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THE DUAL NOTION OF r-SUBMODULES OF MODULES

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Abstract. Let R be a commutative ring with identity and let M be an R-module. A proper submodule N of M is said to be an r-submodule if $am \in N$ with $(0 :_M a) = 0$ implies that $m \in N$ for each $a \in R$ and $m \in M$. The purpose of this paper is to introduce and investigate the dual notion of r-submodules of M.

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1. Introduction

Throughout this paper, R will denote a commutative ring with identity and $\mathbb{Z}$ will denote the ring of integers.

Let $Z(R)$ be the set of all zero divisors of R. A proper ideal P of R is said to be an *r-ideal* if whenever $ab \in P$ and $a \in R \setminus Z(R)$ for some $a, b \in R$, then $b \in P$ [11].

Let M be an R-module. The set of all zero divisors of R on M is $Z_R(M) = \{r \in R \mid rm = 0 \ for\ some\ nonzero\ m \in M\}$.

The authors of [10] extend the concept of r-ideals to r-modules and they investigate some properties of this class of modules. A proper submodule N of M is said to be an *r-submodule* if $am \in N$ with $(0 :_M a) = 0$ (i.e. $a \in R \setminus Z_R(M)$) implies that $m \in N$ for each $a \in R$ and $m \in M$ [10].

The authors of [2] and [3], recently defined r-Noetherian and r-Artinian modules. An R-module M is said to be an r-Noetherian module if every r-submodule of M is finitely generated [2]. They showed that every finitely generated r-Noetherian R-module satisfies the ascending chain condition on r-submodules [2, Lemma 2.1]. Also, M is said to be an r-Artinian module if the set of r-submodules of M satisfies the descending chain condition [3].

In Section 2 of this paper, we define *co*-r-submodules of an R-module M as a dual notion of r-submodules and obtain some properties of this class of modules.

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In Section 3, we define and investigate the notions of *co*-r-Noetherian and *co*-r-Artinian modules.

2. Co-r-submodules of R-modules

Let M be an R-module. The subset $W_R(M)$ of R, the set of all cozero divisors of R (that is the dual notion of $Z_R(M)$), is defined by $\{r \in R \mid rM \neq M\}$ [14].

Definition 2.1. We say that a non-zero submodule N of an R-module M is a *co-r-submodule of* M if for $a \in R$ and submodule K of M, whenever $aN \subseteq K$ and $a \in R \setminus W_R(M)$, then $N \subseteq K$. This can be regarded as a dual notion of r-submodules.

Example 2.2. Let V be a vector space over a field F. Then every non-zero subspace N of V is a *co*-r-submodule.

A non-zero submodule S of an R-module M is said to be *second* if for each $a \in R$, the homomorphism $S \xrightarrow{a} S$ is either surjective or zero [15].

Remark 2.3. A non-zero submodule N of an R-module M is a *co*-r-submodule means that $W(N) \subseteq W(M)$. Thus if N is a *co*-r-submodule of M, then $Ann_R(N) \subseteq W(M)$. In particular, if N is a second submodule of M, then N is a *co*-r-submodule of M if and only if $Ann_R(N) \subseteq W(M)$.

An R-module M is said to be a *multiplication module* (resp. *comultiplication module*) if for every submodule N of M there exists an ideal I of R such that $N = IM$ [7] (resp. $N = (0 :_M I)$ [4]).

Theorem 2.4. (a) *Let M be a multiplication R-module. Then every non-zero submodule N of M is a co-r-submodule.*

(b) *Let M be a comultiplication R-module. Then every proper submodule N of M is an r-submodule.*

Proof. (a) Let $aN \subseteq K$ with $aM = M$ for $a \in R$ and a submodule K of M. As M is a multiplication module, there is an ideal I of R such that $N = IM$. Thus we have $N = IM = IaM = aIM = aN \subseteq K$.

(b) Let $am \in N$ with $a \in R \setminus Z_R(M)$ for $m \in M$. Since M is a comultiplication R-module, there exists an ideal I of R such that $N = (0 :_M I)$. Therefore, $m \in (N :_M a) = (0 :_M aI) = ((0 :_M a) :_M I) = (0 :_M I) = N$. □

The following example shows that the concepts of r-submodules and *co*-r-submodules are different, in general.

Example 2.5. (a) Every non-zero proper submodule of the $\mathbb{Z}$-module $\mathbb{Z}$ is not an r-submodule but it is a *co*-r-submodule.

(b) Let p be a prime number. Every non-zero proper submodule of the $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$ is an r-submodule but it is not a *co*-r-submodule.

Proposition 2.6. *Let M be an R-module. Then we have the following.*

(a) *M is a co-r-submodule of M.*

(b) *The sum of an arbitrary non-empty set of co-r-submodules of M is a co-r-submodule of M.*

Proof. (a) This is clear.

(b) Let N_i be a *co*-r-submodule of M for every $i \in I$. Assume that $a \sum_{i \in I} N_i \subseteq K$ with $aM = M$ for $a \in R$ and submodule K of M. This implies that $aN_i \subseteq K$ for every $i \in I$. As N_i is a *co*-r-submodule of M, we conclude that $N_i \subseteq K$ for every $i \in I$. Hence $\sum_{i \in I} N_i \subseteq K$, as needed. □

The following example shows that the intersection of two *co*-r-submodules need not be a *co*-r-submodule, in general.

Example 2.7. Consider the $\mathbb{Z}$-module $\mathbb{Z}_n$. Then as $\mathbb{Z}_n$ is a multiplication $\mathbb{Z}$-module, $\bar{u}\mathbb{Z}_n$ and $\bar{v}\mathbb{Z}_n$ are *co*-r-submodules by Theorem 2.4 (a). But if $gcd(u, v) = 1$, then $\bar{u}\mathbb{Z}_n \cap \bar{v}\mathbb{Z}_n = 0$ is not a *co*-r-submodule of $\mathbb{Z}_n$.

If N is a second submodule of an R-module M, then $Ann_R(N)$ is a prime ideal of R by [15]. However, the following example shows that the similar result is not always correct for a *co*-r-submodule.

Example 2.8. Consider the $\mathbb{Z}$-module $\mathbb{Z}_n$. Then for each positive integer k, $\bar{k}\mathbb{Z}_n$ is a *co*-r-submodule of $\mathbb{Z}_n$ but $Ann_{\mathbb{Z}}(\bar{k}\mathbb{Z}_n) = t\mathbb{Z}$, where $n = (t)(k)$ is not an r-ideal of $\mathbb{Z}$.

Proposition 2.9. *Let N be a co-r-submodule of an R-module M and S be a non-empty subset of R with $S \not\subseteq Ann_R(N)$. Then SN is a co-r-submodule of M. In particular, SM is always a co-r-submodule if $S \not\subseteq Ann_R(M)$.*

Proof. Let $aSN \subseteq K$ with $aM = M$ for $a \in R$ and a submodule K of M. Then we have $asN \subseteq K$ for every $s \in S$. Thus $aN \subseteq (K :_M s)$. Since N is a *co*-r-submodule, $sN \subseteq K$ for every $s \in S$ and this yields $SN \subseteq K$, as needed. Now the rest is clear. □

Corollary 2.10. *Let M be an R-module. If $a \in R \setminus Ann_R(M)$, then aM is a co-r-submodule of M. In particular, if M is the only co-r-submodule of M, then M is a second R-module.*

Proposition 2.11. *For a non-zero submodule N of an R-module M the following are equivalent:*

(a) *N is a co-r-submodule of M;*
(b) *$aN = N$ for each $a \in R \setminus W_R(M)$;*
(c) *$(N :_M a) = N + (0 :_M a)$ for each $a \in R \setminus W_R(M)$.*

Proof. $(a) \Rightarrow (b)$ Let $a \in R \setminus W_R(M)$. Then by part (a), $aN \subseteq aN$ implies that $N \subseteq aN$. Thus $aN = N$ because the reverse inclusion is clear.

$(b) \Rightarrow (a)$ This is clear.

$(b) \Rightarrow (c)$ For every $a \in R$, the inclusion $N + (0 :_M a) \subseteq (N :_M a)$ always holds. Let $a \in R$ with $aM = M$ and $x \in (N :_M a)$. Then $ax \in N = aN$. Thus $ax = an$ for some $n \in N$. Therefore, $x = x - n + n \in N + (0 :_M a)$. This implies that $(N :_M a) \subseteq N + (0 :_M a)$.

$(c) \Rightarrow (b)$ Clearly, $aN \subseteq N$ for every $a \in R$. Let $a \in R \setminus W_R(M)$ and $x \in N$. Then $aM = M$ implies that $x = am$ for some $m \in M$. Thus $m \in (N :_M a) = N + (0 :_M a)$. It follows that $x = am \in aN$, as needed. □

A submodule N of an R-module M is said to be *copure* if $(N :_M I) = N + (0 :_M I)$ for every ideal I of R [5]. By Proposition 2.11, every copure submodule is a *co-r*-submodule. However, the following example shows that the converse is not true in general.

Example 2.12. Consider the $\mathbb{Z}$-module $\mathbb{Z}_{16}$. Then $\bar{2}\mathbb{Z}_{16}$ is a *co-r*-submodule of $\mathbb{Z}_{16}$. But one can see that $\bar{2}\mathbb{Z}_{16}$ is not a copure submodule of $\mathbb{Z}_{16}$.

Lemma 2.13. *Let N be a submodule of an R-module M and $a \in R$. Then $(N :_M a) = N + (0 :_M a)$ if and only if $aN = N \cap aM$.*

Proof. This follows from the proof of [5, Theorem 2.12 (a)]. □

Recall that an R-module M is said to be *Hopfian* (resp. *co-Hopfian*) if every surjective (resp. injective) endomorphism f of M is an isomorphism.

A submodule N of an R-module M is said to be *idempotent* if $N = (N :_R M)^2M$ [6]. M is said to be *fully idempotent* if every submodule of M is idempotent [6].

A submodule N of an R-module M is said to be *coidempotent* if $N = (0 :_M Ann^2_R(N))$ [6]. Also, an R-module M is said to be *fully coidempotent* if every submodule of M is coidempotent [6].

Remark 2.14. If M is an R-module such that $Z_R(M) = W_R(M)$, then a proper non-zero submodule N of M is a *co*-r-submodule of M if and only if N is an r-submodule of M by Lemma 2.13, Proposition 2.11, and [10, Proposition 4]. For example, if M is a Hopfian and co-Hopfian R-module (in particular, M has finite length or M is a fully idempotent [6, Proposition 2.7] or M is fully coidempotent [6, Proposition 3.5 and Theorem 3.9]), then $Z_R(M) = W_R(M)$. It should be note that every multiplication R-module is Hopfian and every comultiplication R-module is co-Hopfian.

Recall that a submodule N of an R-module M is *small* if for any submodule X of M, $X + N = M$ implies that $X = M$.

Proposition 2.15. *Let N and K be two submodules of an R-module M such that $0 \neq N \subseteq K \subseteq M$. Then we have the following.*

(a) *If N is a co-r-submodule of M and K/N is a co-r-submodule of M/N, then K is a co-r-submodule of M.*

(b) *If N is a small submodule of K and K/N is a co-r-submodule of M/N, then K is a co-r-submodule of M.*

Proof. (a) Let $a \in R \setminus W_R(M)$. Then $a \in R \setminus W_R(M/N)$. Thus by Proposition 2.11, $aN = N$ and $a(K/N) = K/N$. Hence $aN = N$ and $aK + N = K$. Therefore, $aK = a(N + K) = aK + N = K$ as needed.

(b) Let $a \in R \setminus W_R(M)$. Then $a \in R \setminus W_R(M/N)$. Thus by Proposition 2.11, $a(K/N) = K/N$. It follows that $aK + N = K$. Therefore, $aK = K$ since N is a small submodule of K. So K is a *co*-r-submodule of M. □

Theorem 2.16. *Let $S_1, S_2, \dots, S_n$ be second submodules of an R-module M such that $Ann_R(S_i)$ s are not comparable. If $\sum_{i=1}^n S_i$ is a co-r-submodule of M, then S_i is a co-r-submodule of M for each $i \in \{1, 2, \dots, n\}$.*

Proof. Suppose that $\sum_{i=1}^n S_i$ is a *co*-r-submodule of M. Let $aS_j \subseteq K$ with $aM = M$ for $a \in R$ and submodule K of M. Since $Ann_R(S_i)$ s are not comparable, we have $b \in \bigcap_{i=1, i\neq j}^n Ann_R(S_i) \setminus Ann_R(S_j)$ for some $b \in R$. Then we have $ab \sum_{i=1}^n S_i = abS_j \subseteq K$ and so $a \sum_{i=1}^n S_i \subseteq (K :_M b)$. As $\sum_{i=1}^n S_i$ is a *co*-r-submodule of M, we have $\sum_{i=1}^n S_i \subseteq (K :_M b)$. This implies that $S_j = bS_j \subseteq K$ because S_j is a second submodule of M and $b \notin Ann_(S_j)$. Hence, S_j is a *co*-r-submodule of M. □

Definition 2.17. We say that a *co*-r-submodule N of an R-module M is a *minimal co-r-submodule* of M if there does not exist a *co*-r-submodule T of M such that $T \subset N$.

Proposition 2.18. *If N is a minimal co-r-submodule of an R-module M, then N is a second submodule.*

Proof. Let $aN \subseteq K$ and $N \not\subseteq K$, we show that $a \in Ann_R(N)$. Assume that $a \notin Ann_R(N)$. Then aN is a *co*-r-submodule by Proposition 2.9. Since N is a minimal *co*-r-submodule, we conclude that $aN = N \subseteq K$, a contradiction. Thus, we have $a \in Ann_R(N)$, as needed. □

Theorem 2.19. *Let M be an R-module. Then every non-zero submodule of M is a co-r-submodule if and only if for every submodule N of M, $(N :_M a) = N$ for each $a \in R \setminus W_R(M)$.*

Proof. Suppose that every non-zero submodule of M is a *co*-r-submodule. Let N be a submodule and $a \in R \setminus W_R(M)$. Assume that $N = 0$. If $(0 :_M a) \neq 0$, then $(0 :_M a)$ is a *co*-r-submodule of M. Thus $a(0 :_M a) = 0$ and $aM = M$ implies that $(0 :_M a) = 0$, which is a contradiction. So, $(0 :_M a) = 0$. Now assume that N is a non-zero submodule of M. Then $0 \neq N \subseteq (N :_M a)$ and so $(N :_M a)$ is a *co*-r-submodule of M. Since $a(N :_M a) \subseteq N$, we get that $(N :_M a) = N$. Conversely, suppose that $(N :_M a) = N$ for every submodule N of M and every $a \in R \setminus W_R(M)$. Let N be a non-zero submodule of M and $a \in R \setminus W_R(M)$. Then we have $(N :_M a) = N + (0 :_M a)$, and so by Proposition 2.11, N is a *co*-r-submodule of M. □

Let R_i be a commutative ring with identity, M_i be an R_i-module for each $i = 1, 2, \ldots, n$, and $n \in \mathbb{N}$. Assume that $M = M_1 \times M_2 \times \cdots \times M_n$ and $R = R_1 \times R_2 \times \cdots \times R_n$. Then M is an R-module with componentwise addition and scalar multiplication. Also, each submodule N of M is of the form $N = N_1 \times N_2 \times \cdots \times N_n$, where N_i is a submodule of M_i.

Lemma 2.20. *Let $R = R_1 \times R_2$ and $M = M_1 \times M_2$, where M_1 is an R_1-module and M_2 is an R_2-module. Suppose that $N = N_1 \times N_2$ is a submodule of M. Then the following are equivalent:*

(a) *N is a co-r-submodule of M;*

(b) *$N_1 = 0$ and N_2 is a co-r-submodule of M_2 or N_1 is a co-r-submodule of M_1 and $N_2 = 0$ or N_1, N_2 are co-r-submodules of M_1 and M_2, respectively.*

Proof. $(a) \Rightarrow (b)$ First note that

$$W_R(N) = W_{R_1 \times R_2}(N_1 \times N_2) = (W_{R_1}(N_1) \times R_2) \cup (R_1 \times W_{R_2}(N_2)).$$

Suppose that N is a *co-r*-submodule of M and assume that $N_1 = 0$. Since N is a non-zero submodule of M, $N_2 \neq 0$. Then $R_1 \times W_{R_2}(N_2) = W_R(N) \subseteq W_R(M)$ and so $W_{R_2}(N_2) \subseteq W_{R_2}(M_2)$. This implies that N_2 is a *co-r*-submodule of M_2. In other cases, a similar argument shows that (a) implies (b).

$(b) \Rightarrow (a)$ Assume that N_1, N_2 are *co-r*-submodules of M_1 and M_2, respectively. Then $W_{R_1}(N_1) \subseteq W_{R_1}(M_1)$ and $W_{R_2}(N_2) \subseteq W_{R_2}(M_2)$. This implies that

$$W_R(N) = W_{R_1 \times R_2}(N_1 \times N_2) = (W_{R_1}(N_1) \times R_2) \cup (R_1 \times W_{R_2}(N_2))$$
$$\subseteq (W_{R_1}(M_1) \times R_2) \cup (R_1 \times W_{R_2}(M_2)) = W_R(M),$$

i.e. N is a *co-r*-submodule of M. In other cases, one can similarly prove that N is a *co-r*-submodule of M. □

Theorem 2.21. *Suppose that $R = R_1 \times R_2 \times \cdots \times R_n$ and $M = M_1 \times M_2 \times ... \times M_n$, where M_i is an R_i-module for $n \geq 1$. Let $N = N_1 \times N_2 \times \cdots \times N_n$ be a submodule of M. Then the following are equivalent:*

(a) *N is a co-r-submodule of M;*

(b) *$N_i = 0$ for $i \in \{t_1, t_2, \dots, t_k : k < n\} \subseteq \{1, 2, 3, \dots, n\}$ and N_i is a co-r-submodule of M_i for $i \in \{1, 2, ..., n\} \setminus \{t_1, t_2, \dots, t_k\}$.*

Proof. To prove the claim, we use induction on n. If $n = 1$, then (a) and (b) are equivalent. If $n = 2$, by Lemma 2.20, (a) and (b) are equal. Assume that $n \geq 3$ and the claim is valid when $K = M_1 \times M_2 \times \cdots \times M_{n-1}$. We prove that the claim is true when $M = K \times M_n$. Then by Lemma 2.20 we get the result that N is a *co-r*-submodule if and only if $N = 0 \times N_n$ for some *co-r*-submodule N_n of M_n or $N = L \times 0$ for some *co-r*-submodule L of K or $N = L \times N_n$ for some *co-r*-submodule L of K and some *co-r*-submodule N_n of M_n. By induction hypothesis, the result is valid in three cases. □

Theorem 2.22. *For a non-zero submodule N of an R-module M we have the following.*

(a) *N is a co-r-submodule of M if and only if whenever I is an ideal of R such that $I \cap (R \setminus W_R(M)) \neq \emptyset$ and K is a submodule of M with $IN \subseteq K$, then $N \subseteq K$.*

(b) *If $Ann_R(N) \subseteq W_R(M)$ and N is not a co-r-submodule of M, then there exist an ideal I of R and a submodule K of M such that $I \cap (R \setminus W_R(M)) \neq \emptyset$, $K \subset N$, $Ann_R(N) \subset I$, and $IN \subseteq K$.*

Proof. (a) Suppose that N is a *co-r*-submodule, $IN \subseteq K$ for some ideal I of R with $I \cap (R \setminus W_R(M)) \neq \emptyset$, and submodule K of M. Then there exists $a \in I$

such that $aM = M$. Since N is a *co*-r-submodule, $N \subseteq K$. For the converse, let $aN \subseteq K$, $aM = M$ for $a \in R$, and submodule K of M. We take $I = aR$. Note that $I \cap (R \setminus W_R(M)) \neq \emptyset$. Then by assumption we have $N \subseteq K$, and so N is a *co*-r-submodule of M.

(b) Since N is not a *co*-r-submodule of M, there exist $a \in R$ and submodule K of M such that $aN \subseteq K$ with $aM = M$ and $N \not\subseteq K$. We take $I = (K :_R N)$. Note that $a \in I$ and $a \notin Ann_R(N)$ since $aM = M$. Thus, $Ann_R(N) \subset I$. Now we take $K = IN$. Since $N \not\subseteq K$, we have $K \subset N$. Hence, we get $K \subset N$, $Ann_R(N) \subset I$, and $IN = (IN :_M I) \subseteq K$. □

Theorem 2.23. *Let K_1, K_2, K be submodules of an R-module M and I be an ideal of R with $I \cap (R \setminus W_R(M)) \neq \emptyset$. Then the following hold.*

(a) *If K_1, K_2 are co-r-submodules of M with $(K_1 :_M I) = (K_2 :_M I)$, then $K_1 = K_2$.*

(b) *If $(K :_M I)$ is a co-r-submodule, then $(K :_M I) = K$. In particular, K is a co-r-submodule.*

Proof. (a) Since $IK_1 \subseteq K_2$ and K_1 is a *co*-r-submodule, we have $K_1 \subseteq K_2$ by Theorem 2.22 (a). Similarly, we have $K_2 \subseteq K_1$, and so $K_1 = K_2$.

(b) As $(K :_M I)$ is a *co*-r-submodule and $I(K :_M I) \subseteq K$, we have $(K :_M I) \subseteq K$ by Theorem 2.22 (a). Hence, $(K :_M I) = K$ since the reverse inclusion is clear. □

A proper submodule N of an R-module M is called an *n-submodule* if for $a \in R$, $m \in M$, $am \in N$ with $a \notin \sqrt{Ann_R(M)}$, then $m \in N$ [13].

A non-zero submodule N of an R-module M is a *co-n-submodule of M* if for $a \in R$ and submodule K of M, whenever $aN \subseteq K$ and $a \notin \sqrt{Ann_R(M)}$, then $N \subseteq K$ [8].

Proposition 2.24. *Let N be a co-n-submodule of an R-module M. Then N is a co-r-submodule of M.*

Proof. As M is a *co*-n-submodule of M, $N \neq 0$. Let $aN \subseteq K$ with $aM = M$ for $a \in R$ and a submodule K of M. If $a \in \sqrt{Ann_R(M)}$, then there exists a positive integer t such that $a^tM = 0$ and $a^{t-1}M \neq 0$. Now, $aM = M$ implies that $0 = a^tM = a^{t-1}M$, which is a contradiction. Thus $a \notin \sqrt{Ann_R(M)}$. Now, as M is a *co*-n-submodule of M, we have $N \subseteq K$ as required. □

The following example shows that the converse of Proposition 2.24 is not true in general.

Example 2.25. The submodule $\bar{3}\mathbb{Z}_6$ of the $\mathbb{Z}$-module $\mathbb{Z}_6$ is a *co-r*-submodule but it is not a *co-n*-submodule.

Let S be a multiplicatively closed subset of R and P be a submodule of an R-module M with $\sqrt{(P :_R M)} \cap S = \emptyset$. Then P is said to be an *S-primary submodule* if there exists a fixed $s \in S$ and whenever $am \in P$, then either $sa \in \sqrt{(P :_R M)}$ or $sm \in P$ for each $a \in R$ and $m \in M$ [9].

Let S be a multiplicatively closed subset of R and N be a submodule of an R-module M with $\sqrt{Ann_R(N)} \cap S = \emptyset$. Then N is said to be an *S-secondary submodule* if there exists a fixed $t \in S$ and whenever $aN \subseteq K$, then either $ta \in \sqrt{Ann_R(N)}$ or $tN \subseteq K$ for each $a \in R$ and a submodule K of M [9].

Remark 2.26. Let S be a multiplicatively closed subset of R and N be a submodule of a finitely generated R-module M. Then we have the following.

(a) If M is a multiplication R-module with $\sqrt{Ann_R(M)} \cap S = \emptyset$ and each proper submodule of M is S-primary, then $Z_R(M) = \sqrt{Ann_R(M)}$ [9, Theorem 4.7]. Thus N is an n-submodule of M if and only if N is an r-submodule of M.

(b) If M is a comultiplication R-module with $\sqrt{Ann_R(M)} \cap S = \emptyset$ and each non-zero submodule of M is S-secondary, then $W_R(M) = \sqrt{Ann_R(M)}$ [9, Theorem 4.5]. Thus N is a *co-n*-submodule of M if and only if N is a *co-r*-submodule of M.

Lemma 2.27. [9, Lemma 4.2] *Let M be an R-module, S a multiplicatively closed subset of R, and N be a finitely generated submodule of M. If $S^{-1}N \subseteq S^{-1}K$ for a submodule K of M, then there exists an $s \in S$ such that $sN \subseteq K$. In particular, if $S = R \setminus W_R(M)$ and N is a co-r-submodule of M, then $N \subseteq K$.*

Theorem 2.28. *Let N be a finitely generated submodule of a finitely generated R-module M and $S = R \setminus W_R(M)$. Then the following are equivalent:*

(a) *N is a co-r-submodule of M;*

(b) *$S^{-1}N$ is a co-r-submodule of $S^{-1}M$.*

Proof. $(a) \Rightarrow (b)$ If $S^{-1}N = 0$, then Lemma 2.27 implies that $N = 0$, which is a contradiction. Thus $S^{-1}N \neq 0$. Now let $r/t \in S^{-1}R \setminus W_{S^{-1}R}(S^{-1}M)$. Then $S^{-1}(rM) = (r/t)(S^{-1}M) = S^{-1}M$. By using Lemma 2.27, $rM = M$ and so $r \in R \setminus W_R(M)$. Now as N is a *co-r*-submodule of M, we have $rN = N$ by Proposition 2.11. This implies that $(r/s)(S^{-1}N) = S^{-1}N$, as needed.

$(b) \Rightarrow (a)$ Let $aN \subseteq K$ for some $a \in R \setminus W_R(M)$ and a submodule K of M. Then $(a/1)(S^{-1}N) \subseteq S^{-1}K$ and $a/1 \in S^{-1}R \setminus W_{S^{-1}R}(S^{-1}M)$. Thus by part (b), $S^{-1}N \subseteq S^{-1}K$. Hence by Lemma 2.27, $N \subseteq K$. Thus N is a *co*-r-submodule of M. □

3. Ascending and descending chain conditions on *co*-r-submodules

Definition 3.1. We say that an R-module M is a *co-r-Noetherian module* if the set of *co*-r-submodules of M satisfies the ascending chain condition.

Definition 3.2. We say that an R-module M is a *co-r-Artinian module* if the set of *co*-r-submodules of M satisfies the descending chain condition.

Proposition 3.3. (a) *If N is a co-r-submodule of a co-r-Noetherian (resp. co-r-Artinian) R-module M, then M/N is a co-r-Noetherian (resp. co-r-Artinian) R-module.*

(b) *Every Noetherian (resp. Artinian) R-module is a co-r-Noetherian (resp. co-r-Artinian) R-module.*

Proof. (a) This follows from Proposition 2.15 (a).

(b) These are clear. □

The following theorem provides characterizations for *co*-r-Artinian R-modules when M is a Noetherian R-module.

Theorem 3.4. *Let M be a Noetherian R-module and $S = R \setminus W_R(M)$. The following statements are equivalent:*

(a) *M is a co-r-Artinian R-module;*

(b) *$S^{-1}M$ is an Artinian $S^{-1}R$-module.*

Proof. This follows from Lemma 2.27 and Theorem 2.28. □

Let S be a multiplicatively closed subset of R. An R-module M is called S-finite if $sM \subseteq F$ for some finitely generated submodule F of M and some $s \in S$. The module M is called S-Noetherian if each submodule of M is S-finite [1].

Definition 3.5. Let S be a multiplicatively closed subset of R. We say that an R-module M is a *strongly S-Noetherian R-module* if for any ascending chain of submodules

$$N_1 \subseteq N_2 \subseteq \cdots \subseteq N_k \subseteq \cdots$$

of M, there exist $s \in S$ and $k \in \mathbb{N}$ such that $sN_n \subseteq N_k$ for every $n \geq k$.

Let S be a multiplicatively closed subset of R. Clearly, every strongly S-Noetherian R-module is an S-Noetherian R-module. But Example 3.7 shows that the converse is not true in general for every multiplicatively closed subset S of R..

Let S be a multiplicatively closed subset of R. An R-module M is said to be an *S-Artinian R-module* if for any descending chain of submodules

$$N_1 \supseteq N_2 \supseteq \cdots \supseteq N_k \supseteq \cdots$$

of M, there exist $s \in S$ and $k \in \mathbb{N}$ such that $sN_k \subseteq N_n$ for every $n \geq k$ [12].

Proposition 3.6. *Let S be a multiplicatively closed subset of R such that $S \cap W_R(M) = \emptyset$. Then every strongly S-Noetherian (resp. S-Artinian) R-module is a co-r-Noetherian (resp. co-r-Artinian) R-module.*

Proof. This follows from the fact that for each *co*-r-submodule N of M and $s \in S$, we have $sN = N$ by Proposition 2.11. □

The following is an example of a *co*-r-Noetherian module that is not S-Noetherian for every multiplicatively closed subset S of R.

Example 3.7. Let p be a prime number. Consider $R := \mathbb{Z}$ and $M := \mathbb{Z}_{p^\infty}$. Then M is a *co*-r-Noetherian R-module by Example 2.5 (b). Also, M is an S-Noetherian R-module. However, M is not a strongly S-Noetherian R-module for every multiplicatively closed subset S of R. It sufices to verify that M is not a strongly S-Noetherian R-module, where $S = \mathbb{Z} \setminus \{0\}$. Indeed, consider the following ascending chain of submodules of M

$$\langle 1/p + \mathbb{Z}\rangle \subseteq \langle 1/p^2 + \mathbb{Z}\rangle \subseteq \langle 1/p^3 + \mathbb{Z}\rangle \subseteq \cdots \subseteq \langle 1/p^n + \mathbb{Z}\rangle \subseteq \cdots.$$

If $s \in S$, then $s = p^m t$ for some $m \in \mathbb{N} \cup \{0\}$ and $t \in \mathbb{Z}$ with $gcd(t,p) = 1$. Now, we let $k \in \mathbb{N}$. Then, $s\langle 1/p^{m+k+1} + \mathbb{Z}\rangle \not\subseteq \langle 1/p^k + \mathbb{Z}\rangle$ and thus M is not a strongly S-Noetherian R-module.

Lemma 3.8. *Let M be a multiplication R-module with $W_R(M) \subseteq Z(R)$. If N is a non-zero submodule of M, then $(N :_R M)$ is an r-ideal of R.*

Proof. As M is a multiplication R-module, we have $N = (N :_R M)M$. Let $ab \in (N :_R M)$ with $a \notin Z(R)$ for some $a, b \in R$. Then by assumption, $aM = M$. Thus

$$bN = b(N :_R M)M = b(N :_R M)aM = ab(N :_R M)M = abN \subseteq M,$$

as needed. □

Theorem 3.9. *Let M be a multiplication R-module with $W_R(M) \subseteq Z(R)$ and R satisfy ascending chain condition on r-ideals of R. Then M is a Noetherian R-module.*

Proof. Let $N_1 \subseteq N_2 \subseteq \cdots \subseteq N_k \subseteq \cdots$ be an ascending chain of submodules of M. By Lemma 3.8, for each i, $(N_i :_R M)$ is an r-ideal of R. So

$$(N_1 :_R M) \subseteq (N_2 :_R M) \subseteq \cdots \subseteq (N_k :_R M) \subseteq \cdots$$

is an ascending chain of r-ideals of R. Since R satisfies ascending chain condition on r-ideals, there exists $t \in \mathbb{N}$ such that $(N_i :_R M) = (N_t :_R M)$ for each $i \geq t$. Therefore, $N_i = (N_i :_R M)M = (N_t :_R M)M = N_t$ for each $i \geq t$. It follows that M is a Noetherian module. □

Lemma 3.10. *Let $f : M \to \acute{M}$ be an epimorphism of R-modules. If N is a co-r-submodule of $\acute{M}$ and $Ker(f)$ is a co-r-submodule of M, then $f^{-1}(N)$ is a co-r-submodule M.*

Proof. Since $Ker(f)$ is a *co*-r-submodule of M, we have $Ker(f) \neq 0$. So $f^{-1}(N) \neq 0$. Now let $a \in R \setminus W_R(M)$ and $af^{-1}(N) \subseteq K$ for some submodule K of M. Then $aKer(f) \subseteq K$ and so by assumption, $Ker(f) \subseteq K$. Clearly $a \in R \setminus W_R(\acute{M})$. Thus $aN = aN \cap \acute{M} = aN \cap f(M) = f(f^{-1}(aN)) \subseteq f(K)$ implies that $N \subseteq f(K)$. Thus $f^{-1}(N) \subseteq K + Ker(f) = K$, as needed. □

Theorem 3.11. *Let $0 \longrightarrow M_1 \stackrel{\psi}{\longrightarrow} M_2 \stackrel{\phi}{\longrightarrow} M_3 \longrightarrow 0$ be an exact sequence of R-modules. Then we have the following.*

(a) *Assume that $W_R(M_1) \subseteq W_R(M_2)$. If M_2 is a co-r-Noetherian R-module, then so is M_1.*

(b) *Suppose that $W_R(M_2) \subseteq W_R(M_3)$. If M_3 is a co-r-Noetherian R-module and M_1 is a strongly S-Noetherian R-module where $S := R \setminus W_R(M_2)$, then M_2 is a co-r-Noetherian R-module.*

(c) *If M_2 is a co-r-Noetherian R-module and $Ker(\phi)$ is a co-r-submodule of M_2, then M_3 is a co-r-Noetherian R-module.*

Proof. (a) As $W_R(M_1) \subseteq W_R(M_2)$, we conclude that $\psi(N)$ is a *co*-r-submodule of M_2 for every *co*-r-submodule N of M_1. Hence if M_2 is a *co*-r-Noetherian module, then we can easily get M_1 is a *co*-r-Noetherian module.

(b) Let

$$N_1 \subseteq N_2 \subseteq \cdots \subseteq N_n \subseteq \cdots$$

be an ascending chain of *co*-r-submodules of M_2. Since M_1 is an S-Noetherian R-module with $S := R \setminus W_R(M_2)$, then there exist $s \in S$ and $k_1 \in \mathbb{N}$ such that

$s\psi^{-1}(N_n) \subseteq \psi^{-1}(N_{k_1})$ for each $n \geq k_1$. It follows that $sN_n \cap \psi(M_1) \subseteq N_{k_1}$. On the other hand, we have the ascending chain

$$\phi(N_1) \subseteq \phi(N_2) \subseteq \cdots \subseteq \phi(N_n) \subseteq \cdots$$

of *co*-r-submodules of M_3. As M_3 is a *co*-r-Noetherian module, there exists $k_2 \in \mathbb{N}$ such that $\phi(N_{k_2}) = \phi(N_n)$ for each $n \geq k_2$ This implies that $N_{k_2} + \psi(M_1) = N_n + \psi(M_1)$ for each $n \geq k_2$. Now put $k = max\{k_1, k_2\}$. Then we have $sN_n \cap \psi(M_1) \subseteq N_k$ and $N_k + \psi(M_1) = N_n + \psi(M_1)$ for each $n \geq k$. Now since $N_k \subseteq N_n$, we have

$$sN_n = s(N_n \cap (N_n + \psi(M_1))) = s(N_n \cap (N_k + \psi(M_1))) =$$

$$s((N_n \cap N_k) + (N_n \cap \psi(M_1))) \subseteq N_k + (sN_n \cap \psi(M_1)) \subseteq N_k.$$

Hence $N_n \subseteq N_k$ since N_n is a *co*-r-submodule of M_2. Thus M_2 is a *co*-r-Noetherian R-module.

(c) This follows from Lemma 3.10. □

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