Then the first Hawaiian homology group $\mathbb{H}_1(X, x)$ is cotorsion.*

Proof. Since all 1-loops at x are small, by Theorem 5.2, $\mathcal{H}_1(X, x)$ is Higmann-complete. Also by Theorem 4.1, the group $\mathbb{H}_1(X, x)$ is the epimorphic image of the group $\mathcal{H}_1(X, x)$. Then it is Higmann-complete by [14, Lemma 2]. Moreover, every abelian group is Higmann-complete if and only if it is cotorsion [14, Theorem 3]. Therefore $\mathbb{H}_1(X, x)$ is cotorsion. □

If all n-loops at a point are small, then both nth homotopy and nth Hawaiian groups are Higman-complete. Note that the homotopy groups on path-connected spaces do not depend on the choice of the base point. It implies that $\pi_n(X, x)$ is Higmann-complete if all n-loops at some points of X are small, but it is not true for Hawaiian groups. Some counterexamples are demonstrated in Corollary 5.12, by applying the following lemma.

Lemma 5.4. *Let X be semi-locally strongly contractible at x_0 and let $n \geq 2$. If $H_1(X, x_0)$ is not torsion, then neither $\mathbb{H}_1(X, x_0)$ is cotorsion, nor $\mathcal{H}_1(X, x_0)$ is Higman-complete. Moreover, if $\pi_n(X, x_0)$ is not torsion, then $\mathcal{H}_n(X, x_0)$ is not cotorsion.*

Proof. Since X is semi-locally strongly contractible at x, by Corollary 3.5, the isomorphism $\mathbb{H}_1(X, x) \cong \sum_{\aleph_0} H_1(X)$ holds. If $\mathbb{H}_1(X, x)$ is cotorsion, then $\sum_{\aleph_0} H_1(X)$ is cotorsion. Thus by [12, Chap. 9, Proposition 6.10], $H_1(X)$ is torsion, which is a contradiction. By [2, Theorem 2.5], $\mathcal{H}_1(X, x_0) \cong \prod^W_{\aleph_0} \pi_1(X, x_0)$. If $\mathcal{H}_1(X, x_0)$ is Higman-complete, then $\prod^W_{\aleph_0} \pi_1(X, x_0)$ is Higmann-complete. Thus by [14, Theorem 3], its abelianization $\mathbb{H}_1(X, x) \cong \sum_{\aleph_0} H_1(X)$ is cotorsion, which does not hold. □

Two notions of weak direct product ($\prod^W$) and direct sum ($\sum$) are defined for families of groups, and they are almost the same, but direct sum is used only if the given groups are abelian and weak direct product is used for any family $\{G_i\}_i$ of groups and denoted by $\prod^W_i G_i$ being the subgroup of $\prod_{i \in I} G_i$ consisting of all elements $\{g_i\}_{i \in I}$ of $\prod_{i \in I} G_i$ such that $g_i = e_i$, for all $i \in I$ except a finite number..

CW spaces are semi-locally strongly contractible at any point, and then by Lemma 5.4, the following corollary is obtained.

Corollary 5.5. *Let X be a CW space with torsion-free first homology group. Then $\mathcal{H}_1(X, x)$ is not Higman-complete for all $x \in X$. Moreover, if $H_n(X)$ is torsion-free, then $\mathbb{H}_n(X, x)$ is not cotorsion for $x \in X$.*

If $\mathcal{H}_n(X, x)$ is Higman-complete for some $x \in X$, then so is $\pi_n(X, x)$ as an epimorphic image of $\mathcal{H}_n(X, x)$, but the converse statement does not hold. There are spaces such that their homotopy groups are Higman-complete but neither of their Hawaiian group nor Hawaiian homology groups are; see the following example.

Example 5.6. Consider a Higman-complete torsion-free group, namely $\mathbb{Q}$, and $X = K(\mathbb{Q}, 1)$ as the Eilenberg–MacLane space corresponded to $\mathbb{Q}$. Then $H_1(X) \cong \pi_1(X) \cong \mathbb{Q}$, which is Higman-complete because it is cotorsion. Moreover, since X is semi-locally strongly contractible at any point x, $\mathcal{H}_1(X, x) \cong \sum_{\aleph_0} \pi_1(X, x) \cong \sum_{\aleph_0} \mathbb{Q}$ by [2, Theorem 2.5]. Since X is a CW space and $H_1(X)$ is torsion-free, by Corollary 5.5, $\mathcal{H}_1(X, x)$ is not

Higman-complete for each $x \in X$. Also, $\mathbb{H}_1(X,x) \cong \sum_{\aleph_0} H_1(X) \cong \sum_{\aleph_0} \mathbb{Q}$ by Corollary 3.5, and then $\mathbb{H}_1(X,x)$ is not cotorsion for each $x \in X$ by Corollary 5.5.

Some pseudomanifolds satisfy the conditions of Corollary 5.5; see [18, Theorem 8.2].

Corollary 5.7. *The first Hawaiian group of an orientable two-dimensional pseudomanifold X is not Higman-complete. Also its first Hawaiian homology group $\mathbb{H}_1(X,x)$ is not cotorsion for all $x \in X$.*

Archipelago groups are the first examples of Higman-complete groups. They are introduced as the fundamental groups of archipelago spaces. The well-known Harmonic archipelago is a non-locally Euclidean space, which was introduced to study non-semi-locally simply connected spaces and it is the first example of spaces having small loops. This space was generalized to archipelago space [7]. Herfort and Hojka [14] computed the singular homology groups of archipelago spaces. In this paper, we use a similar argument to present Hawaiian homology groups of the archipelago spaces.

Definition 5.8 ([7])**.** Let $\{(X_i,x_i)\}_{i\in I}$ be a family of pointed spaces. Then the archipelago space $\mathbb{A} = \mathbb{A}(\{X_i\}_{i\in I})$ on the given family is defined as the mapping cone on the natural continuous bijection $f : \bigvee_{i\in I}(X_i,x_i) \to \widetilde{\bigvee}_{i\in I}(X_i,x_i)$. Let $X = \bigvee_{i\in I}(X_i,x_i)$ and $Y = \widetilde{\bigvee}_{i\in I}(X_i,x_i)$ and x_* be join point of Y. The quotient space C_f is obtained from $(X\times\mathbb{I})\cup Y$ by identifying $(x,0)$ with $f(x)$ for any $x \in X$ and $X\times\{1\}$ is collapsed to a point.

Archipelago spaces are examples of wild spaces. Free σ-groups are also fundamental groups of some wild spaces [9, Theorem A.1]. Indeed free σ-groups are not Higman-complete with behaves similar to free abelian groups, which are not cotorsion.

Remark 5.9. If $\{G_i\}_I$ is a family of groups such that $Ab(G_i)$ is not cotorsion for some $i \in I$, then the free product $*_I G_i$ and free σ-product $\circledast_I G_i$ are not Higman-complete. In fact, there are natural epimorphisms $\circledast_I G_i \to \prod_I G_i$ and $*_I G_i \to \prod_I^W G_i$. Then G_i and hence $Ab(G_i)$ are their epimorphic images for all $i \in I$. If either $\circledast_I G_i$ or $*_I G_i$ is Higman-complete, then so is G_i for all $i \in I$ by [14, Lemma 2]. Thus $Ab(G_i)$ is cotorsion for all $i \in I$ by [14, Theorem 3].

By Lemma 5.4, Hawaiian groups of semi-locally strongly contractible spaces are not Higman-complete. In the following result, we see that there are spaces, such as the one-dimensional Hawaiian earring, that are not semi-locally strongly contractible and their Hawaiian groups are Higman-complete.

Theorem 5.10. *Let $\{X_i\}_{i\in I}$ be a family of connected spaces, let $X = \widetilde{\bigvee}_I X_i$, and let x be the common point. If $H_1(X_i)$ is not cotorsion for some $i \in I$, then $\mathbb{H}_1(X,x)$ is not cotorsion, and moreover, $\pi_1(X,x)$ and $\mathcal{H}_1(X,x)$ are not Higman-complete.*

Proof. Since X_i is a retract of the space X, there exits a natural epimorphism $\mathbb{H}_1(X,x) \to \mathbb{H}_1(X_i,x_i)$ induced by the Hawaiian homology functor. Also by [9, Theorem A.1], we have $\pi_1(X,x) \cong \circledast_I G_i$, where $G_i = \pi_1(X_i,x_i)$. If $\mathcal{H}_1(X,x)$ is Higman-complete, then so is its epimorphic image $H_1(X_i)$ by [14, Lemma 2] for all $i \in I$. Thus, since $H_1(X_i)$ is abelian by [14, Theorem 3], it must be cotorsion for all $i \in I$. □

Theorem 5.10 and Lemma 5.2 imply Corollary 5.12 which compares the first Hawaiian groups of some weak join space and the Griffiths space over the space. One can generalize Griffiths space $\mathcal{G}(X_i)$ over an arbitrary family of spaces $\{X_i\}_{\mathbb{N}}$ as the wedge of two cones on $\widetilde{\bigvee}_{\mathbb{N}} X_i$.

Definition 5.11. Let $\{(X_i,x_i)\}_{i\in I}$ be a family of pointed spaces, $(X,x_*) = \widetilde{\bigvee}(X_i,x_i)$ be the weak join space, and C_1 and C_2 be two copies of the cone over X. Also let $y_1 = (x_*,0) \in C_1$ and $y_2 = (x_*,0) \in C_2$ be the base points in corresponded cones. Then the Griffiths space $\mathcal{G} = \mathcal{G}(X_i)$ over the family $\{X_i\}_{i\in I}$ is defined as the wedge of two spaces (C_1, y_1) and (C_2, y_2) and denoted by $\mathcal{G}(X_i)$; That $\mathcal{G} = \mathcal{G}(X_i)$ is the disjoint union of C_1 and C_2 with y_1 and y_2 are identified to a point.

Corollary 5.12. *Let $\{X_i\}_{i\in I}$ be as in Theorem 5.10, let $X = \widetilde{\bigvee}_{i\in I}$ be the week join space, let $\mathbb{A}(X_i)$ be the archipelago space, and let $\mathcal{G}(X_i)$ be the Griffiths space on X. Then $\mathcal{H}_1(\mathbb{A}(X_i), x_*) \ncong \mathcal{H}_1(X, x_*)$, and also, $\mathcal{H}_1(\mathcal{G}(X_i), x_*) \ncong \mathcal{H}_1(X, x_*)$.*

In the following theorem, we prove that for two points of the archipelago spaces, the Hawaiian groups are not isomorphic and depend on the base points. Thus archipelago spaces are other examples of path-connected spaces whose Hawaiian groups depend on the base points at different points.

Theorem 5.13. *Let $\{X_i\}_{i\in I}$ be a family of spaces such that X_i is locally strongly contractible at its base point x_i, and let $\mathbb{A} = \mathbb{A}_{i\in I}(X_i)$ be the archipelago space over the family $\{(X_i, x_i)\}$. If $a \in \mathbb{A}$ is any point other than common point x, then $\mathcal{H}_1(\mathbb{A}, x) \ncong \mathcal{H}_1(\mathbb{A}, a)$.*

Proof. By [14, Theorem 8], $H_1(\mathbb{A}) \cong \dfrac{\prod_{\aleph_0}\mathbb{Z}}{\sum_{\aleph_0}\mathbb{Z}}$, which is torsion-free by [5]. Since $\mathbb{A}$ is semi-locally strongly contractible at a, by Lemma 5.4, $\mathcal{H}_1(\mathbb{A}, a)$ is not Higman-complete. Also, since all 1-loops at x are small (see [14, Lemma 7]), by Lemma 5.2 $\mathcal{H}_1(X, x)$ is Higman-complete, and then it is not isomorphic to $\mathcal{H}_1(\mathbb{A}, a)$. □

Since the Griffiths space is not necessarily semi-locally strongly contractible at a, Lemma 5.4 cannot be used.

Theorem 5.14. *Let $\{X_i\}_{i\in I}$ be a family of spaces such that X_i is locally strongly contractible at its base point x_i. If $a \in \mathcal{G} = \mathcal{G}(\{X_i\})$ is any point other than common point x, then $\mathcal{H}_1(\mathcal{G}, x) \ncong \mathcal{H}_1(\mathcal{G}, a)$.*

Proof. We show that $\mathcal{H}_1(\mathcal{G}, a)$ is not Higman-complete and $\mathcal{H}_1(\mathcal{G}, x)$ is Higman-complete, and hence $\mathcal{H}_1(\mathcal{G}, a) \ncong \mathcal{H}_1(\mathcal{G}, x)$ By [8], $\mathcal{G}$ satisfies the premises of Lemma 5.2, and then $\mathcal{H}_1(\mathcal{G}, x)$ is Higman-complete. If $a \neq x_*$, then by [3, Theorem 3.2], $\mathcal{H}_1(\mathcal{G}, a)$ has $\prod_{\aleph_0}^{W} \pi_1(\mathcal{G}, a)$ as a direct factor. Therefore $\sum_{\aleph_0} H_1(\mathcal{G})$ is the epimorphic image of $\mathcal{H}_1(\mathcal{G}, a)$, and by [14, Lemma 2 and Theorem 3], it is cotorsion. By [12, Proposition 6.10], $H_1(\mathcal{G})$ should be torsion, which contradicts [16, page 2]. □

For both archipelago and Griffiths spaces, by the same argument as Theorems 5.13 and 5.14, we can compare $L_1(\mathbb{A}(X_i))$ and $L_1(\mathcal{G}(X_i))$ at different points. Then L_1 groups of both of archipelago and Griffiths spaces are not isomorphic at different points. The spaces are path-connected, which implies that paths do not transfer Hawaiian groups and L_n groups isomorphically. In the following theorem, we present an equivalent condition for paths to transfer Hawaiian groups isomorphically. The condition n-SLT introduced in [4], is sufficient but not necessary. Recall that a path γ from x_0 to x_1 in a space X is called a *small n-loop transfer* (abbreviated to n-SLT), if for every open neighborhood U of x_0, there exists an open neighborhood V of x_1, such that for every n-loop $\beta : (\mathbb{I}^n, \dot{\mathbb{I}}^n) \to (V, x_1)$, there is an n-loop $\alpha : (\mathbb{I}^n, \dot{\mathbb{I}}^n) \to (U, x_0)$ being homotopic to $\gamma_{\#}^{-1}(\beta)$, where $\gamma_{\#}$ is the base point change isomorphism from $\pi_n(X, x_0)$ to $\pi_n(X, x_1)$ induced by γ (see [21, p. 381]).

Theorem 5.15. *Let X have local nested bases at x_0 and x_1. A path γ from x_0 to x_1 in X, induces the isomorphism $\Gamma_\gamma : \mathcal{H}_n(X, x_0) \to \mathcal{H}_n(X, x_1)$ if and only if γ and γ^{-1} are n-SLT paths preserving null-convergent homotopies of n-loops.*

Proof. First assume that Γ_γ is an isomorphism. We show that γ and the inverse path γ^{-1} are n-SLT paths. Let $\{U_m\}$ and $\{V_m\}$ be local nested bases at x_0 and x_1, respectively. Assume that U is an open neighborhood of x_0 such that for each $m \in \mathbb{N}$, there is an

n-loop β_m in V_m at x_1 without any homotopic n-loop to $\gamma_{\#}^{-1}(\beta_m)$ in U. Define $\beta : (\mathbb{HE}^n, \theta) \to (X, x_1)$ by $\beta|_{\mathbb{S}^n_m} = \beta_m$. From [2, Lemma 2.2], β is continuous. Since Γ_γ is an epimorphism, there is $\alpha : (\mathbb{HE}^n, \theta) \to (X, x_0)$ such that $\Gamma_\gamma([\alpha]) = [\beta]$. Then $\beta_k \simeq \gamma_{\#}(\alpha|_{\mathbb{S}^n_k})$ or equivalently $\gamma_{\#}^{-1}(\beta|_{\mathbb{S}^n_k}) \simeq \alpha_k = \alpha|_{\mathbb{S}^n_k}$. Since α is continuous and U is open, there is $K \in \mathbb{N}$ such that if $k \geq K$, then $im(\alpha|_{\mathbb{S}^n_k}) \subseteq U$. This contradicts the definition of β. Therefore, γ is an n-SLT path. Similarly since Γ_γ^{-1} is an epimorphism, γ^{-1} is an n-SLT path. Therefore, for each open neighborhood U of x_0, there is an open neighborhood V of x_1 such that for any two n-loops α and α' in U, there are n-loops β and β' in V homotopic to $\gamma_{\#}(\alpha)$ and $\gamma_{\#}(\alpha')$, respectively. Let $\{H_k\}$ be a null-convergent sequence of homotopies of n-loops. That is, there is an increasing sequence of natural numbers, namely $\{K_m\}$, such that for $K_m \leq k < K_{m+1}$, $im(H_k) \subseteq U_m$. Since γ^{-1} is an n-SLT path, there are n-loops β_k and β'_k with $im(\beta_k), im(\beta'_k) \subseteq V_m$ such that $\beta_k \simeq \gamma_{\#}(H_k(-,0))$ and $\beta'_k \simeq \gamma_{\#}(H_k(-,1))$ whenever $K_m \leq k < K_{m+1}$. Define $f, f' : (\mathbb{HE}^n, \theta) \to (X, x_0)$ and $f, g : (\mathbb{HE}^n, \theta) \to (X, x_1)$ by $f|_{\mathbb{S}^n_k} = H_k(-,0)$, $f'|_{\mathbb{S}^n_k} = H_k(-,1)$, $g|_{\mathbb{S}^n_k} = \beta_k$, and $g'|_{\mathbb{S}^n_k} = \beta'_k$. Now since $\{H_k\}$ is null-convergent, we have $[f] = [f']$ by [2, Lemma 2.2]. Also since γ induces a well-defined homomorphism, $\Gamma_\gamma([f]) = \Gamma_\gamma([f'])$. Therefore, $[g] = [g']$, or equivalently there is $G : g \simeq g'$. Since $G : \mathbb{HE}^n \times \mathbb{I} \to X$ is continuous and $\mathbb{I}$ is compact, $G|_{\mathbb{S}^n_k \times \mathbb{I}}$ is null-convergent. Then $\{\beta_k\}$ and $\{\beta'_k\}$ are homotopic by a null-convergent sequence of homotopies. Similarly since γ^{-1} induces the well-defined homomorphism Γ_γ^{-1}, one can show that γ^{-1} preserves null-convergent homotopies of n-loops.

The proof of the converse statement is the same as [17, Theorem 5.5]. □

6. The first Hawaiian homology of harmonic archipelago

In this section, we obtain the structure of the first Hawaiian homology of the Harmonic archipelago spaces, up to isomorphism. In fact, the group is isomorphic to its first singular homology group. First, we prove that it is locally free; that is, all of its finitely generated subgroups are free.

Theorem 6.1. *Let $\mathbb{A} = \mathbb{A}(\{X_i\})$ be the archipelago space on $\{X_i\}_{i \in \mathbb{N}}$, where each X_i is locally strongly contractible and let x_* be the common point. If $\pi_1(X_i)$ is free for all $i \in I$ except finitely many indices i, then $\mathcal{H}_1(\mathbb{A}, x_*)$ is locally free.*

Proof. Since X_i is locally strongly contractible and first countable at x_i, for $i \in \mathbb{N}$, there exists a nested local basis $\{V^i_j\}_{j \in \mathbb{N}}$ at x_i such that the inclusion mapping $V^i_j \hookrightarrow V^i_{j-1}$ is null homotopic to x_i in V^i_{j-1}. Let $\{U_m\}_{m \in \mathbb{N}}$ be the local basis at x_* obtained by $U_m = (\bigvee_{i<m} V^i_m) \vee \widetilde{\bigvee}_{i \geq m} X_i$. Therefore, the inclusion map $U_{m+1} \hookrightarrow U_m$ is homotopic to the map retracting each point of X_i onto x_i for $i < m$ and identity for the others. The topology of archipelago spaces implies that the natural basis at the common point can be retracted onto the neighborhoods $\{U_m\}$. Consider M as the minimum number such that $\pi_1(X_i)$ is free for all $i \geq M$. Let $G = \langle \{[f^1], [f^2], \ldots, [f^m]\} \rangle$ be a subgroup of $\mathcal{H}_1(\mathbb{A}, x_*)$. Also let w be a word such that $w([f^1], [f^2], \ldots, [f^l])$ is the trivial element. Hence there exists a homotopy mapping $H : w(f^1, \ldots, f^l) \simeq C_{x_*}$. Since H is continuous, for each $m \in \mathbb{N}$, there exists $K_m \in \mathbb{N}$ such that $im(H|_{\mathbb{S}^1_k \times \mathbb{I}})$ is a subset of U_{m+1} if $k \geq K_m$. Then, $im(w(f|^1_{\mathbb{S}^1_k}, f|^2_{\mathbb{S}^1_k}, \ldots, f|^l_{\mathbb{S}^1_k})) \subseteq U_m$ for $k \geq K_m$. Thus $r(w(f|^1_{\mathbb{S}^1_k}, f|^2_{\mathbb{S}^1_k}, \ldots, f|^l_{\mathbb{S}^1_k}))$ induces a word of 1-loops at x_* in $\widetilde{\bigvee}_{i \geq m} X_i$ for $k \geq K_m$, where $r : \widetilde{\bigvee}_{\aleph_0} X_i \to \widetilde{\bigvee}_{i \geq m} X_i$ is the natural retraction mapping each X_i on to x_* if $i \leq m$ and being the identity on the other indices. Also, $r \circ H|_{\mathbb{S}^1_k \times \mathbb{I}} : r(w(f|^1_{\mathbb{S}^1_k}, f|^2_{\mathbb{S}^1_k}, \ldots, f|^l_{\mathbb{S}^1_k})) \simeq c_{x_*}$ in $\widetilde{\bigvee}_{i \geq m} X_i$ for $k \geq K_m$. Since $\pi_1(\widetilde{\bigvee}_{i \geq m} X_i)$ is locally free, all 1-loops $f^j_{\mathbb{S}^1_k}$ are null-homotopic in $\widetilde{\bigvee}_{i \geq m} X_i$ for $1 \leq j \leq l$.

Now by using [2, Lemma 2.2], one can obtain a continuous homotopy to make f^j null-homotopic. It implies that if a finite reduced word is trivial, then all of its generating elements are trivial, or equivalently the group $\mathcal{H}_1(\mathbb{A}, x_*)$ is locally free. □

In the following theorem, assume that for each i, X_i satisfies the assumption $2 \leq card(\pi_1(X_i, x_i)) \leq c$. Then we present the structure of the first Hawaiian homology group of some archipelago spaces by using the fact that this group is cotorsion.

Theorem 6.2. *Let $\mathbb{A} = \mathbb{A}(\{X_i\})$ be the archipelago space on $\{X_i\}_{i\in\mathbb{N}}$, where each X_i is locally strongly contractible, and let x_* be the join point. Also let $\pi_1(X_i)$ be free for all $i \in I$ except a finite number. Then*

$$\mathbb{H}_1(\mathbb{A}, x_*) \cong \sum_c \mathbb{Q} \oplus \prod_{p\in P} \widehat{\sum_c \mathbb{J}_p}^p \cong \frac{\prod_{\aleph_0} \mathbb{Z}}{\sum_{\aleph_0} \mathbb{Z}}.$$

Proof. Since all 1-loops at x_* are small, by Lemma 5.2, $\mathcal{H}_1(\mathbb{A}, x_*)$ is Higman-complete. Then by [14, Lemma 2], $Ab(\mathcal{H}_1(\mathbb{A}, x_*))$ is Higman-complete. Thus by [14, Theorem 3], $Ab(\mathcal{H}_1(\mathbb{A}, x_*))$ is cotorsion. Moreover, by Theorem 6.1, $\mathcal{H}_1(\mathbb{A}, x_*)$ is locally free. It was shown that the abelianization of locally free group is torsion-free [14, Lemma 12]. Therefore, the group $Ab(\mathcal{H}_1(\mathbb{A}, x_*))$ is a torsion-free cotorsion group, and then it is algebraically compact. Also, it is torsion-free algebraically compact, and then it is isomorphic to $\sum_{m_0} \mathbb{Q} \oplus \prod_{p\in P} \widehat{\sum_{m_p} \mathbb{J}_p}^p$ for some cardinalities m_0 and m_p $(p \in P)$ [11, pp. 105,169]. Hence, it remains to obtain the cardinalities m_0 and m_p $(p \in P)$. Moreover, since X_i is locally strongly contractible, it is locally path-connected, and then the archipelago space $\mathbb{A}(X_i)$ is locally path-connected. Therefore by Theorem 4.1,

$$\mathbb{H}_1(\mathbb{A}, x_*) \cong Ab(\mathcal{H}_1(\mathbb{A}, x_*)) \cong \sum_{m_0} \mathbb{Q} \oplus \prod_{p\in P} \widehat{\sum_{m_p} \mathbb{J}_p}^p.$$

First, we show that $card(\mathbb{H}_1(\mathbb{A}, x_*)) \leq c$. Thus $m_0, m_p \leq c$, where c is the continuum cardinality and p is a prime. Let $X = \widetilde{\bigvee}_{i\in I} X_i$. One can generalize the proof of [17, Theorem 2.8] for the homomorphism $i_* : \mathcal{H}_1(X, x_*)) \to \mathcal{H}_1(\mathbb{A}, x_*)$, induced by the inclusion, to be an epimorphism. Therefore, we have $card(\mathcal{H}_1(X, x_*)) \geq card(\mathcal{H}_1(\mathbb{A}, x_*))$. By [2, Theorem 2.9], $card(\mathcal{H}_1(X, x_*)) \leq \prod_{i\in\mathbb{N}} card(\pi_1(X_i, x_i)) \leq c^{\aleph_0} = c$. Thus $card(\mathcal{H}_1(\mathbb{A}, x_*)) \leq c$. Moreover, for an arbitrary pointed space (X, x_0), the group $\pi_1(X, x_0)$ can be considered as a subgroup of $\mathcal{H}_1(X, x_0)$. It implies that $card(\mathcal{H}_1(X, x_0)) \geq card(\pi_1(X, x_0))$. We know that $card(\pi_1(\mathbb{A}, x_*)) = c$ and then $card(\mathcal{H}_1(\mathbb{A}, x_*)) \geq c$. Therefore $card(\mathcal{H}_1(\mathbb{A}, x_*)) = c$, and then we have $card(\mathbb{H}_1(\mathbb{A}, x_*)) \leq c$, by Theorem 4.1. Thus $m_0, m_p \leq c$. Now we show that $m_0, m_p \geq c$ by constructing an epimorphism.

Since all 1-loops at x_* are small, by [4, Corollary 4.3], there is an epimorphism $\varphi : \mathcal{H}_1(\mathbb{A}, x_*) \to \prod_{\aleph_0} \pi_1(\mathbb{A}, x_*)$. Also there is an isomorphism $\pi_1(\mathbb{A}, \theta) \to \frac{\times^{\sigma}_{\aleph_0} G}{*_{\aleph_0} G}$, where G is either $\mathbb{Z}$ or $\mathbb{Z}_2$ by [7, Theorem A]. Thus there is an epimorphism $\mathcal{H}_1(\mathbb{H}\mathbb{A}, x_*) \to \prod_{\aleph_0} \frac{\times^{\sigma}_{\aleph_0} G}{*_{\aleph_0} G}$. This induces the epimorphism $Ab(\mathcal{H}_1(\mathbb{A}, x_*)) \to Ab(\prod_{\aleph_0} \frac{\times^{\sigma}_{\aleph_0} G}{*_{\aleph_0} G})$. Moreover, for any family of groups $\{G_i\}_{i\in I}$, there is an epimorphism $Ab(\prod_{i\in I} G_i) \to \prod_{i\in I} Ab(G_i)$. Since $\mathbb{H}_1(\mathbb{A}, x_*) \cong Ab(\mathcal{H}_1(\mathbb{A}, x_*))$, there is an epimorphism $\mathbb{H}_1(\mathbb{A}, x_*) \to \prod_{\aleph_0} Ab(\frac{\times^{\sigma}_{\aleph_0} G}{*_{\aleph_0} G})$. Also, $Ab(\frac{\times^{\sigma}_{\aleph_0} G}{*_{\aleph_0} G})$ is isomorphic to $\frac{\prod_{\aleph_0} \mathbb{Z}}{\sum_{\aleph_0} \mathbb{Z}}$ if G is either $\mathbb{Z}$ or $\mathbb{Z}_2$ by [14, Theorem 8]. Hence, we have an epimorphism $\mathbb{H}_1(\mathbb{A}, x_*) \to \prod_{\aleph_0} \frac{\prod_{\aleph_0} \mathbb{Z}}{\sum_{\aleph_0} \mathbb{Z}}$. By [11, Corollary 42.2], $\frac{\prod_{\aleph_0} \mathbb{Z}}{\sum_{\aleph_0} \mathbb{Z}}$ is algebraically compact and it is isomorphic to $\sum_c \mathbb{Q} \oplus \prod_{p\in P} \widehat{\sum_c \mathbb{J}_p}^p$ by [5]. Then there is an epimorphism $\mathbb{H}_1(\mathbb{A},x_*)\to\prod_{\aleph_0}\sum_c\mathbb{Q}\oplus\prod_{p\in P}\widehat{\sum_c\mathbb{J}_p}^p$. Equivalently there is an epimorphism $\sum_{m_0}\mathbb{Q}\oplus\prod_{p\in P}\widehat{\sum_{m_p}\mathbb{J}_p}^p\to\prod_{\aleph_0}\left(\sum_c\mathbb{Q}\oplus\prod_{p\in P}\widehat{\sum_c\mathbb{J}_p}^p\right)$. Therefore, $m_0,m_p\geq c$, and then $m_0,m_p=c$ for $p\in P$. Hence $\mathbb{H}_1(\mathbb{A},x_*)\cong\sum_c\mathbb{Q}\oplus\prod_{p\in P}\widehat{\sum_c\mathbb{J}_p}^p$, and by [11, Corollary 42.2], $\mathbb{H}_1(\mathbb{A},x_*)\cong\dfrac{\prod_{\aleph_0}\mathbb{Z}}{\sum_{\aleph_0}\mathbb{Z}}$. □

Herfort and Hojka [14] proved that the first singular homology group of any archipelago space is isomorphic to $\dfrac{\prod_{\aleph_0}\mathbb{Z}}{\sum_{\aleph_0}\mathbb{Z}}$. Now by Theorem 6.2, the first singular homology and the first Hawaiian homology groups are isomorphic for archipelago spaces. It does not hold for many classes of spaces; for instance, see Example 5.6.

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