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Research Article

Left-ray right-ray hybrid topologies on the real line

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Abstract

Given a non-empty set $A \subseteq \mathbb{R}$, we consider the smallest topology on $\mathbb{R}$ which contains the open left rays containing points $a \in A$ and the open right rays containing points $b \in \mathbb{R} - A$. We present a natural model for this hybrid topology and show that it is quasi-metrizable. We investigate other variations of this topology.

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1. Introduction

Given a set $A \subseteq \mathbb{R}$, the Hattori space $H(A)$ is $\mathbb{R}$ with the topology having the base $\{(a-\varepsilon, a+\varepsilon) : a \in A, \varepsilon > 0\} \cup \{[b, b+\varepsilon) : b \in \mathbb{R} - A, \varepsilon > 0\}$. These spaces were introduced in [7] and studied further in [1–4, 9, 12]. The Hattori topology may be viewed as a hybrid of the Euclidean and lower-limit topologies on $\mathbb{R}$. In [12], hybrid topologies based on various combinations of the lower-limit, upper-limit, left-ray, discrete, and Euclidean topologies were studied. Here, we consider hybrid topologies of the left-ray and right-ray topologies. After some basic properties in Section 1, in Section 2 we investigate the subposet of the hybrids of left-ray and right-ray topologies in the lattice $TOP(\mathbb{R})$ of all topologies on $\mathbb{R}$. The left-ray and right-ray topologies each arise from a quasi-metric. In Section 3, we show that our hybrids of left-ray and right-ray topologies arise from hybrid quasi-metrics. In Section 4, we consider variations using closed rays and using rays $(-\infty, a+\varepsilon)$ where $\varepsilon > 0$ is bounded above by 1. In the last section, we consider hybrids of the Euclidean and right-ray topologies, which are closely related.

A quasi-metric on X is a function $q : X \times X \to [0, \infty)$ satisfying (a) $x = y$ if and only if $q(x, y) = 0 = q(y, x)$ and (b) $q(x, y) + q(y, z) \geq q(x, z)$, for all $x, y, z \in X$. The left-ray topology arises from the quasi-metric $q_{lr}(x, y) = y - x$ if $y \geq x$ and $q_{lr}(x, y) = 0$ if $y < x$. The right-ray quasi-metric is defined similarly. There are several quasi-metrization theorems [5, 6, 8, 10], but it is often difficult to exhibit a specific quasi-metric for a quasi-metrizable space. For example, Fletcher and Lindgren [5] show that the Niemytzki tangent

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disk topology on the closed half-plane (see [14]) is quasi-metrizable, but to our knowledge, no explicit quasi-metric has been exhibited. Further results on quasi-metrics can be found in the works of H.-P. Künzi and of F. Yildiz, including [11]. Standard topological concepts can be found in [13]. Recall that a set C in a poset P is *order dense* if $a, b \in P$ and $a < b$ imply that there exists $c \in C$ with $a < c < b$. For the poset $(\mathbb{R}, \leq)$, C is order dense in $\mathbb{R}$ if and only if C is dense in $\mathbb{R}$ with the Euclidean topology. Throughout, we assume $A \subseteq \mathbb{R}$ and $B = \mathbb{R} - A$. We use $C - D$ to represent the relative complement of D in C.

2. Basic Properties

Given $A \subseteq \mathbb{R}$, by $L(A)$, we denote the hybrid of the left-ray and right-ray topology on $\mathbb{R}$ having a subbasis

$$\mathcal{S} = \{(-\infty, a + \varepsilon) : a \in A, \varepsilon > 0\} \cup \{(b - \varepsilon, \infty) : b \in \mathbb{R} - A, \varepsilon > 0\}.$$

Note that $\mathcal{S}$ is not a basis, since the intersection of two oppositely directed rays will never contain a ray.

Let $B = \mathbb{R} - A$. Assume that A, B are non-empty (or else we simply have the right-ray or left-ray topology). If there exists $a < b$ with $a \in A, b \in B$, then a has the neighborhood base $(a - \varepsilon, a + \varepsilon) = (b - (b - a + \varepsilon), \infty) \cap (-\infty, a + \varepsilon)$ and similarly, b has the neighborhood base $(b - \varepsilon, b + \varepsilon)$. If $[a, \infty) \subseteq A$ (so $a \in A, a \geq \sup B$), then a has the neighborhood base $\{(b - \varepsilon, a + \varepsilon) : b < a, b \in B, \varepsilon > 0\}$. If $(-\infty, b] \subseteq B$ (so $b \leq \inf A$), then b has the neighborhood base $\{(b - \varepsilon, a + \varepsilon) : b < a, a \in A, \varepsilon > 0\}$. We tabulate this in Table 1. Figure 1 suggests the neighorhoods.

$x = a \in A$ or $x = b \in B$	form of the neighborhood base at x ($\varepsilon > 0$)
$x = a < \sup B$	$(x - \varepsilon, x + \varepsilon)$
$x = a \geq \sup B$	$(b - \varepsilon, x + \varepsilon)$, $b \leq \sup B \leq a = x$
$\inf A < b = x$	$(x - \varepsilon, x + \varepsilon)$
$x = b \leq \inf A$	$(x - \varepsilon, \inf A + \varepsilon)$

Table 1. Neighborhoods of points in $L(A)$.

Some of the results below also hold for $A = \emptyset$ or $A = \mathbb{R}$, but care must be taken in these cases. Since $B = \mathbb{R} - A$, if $\emptyset \subset A \subset \mathbb{R}$, then $\inf A \leq \sup B$. However, if $A = \emptyset$, then $\inf A = \infty \geq \sup B$ and if $A = \mathbb{R}$, $\sup B = -\infty \leq \inf A$.

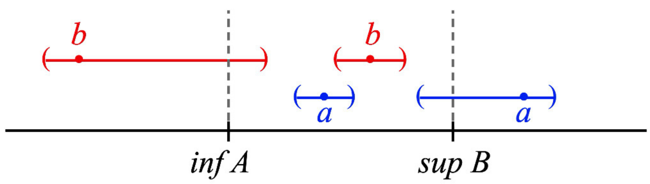

Figure 1. $L(A)$ neighborhoods of $a \in A$ and $b \in B$, based on the position of a, b.

Example 2.1. A model for $L(A)$. For real numbers $i < j$, consider the set $X = \{(i, y) \in \mathbb{R}^2 : y \geq 0\} \cup \{(x, 0) \in \mathbb{R}^2 : i \leq x \leq j\} \cup \{(j, y) \in \mathbb{R}^2 : y \geq 0\}$ as suggested in Figure 2. Think of the x-axis as being on the ground and the y-axis extending vertically above the ground. It takes energy to travel on the ground (horizontally) and upward, but no added energy to travel downward—gravity supplies that required force. This suggests a quasi-metric on X defined by taking the ε-ball around $(r,s) \in X$ to be the set of points accessible from (r,s) if you have enough energy to move a distance of ε units. We will delay the verification that this is a quasi-metric until Section 4, but we can recognize that the topology required is $L(A)$, with $A = \{i\} \cup [j,\infty)$, so $[i,j] = [\inf A, \sup B]$. If exactly one of $\inf A$ and $\sup B$ is infinite, then only one of the vertical poles of Figure 2 is present.

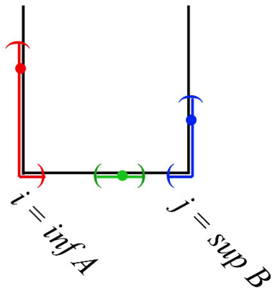

Figure 2. $L(A)$ models distances required to move overland or upward.

We list some immediate facts about $L(A)$ for reference.

Theorem 2.2. (a) *$L(A)$ is the Euclidean topology if and only if $A \neq \mathbb{R}$, $\inf A = -\infty$, and $\sup B = \infty$.*
(b) *For any $A \subseteq \mathbb{R}$, $L(A)$ is T_0.*
(c) *$L(A)$ is T_1 if and only if $A \neq \mathbb{R}$, $\inf A = -\infty$, and $\sup B = \infty$.*
(d) *If $\emptyset \subset A \subset \mathbb{R}$, the constant sequence $(x)_{n=1}^{\infty}$ in $L(A)$ has the unique limit x if $x \in (\inf A, \sup B)$, converges to all $y \in (-\infty, x]$ if $x \leq \inf A$, and converges to all $z \in [x,\infty)$ if $x \geq \sup B$.*
(e) *$L(A)$ is always connected.*
(f) *$L(A)$ is separable.*

Proof. (a) follows from the definition of $L(A)$.

(b) If $x \neq y$ then either $(-\infty, \frac{x+y}{2})$ or $(\frac{x+y}{2}, \infty)$ is a neighborhood of one of the points which excludes the other.

(c) If $\inf A = -\infty$ and $\sup B = \infty$, then by (a), $L(A)$ is Euclidean, which is T_1. If $\inf A > -\infty$, then for $x < y < \inf A$, every neighborhood of x includes y, so $L(A)$ is not T_1. The dual argument shows that if $\sup B < \infty$, $L(A)$ is not T_1.

(d) is immediate.

(e) If $A = \emptyset$ or $\mathbb{R}$, then $L(A)$ is the left- or right-ray topology, which is connected. Otherwise, $(\inf A, \sup B)$ inherits the Euclidean topology and is connected. If U, V is a separation of $L(A)$, for $y \in (\inf A, \sup B) \cap U$, U must contain the connected set $(\inf A, \sup B)$. If $x \leq \inf A$, every open set containing x must intersect $(\inf A, \sup B)$, so $x \in U$. Similarly, every open neighborhood of $y > \sup B$ must intersect $(\inf A, \sup B)$, so $y \in U$. This gives the contradiction that $U = \mathbb{R}$.

(f) In $L(A)$, every open set intersects $\mathbb{Q}$. □

Combining (a) and (c), we see that for $A \neq \mathbb{R}$, $L(A)$ is T_1 if and only if it is T_j for any $j \in \{2, 3, 3.5, 4\}$. Note that if A and B are order dense (or equivalently, dense in the Euclidean topology), then $L(A)$ is the Euclidean topology.

From Figure 1, we see that if $i = \inf A < \sup B = j$, the closed and bounded sets are arbitrary intersections of Euclidean closed sets $[i, c_n] \cup [d_n, j] \subseteq [i,j]$ which are contained in the Euclidean subspace $[i,j]$ of $L(A)$, and thus are compact. If $i = \inf A = \sup B = j$, the closed sets are of form $\emptyset, \mathbb{R}$, $(-\infty, c]$ for $c < i = j$, and $[d, \infty)$ for $d > i = j$, so $\emptyset$ is the only closed and bounded set. Thus, in all cases, the closed and bounded sets in $L(A)$ are compact. The converse fails. In $L(A)$, compact sets need not be closed, as the following example shows.

Example 2.3. Let $A = (-1, 0)$. For $x < \inf A = -1$, $cl\{x\} = (-\infty, x]$ so $\{x\}$ is not closed. As a finite set, $\{x\}$ is compact.

Or, with $A = (0, \infty)$, consider $S = (2, 3]$. Any open set covering 3 covers $[0, 3] = [\sup B, 3]$ and thus covers S. Thus, S is compact, but S is not closed in $L(A)$ nor in the Euclidean topology.

A compact set S in $L(A)$ must be bounded. In particular, $\mathbb{R}$ is not $L(A)$-compact. This is implied by the following characterization of the compact subsets of $L(A)$.

Theorem 2.4. *Suppose $A \neq \emptyset$ and $B \neq \emptyset$. Then a non-empty set $S \subseteq \mathbb{R}$ is $L(A)$-compact if and only if $(-\infty, \sup B) \cap S$ is empty or has a smallest element, $(\inf A, \infty) \cap S$ is empty or has a largest element, and there is no sequence (s_n) in S converging (in the Euclidean topology) to $x \in (\inf A, \sup B) - S$.*

Proof. Suppose $(-\infty, \sup B) \cap S \neq \emptyset$ and has no smallest element. Then there exists a strictly decreasing sequence (s_n) in S with $S \subseteq \bigcup\{(s_n, \infty) : n \geq 1\}$ and $s_n < b$ for some $b \in B$ and all $n \in \mathbb{N}$. Now $\{(s_n, \infty) : n \in \mathbb{N}\}$ is an open cover of S with no finite subcover. The dual argument covers the case $(\inf A, \infty) \cap S \neq \emptyset$ and has no largest element. Suppose there is a sequence (s_n) in S converging to $x \in (\inf A, \sup B) - S$. Now there exist $a \in A, b \in B$ with $a < x < b$. Without loss of generality (dropping to a subsequence, if it is necessary), we will assume (s_n) is a strictly monotone sequence in (a, b). If (s_n) is strictly decreasing, $\{(-\infty, x)\} \cup \{(s_n, \infty) : n \in \mathbb{N}\}$ is an open cover with no finite subcover. The dual construction applies if (s_n) is strictly increasing. Thus, compactness of S implies the conditions listed in the theorem.

Now suppose the conditions listed are satisfied and $\mathcal{C}$ is an open cover of S. Recall that $\inf A \leq \sup B$. Now either

(1) $(-\infty, \sup B) \cap S = \emptyset$,

(2) $\min((-\infty, \sup B) \cap S) = s' \leq \inf A$, or

(3) $\min((-\infty, \sup B) \cap S) = s' \in (\inf A, \sup B)$

and either

(a) $(\inf A, \infty) \cap S = \emptyset$,

(b) $\max((\inf A, \infty) \cap S) = s'' \geq \sup B$, or

(c) $\max((\inf A, \infty) \cap S) = s'' \in (\inf A, \sup B)$.

In case (1), we have $S \subseteq [\sup B, \infty)$, and it follows that case (c) cannot occur. Cases (1) and (a) occur if and only if $S = \{\sup B\} = \{\inf A\}$, in which case S is finite and thus compact. If cases (1) and (b) occur, any $C_1 \in \mathcal{C}$ which covers s'' also covers $[\sup B, s'']$ and thus covers S.

If cases (2) and (a) occur, $S \subseteq [s', \inf A]$ and any $C_1 \in \mathcal{C}$ covering s' covers $[s', \inf A]$ and thus covers S. If cases (2) and (b) occur, any $C_1 \in \mathcal{C}$ covering s' covers $[s', \inf A + \varepsilon')$ for some $\varepsilon' > 0$ and any $C_2 \in \mathcal{C}$ covering s'' covers $(\sup B - \varepsilon'', s'']$ for some $\varepsilon'' > 0$. Consider $S' = S \cap [\inf A' + \varepsilon', \sup B' - \varepsilon'']$. If $S' = \emptyset$, then $\{C_1, C_2\}$ covers S. Otherwise, the hypothesis about no sequences in S converging to $x \in (\inf A, \sup B) - S$ implies S' is Euclidean closed. Since $S' \subseteq (\inf A, \sup B)$, the $L(A)$ topology on S' is Euclidean. As a closed and bounded Euclidean set covered by the Euclidean open sets of $\mathcal{C}$, there must be a finite subcover $\{C_k\}_{k=3}^n$ of S'. Now $\{C_k\}_{k=1}^n$ is a finite subcover of S. In case (2) and (c) hold, any $C_1 \in \mathcal{C}$ covering s' covers $[s', \inf A + \varepsilon')$ for some $\varepsilon' > 0$, and $S' = [\inf A + \varepsilon', s'']$

is a Euclidean closed and bounded set in $(\inf A, \sup B)$, and since $L(A)$ agrees with the Euclidean topology in $(\inf A, \sup B)$, there is a finite subcover $\{C_k\}_{k=2}^n$ of S'. Now $\{C_k\}_{k=1}^n$ is a finite subcover of S.

The cases involving (3) are dual to those with (2). □

For $S \subseteq \mathbb{R}$, divide S into its left, middle, and right parts $S_l = S \cap (-\infty, \inf A)$, $S_m = S \cap [\inf A, \sup B]$, and $S_r = S \cap (\sup B, \infty)$. Theorem 2.4 almost characterizes compact sets in $L(A)$ as the sets S for which S_m is closed, S_l is empty or has a least element, and S_r is empty or has a greatest element. However, this potential characterization fails due to some issues around the points $\inf A$ and $\sup B$. For example, with $A = (0, 1)$ and $S = \{-1\} \cup (0, 2]$, S is compact even though S_m is not closed. Note that Theorem 2.4 partly avoids these issues by considering $S \cap (\inf A, \infty)$ instead of $S_r = S \cap (\sup B, \infty)$ and $S \cap (-\infty, \sup B)$ instead of $S_l = (-\infty, \inf A)$.

3. Lattice Properties

Let $\mathcal{L}(\mathbb{R}) = \{L(A) : A \subseteq \mathbb{R}\}$ be the subset of the lattice $TOP(\mathbb{R})$ of all toplogies on $\mathbb{R}$, ordered by $\subseteq$. It is well-known that $TOP(X)$ is a lattice with $\tau \wedge \tau' = \tau \cap \tau'$ and $\tau \vee \tau'$ having $\tau \cup \tau'$ as a subbase (that is, $\tau \vee \tau' = [\tau \cup \tau']$).

Figures 1 and 2 suggest that $L(A)$ depends more on the pair $(i, j) = (\inf A, \sup B)$ than on the set A itself. The proof of the theorem below is straightforward.

Theorem 3.1. *Suppose A, A' are subsets of $\mathbb{R}$ with complements B, B', respectively. Let $(i, j) = (\inf A, \sup B)$ and $(i', j') = (\inf A', \sup B')$.*

(a) *$L(A) = L(A')$ if and only if $(i, j) = (i', j')$.*
(b) *If A, A' are non-empty and proper subsets of $\mathbb{R}$, $L(A) \subseteq L(A')$ if and only if $[i, j] \subseteq [i', j']$.*
(c) *If A, A' are non-empty and proper subsets of $\mathbb{R}$, $L(A) \subset L(A')$ if and only if $[i, j] \subset [i', j']$.*

We will now consider infima in $\mathcal{L}(\mathbb{R})$ and compare them to infima in $TOP(\mathbb{R})$. We start with an instructive example.

Example 3.2. $L(A) \cap L(A') \neq L(A \cap A')$. Let $A = \{0\}$ and $A' = \{3\}$. Now $L(A)$ has basis $\{(y, z) : y < 0 < z\} \cup \{(z, w) : 0 < z < w\}$, $L(A')$ has basis $\{(y, z) : y < 3 < z\} \cup \{(z, w) : 3 < z < w\}$, and $L(A \cap A') = L(\emptyset)$ has basis $\{(b, \infty) : b \in \mathbb{R}\}$. Now $L(A) \cap L(A')$ has basis $\{(y, z) : y < 3 < z\} \cup \{(z, w) : 3 < z < w\}$. In particular, $(-1, 4) \in L(A) \cap L(A')$ but $(-1, 4) \notin L(A \cap A')$.

Indeed observe that $L(\{0\}) = L(\{0, 3\})$ since $\inf\{0, 3\} = \inf\{0\}$ and $\sup(\mathbb{R} - \{0, 3\}) = \sup(\mathbb{R} - \{0\})$. Now $L(\{0\}) \cap L(\{3\}) = L(\{0, 3\}) \cap L(\{3\}) = L(\{3\})$ by Theorem 3.1.

For another example, if $A = \mathbb{Q} \cap [0, 1]$ and $A' = [0, 1] - \mathbb{Q}$, then by Thoerem 3.1, $L(A) = L(A') = L(A) \cap L(A') \neq L(A \cap A') = L(\emptyset)$.

Example 3.2 suggests that the intervals $[i, j]$, $[i', j']$ are more significant in determining $L(A) \cap L(A')$ than the actual sets A, A'. Our next result confirms this and shows that $\mathcal{L}(\mathbb{R})$ is not a lattice.

Theorem 3.3. *Suppose A, A' are non-empty, proper subsets of $\mathbb{R}$, $i = \inf A, j = \sup B$, $i' = \inf A'$, and $j' = \sup B'$.*

(a) *If $[i'', j''] = [i, j] \cap [i', j'] \neq \emptyset$, then in $\mathcal{L}(\mathbb{R})$, $L(A) \wedge L(A') = L(A) \cap L(A') = L(A'')$ where A'' arises from $[i'', j'']$ (that is, A'' is any subset of $\mathbb{R}$ with $\inf A'' = i''$ and $\sup B'' = \sup(\mathbb{R} - A'') = j''$).*
(b) *If $[i, j] \cap [i', j'] = \emptyset$, then there is no topology $\tau \in \mathcal{L}(\mathbb{R})$ with $\tau \subseteq L(A) \cap L(A')$, so in $\mathcal{L}(\mathbb{R})$, $L(A) \wedge L(A')$ fails to exists.*

Proof. (a) By Theorem 3.1, $L(A'') = L(A) \wedge L(A')$ in $\mathcal{L}(\mathbb{R})$. It is easy to verify that $L(A'') = L(A) \cap L(A')$, which is $L(A) \wedge L(A')$ in $TOP(\mathbb{R})$.

(b) Suppose $[i, j] \cap [i', j'] = \emptyset$. Without loss of generality, $i \leq j < i' \leq j'$. We will first describe $L(A) \cap L(A')$.

Suppose $U \in L(A) \cap L(A')$, $x \in U$, and $x \leq i' = \inf A'$. Now every $L(A')$ neighborhood of x must include $[x, i']$, and every $L(A)$-open set containing i' must contain $[j, i']$. Thus, $[j, i'] \subseteq U$. Indeed, every interval $(r, s) \supseteq [j, i'] \cup [x, i']$ is a $L(A) \cap L(A')$ neighborhood of $x \leq i'$.

Suppose $U \in L(A) \cap L(A')$, $x \in U$, and $x > i' = \inf A'$. Now every $L(A)$ neighborhood of x must include $[j, x]$, so $[j, i'] \subseteq [j, x] \subseteq U$. Indeed, every interval $(r, s) \supseteq [j, x]$ is a $L(A) \cap L(A')$ neighborhood of $x > i'$.

The last two paragraphs show that if $i \leq j < i' \leq j'$, the non-empty elements of $L(A) \cap L(A')$ are the intervals (r, s) with $[j, i'] \subseteq (r, s)$ $(r, s \in \mathbb{R} \cup \{\pm\infty\})$. Observe that this topology models the situation of Example 2.1 if no energy is required to move horizontally. In particular, for $j < x < y < i'$, every neighborhood of x contains y and every neighborhood of y contains x, so $L(A) \cap L(A')$ is not T_0 and thus not of form $L(A'')$ for any set A''. Indeed, there is no topology $\tau \subseteq L(A) \cap L(A')$ which is T_0, so in the poset $\mathcal{L}(\mathbb{R})$, $L(A) \wedge L(A')$ does not exist if $i \leq j < i' \leq j'$. □

Now we turn to suprema in $\mathcal{L}(\mathbb{R})$. Recall that in $TOP(\mathbb{R})$, $L(A) \vee L(A') = [L(A) \cup L(A')]$, the topology generated by the basis $L(A) \cup L(A')$.

Example 3.4. $L(A \cup A') \neq [L(A) \cup L(A')]$. Let $A = \mathbb{Q}$ and $A' = \mathbb{R} - \mathbb{Q}$. By Theorem 2.2, $L(A)$ and $L(A')$ are the Euclidean topology, so $L(A) \vee L(A')$ is the Euclidean topology. However, $L(A \cup A') = L(\mathbb{R})$ is the left-ray topology.

Notation: $convS$ is the convex hull of S. If $i = -\infty$ or $j = \infty$, by $[i, j]$ we mean $[i, j] \cap \mathbb{R}$. Infima and suprema are taken in $\mathbb{R} \cup \{\pm\infty\}$.

Theorem 3.5. *$\mathcal{L}(\mathbb{R})$ is an upper sub-semi-lattice of $Top(\mathbb{R})$. If $L(A)$ and $L(A')$ correspond to $[i, j]$ and $[i', j']$, then in $\mathcal{L}(\mathbb{R})$, $L(A) \vee L(A') = [L(A) \cup L(A')] = L(A'')$ where A'' corresponds to $[i'', j''] = [\inf\{i, i'\}, \sup\{j, j'\}] = conv([i, j] \cup [i', j'])$.*

Proof. Given $L(A), L(A')$ corresponding to $[i, j]$ and $[i', j']$, let $[i'', j''] = [\inf\{i, i'\}, \sup\{j, j'\}] = conv([i, j] \cup [i', j'])$ correspond to $L(A'')$. In $L(A'')$, $x < i''$ has a neighborhood base of form $(x - \varepsilon, i'' + \varepsilon)$, which is open in $L(A)$ or $L(A')$, depending on whether $i'' = i$ or $i'' = i'$. In $L(A'')$, $x > j''$ has a neighborhood base of form $(j'' - \varepsilon, x + \varepsilon)$, which is open in $L(A)$ or $L(A')$, depending on whether $j'' = j$ or $j'' = j'$. In $L(A'')$, $x \in [i'', j'']$ has a Euclidean neighborhood base. Now either $[i'', j''] - ([i, j] \cup [i', j'])$ is empty (if $[i, j] \cap [i', j'] \neq \emptyset$) or $[i'', j''] - ([i, j] \cup [i', j'])$ is an interval I. If $x \in [i, j] \cup [i', j']$, x has a Euclidean neighborhood base in $L(A)$ or $L(A')$. If x is in the interval I between $[i, j]$ and $[i', j']$, say $i \leq j < x < i' \leq j'$, then x has $L(A)$ neighborhoods $(j - \varepsilon, x + \varepsilon)$ and $L(A')$ neighborhoods $(x - \varepsilon, i' + \varepsilon)$, and thus has $[L(A) \cup L(A')]$ neighborhoods $(x - \varepsilon, x + \varepsilon)$. This shows that $L(A'') \subseteq [L(A) \cup L(A')]$. Since $[i, j], [i', j'] \subseteq [i'', j'']$, Theorem 3.1 shows that $L(A), L(A') \subseteq L(A'')$, so $[L(A) \cup L(A')] \subseteq L(A'')$. □

4. A quasi-metric for $L(A)$.

Theorem 4.1. *For $a \in A$, $b \in B = \mathbb{R} - A$, and $y \in \mathbb{R}$, let*

$$q(a, y) = \begin{cases} |a - y| & \text{if } a < \sup B & (1) \\ y - a & \text{if } \sup B \leq a \leq y & (2) \\ 0 & \text{if } \sup B \leq y \leq a & (3) \\ \sup B - y & \text{if } y < \sup B \leq a & (4) \end{cases}$$

$$q(b,y)=\begin{cases} |b-y| & \text{if } \inf A<b \quad (5)\\ b-y & \text{if } y<b\le\inf A \quad (6)\\ 0 & \text{if } b<y\le\inf A \quad (7)\\ y-\inf A & \text{if } b\le\inf A<y. \quad (8)\end{cases}$$

Then $q(x,y)$ is a quasi-metric which generates the topology $L(A)$.

Proof. It is easy to see that $q(x,y)\geq 0$ and $x=y$ if and only if $q(x,y)=0=q(y,x)$, and the q-balls around $x\in\mathbb{R}$ match the base of $L(A)$ neighborhoods given in Table 1. Thus, it only remains to show that $q(x,y)+q(y,z)\geq q(x,z)$ for distinct $x,y,z\in\mathbb{R}$.

If line (n) of the definition of q is used to find the distance $q(x,y)$, we will say (x,y) are "in position (n)". Observe that for $(n)=(1)$ or (5), we could more accurately say x (rather than (x,y)) is in position (n), but for consistency, we may still say (x,y) is in position (n).

Case $(x,y)=(a,a')\in A^2$: Suppose $(x,y)=(a,a')$ is in position (1). If (a',z) is in position (1) or (2), then all distances are Euclidean and the triangle inequality holds. If (a',z) is in position (3), we have $a<\sup B\leq z\leq a'$, and $q(a,a')\geq q(a,z)$ since both of these are Euclidean distances. If (a',z) are in position (4), then either (i) $a\leq z<\sup B\leq a'$ or (ii) $z\leq a\leq \sup B\leq a'$. In case (i), then $q(a,a')\geq q(a,z)$ since both of these distances are Euclidean. In case (ii), $q(a',z)=\sup B-z\geq a-z=q(a,z)$.

Suppose $(x,y)=(a,a')$ is in position (2). Note that (a',z) cannot be in position (1). If (a',z) is in position (2), then all distances are Euclidean. If (a',z) is in position (3), either (i) $\sup B\leq z\leq a\leq a'$ or (ii) $\sup B\leq z\leq a'$. In case (i), $q(a,z)=0$ and in case (ii), $q(a,a')=a'-a\geq z-a=q(a,z)$. If (a',z) is in position (4), then $z\leq\sup B\leq a\leq a'$, so $q(a,z)=\sup B-z=q(a',z)$.

Suppose $(x,y)=(a,a')$ are in positioned (3). Now (a',z) cannot be in position (1). If (a',z) is in position (2), then we have (i) $\sup B\leq a'\leq z\leq a$ or (ii) $\sup B\leq a'\leq a\leq z$. In the case (i), (a,z) are in position (3), so $q(a,z)=0$. In case (ii), $q(a',z)$ and $q(a,z)$ are computed by (2), so $q(a',z)\geq q(a,z)$. If (a',z) is in position (3) then $\sup B\leq z\leq a'\leq a$, so $q(a,z)=0$. If (a',z) is in position (4) then $z\leq \sup B\leq a'\leq a$, so $q(a,z)=\sup B-z=q(a',z)$.

Suppose $(x,y)=(a,a')$ is in position (4). Then we have $a'<\sup B\leq a$, so (a',z) is necessarily in position (1). If $z\leq a'$, then $q(a,a')+q(a',z)=\sup B-a'+a'-z=q(a,z)$. If $a'\leq z\leq\sup B\leq a$, then $q(a,a')=\sup B-a'\geq\sup B-z=q(a,z)$. If $a'\leq\sup B\leq z\leq a$, then $q(a,z)=0$. If $a'\leq\sup B\leq a\leq z$ then $q(a',z)=z-a'\geq z-a=q(a,z)$.

Case $(x,y)=(a,b)\in A\times B$: Suppose $(x,y)=(a,b)$ is in position (1). If (b,z) is in position (5) or (6), then all distances are Euclidean. If (b,z) is in position (7), then $b<z\leq\inf A\leq a\leq\sup B$, and $q(a,b)=a-b\geq a-z=q(a,z)$. If (b,z) is in position (8), either (i) $b\leq\inf A<z\leq a$ or (ii) $b\leq\inf A\leq a\leq z$. In case (i), $q(a,b)=a-b\geq a-z=q(a,z)$. In case (ii), $q(b,z)=z-\inf A\geq z-a=q(a,z)$.

Observe that $(x,y)=(a,b)\in A\times B$ cannot be in position (2), since this would imply $\sup B<b\in B$.

Suppose $(x,y)=(a,b)$ is in position (3), so $\sup B=b<a$. Suppose (b,z) is in position (5), so $\inf A<b=\sup B<a$ and $q(b,z)$ is the Euclidean distance. Either (i) $z\leq b<a$ and $q(a,z)=\sup B-z=b-z=q(b,z)$, (ii) $b<z<a$ and $q(a,z)=0$, or (iii) $b<a<z$ and $q(a,z)=z-a\leq z-b=q(b,z)$. Suppose (b,z) is in position (6) so $z<b=\sup B\leq\inf A\leq a$. Then $q(a,z)=b-z=q(b,z)$. Suppose (b,z) is in position (7) so $b=\sup B<z\leq\inf A\leq a$ and thus $q(a,z)=0$. Suppose (b,z) is in position (8) so either (i) $\sup B=b\leq\inf A<z<a$ or (ii) $\sup B=b\leq\inf A\leq a<z$. In case (i), $q(a,z)=0$ and in case (ii) $q(a,z)=z-a\leq y-\inf A=q(b,z)$.

Suppose $(x,y)=(a,b)$ is in position (4) so $b<\sup B\leq a$. These three points give four possible positions for z. If $z<b<\sup B\leq a$, then $q(a,b)=\sup B-b$ and $q(b,z)=b-z$, determined by either (5) or (6). Thus, $q(a,z)=\sup B-z=q(a,b)+q(b,z)$.

If $b < z < \sup B \leq a$, then $q(a,z) = \sup B - z \leq \sup B - b = q(b,z)$. If $b < \sup B \leq z < a$, then $q(a,z) = 0$. If $b < \sup B \leq a < z$, then (b,z) cannot be in positions (6) or (7). If (b,z) is in position (5), then $q(a,z) = z - a \leq z - b = q(b,z)$. If (b,z) is in position (8), $q(a,z) = z - a \leq z - \inf A = q(b,z)$.

The cases $\boldsymbol{(x,y) = (b,b') \in B^2}$ and $\boldsymbol{(x,y) = (b,a) \in B \times A}$ are dual. □

5. Variations

5.1. Bounded neighborhoods: $L^*(A)$

A different topology arises if we replace $\varepsilon > 0$ in the definition of the subbasis $\mathcal{S}$ for $L(A)$ by $\varepsilon \in (0,1]$. Let $L^*(A)$ be the topology on $\mathbb{R}$ having a subbasis

$$\mathcal{S}^* = \{(-\infty, a+\varepsilon) : a \in A, \varepsilon \in (0,1]\} \cup \{(b-\varepsilon, \infty) : b \in \mathbb{R} - A, \varepsilon \in (0,1]\}.$$

For example, if $A = \{0\}$, $(0.9, 1.2) = (-\infty, 0+1.2) \cap (1-0.1, \infty)$ is a $L(A)$-neighborhood of 1 (using $\varepsilon = 1.2$) which is not a $L^*(A)$-neighborhood of 1.

With $A = (-\infty, 0) \cup \{2\}$, for every $a \in A$ ($b \in B$) there exists $b \in B$ ($a \in A$) with $a < b$, so $L(A)$ is the Euclidean topology. Now consider the topology $L^*(A)$. For $a \leq -1$, a has a neighborhood base $(-\infty, a+\varepsilon)$ for $\varepsilon \in (0,1]$. For $x \in (-1,1) \cup [2,3)$, x has a Euclidean neighborhood. For $b \in [1,2)$, b has a neighborhood base $(b-\varepsilon, 2+\varepsilon)$ for $\varepsilon \in (0,1]$. For $b \geq 3$, b has a neighborhood base $(b-\varepsilon, \infty)$ for $\varepsilon \in (0,1]$.

Theorem 5.1. *The following are equivalent.*

(a) $L^*(A)$ *is Euclidean.*

(b) $\forall a \in A, (a, a+1) \cap B \neq \emptyset$ *and* $\forall b \in B, (b-1, b) \cap A \neq \emptyset$.

(c) *For every* $x \in \mathbb{R}$, $[x, x+1) \cap B \neq \emptyset$ *and* $(x-1, x] \cap A \neq \emptyset$.

Proof. (a) $\iff$ (b): If (b) holds, for any $a \in A$, there exists $b \in (a, a+1)$. For all $\varepsilon \in (0, a+1-b)$, $(a-\varepsilon, a+\varepsilon)$ is a neighborhood of a, so a has a Euclidean neighborhood. The dual argument shows that every $b \in B$ has a Euclidean neighborhood. Suppose (b) fails. Suppose there exists $a \in A$ with $[a, a+1) \subseteq A$. Thus, there is no $b \in B$ such that $a \in (b-\varepsilon, \infty)$ for $\varepsilon \in (0,1)$, so every neighborhood of a has form $(-\infty, a+\varepsilon)$, and $L^*(A)$ is not Euclidean. The dual argument covers the case of $(b-1, b] \subseteq B$.

(b) $\iff$ (c): Suppose (c) fails. Then there exists $x \in \mathbb{R}$ such that (i) $[x, x+1) \cap B = \emptyset$ or (ii) $(x-1, x] \cap A = \emptyset$. If $x \in A$, (i) contradicts (b) and if $x \in B$, (ii) contradicts (b). Conversely, suppose (b) fails. Then either there exists $x = a \in A$ with $[a, a-1) \cap B = \emptyset$ or there exists $x = b \in B$ with $(b-1, b] \cap A = \emptyset$, which shows that (c) fails. □

If B and S are subsets of $\mathbb{R}$, let B_S represent B restricted to S. That is, $B_S = B \cap S$. Define A_S analogously. The neighborhoods of a point in $L^*(A)$ are described below.

$x = a \in A$ or $x = b \in B$	form of the neighborhood base at x ($\varepsilon > 0$)
$x = a, B_{(a,a+1)} = \emptyset$	$(\sup B_{(-\infty,a)} - \varepsilon, a + \varepsilon')$, $\varepsilon, \varepsilon' \in (0,1]$
$x = a, B_{(a,a+1)} \neq \emptyset$	$(a - \varepsilon, a + \varepsilon')$, $\varepsilon \in (0, a + 1 - \inf B_{(a,a+1)}]$ $\varepsilon' \in (0,1]$
$x = b, A_{(b-1,b)} = \emptyset$	$(b - \varepsilon', b + \inf A_{(b,\infty)} + \varepsilon)$, $\varepsilon, \varepsilon' \in (0,1]$
$x = b, A_{(b-1,b)} \neq \emptyset$	$(b - \varepsilon', \sup A_{(b-1,b)} + \varepsilon)$, $\varepsilon, \varepsilon' \in (0,1]$ $= (b - \varepsilon', b + \varepsilon)$, $\varepsilon \in (0, \sup A_{(b-1,b)} + 1 - b]$, $\varepsilon' \in (0,1]$

5.2. Hybrids of Closed Left-Ray Right-Ray Topologies

If $\leq$ is the usual order on $\mathbb{R}$, in the specialization topology, the smallest neighborhood of x is $N(x) = \uparrow x = [x, \infty)$. This is the closed right-ray topology. The closed left-ray topology is defined dually. Note that the closed right-ray topology is finer than the (open) right ray topology, since $(a, \infty) = \bigcup_{x>a}[x, \infty)$. Let $L^c(A)$ be the topology generated by the subbase

$$\mathcal{S} = \{(-\infty, a] : a \in A\} \cup \{[b, \infty) : b \in B = \mathbb{R} - A\}.$$

A base of $L^c(A)$ neighborhoods of $a \in A$ is $\{(-\infty, a]\} \cup \{[b, a], b \in B, b < a\}$, and a base of $L^c(A)$ neighborhoods of $b \in B$ is $\{[b, \infty)\} \cup \{[b, a]], a \in A, b < a\}$.

This is modeled if each point can emit a particle in one direction only, and may block oppositely directed particles. The points of A on the real line are the points which emit particles to the left and may absorb particles arriving from the right.

Theorem 5.2. (a) *$L^c(A)$ is never Euclidean. $L^c(A)$ is finer than the Euclidean topology if and only if A and B are both order dense in $\mathbb{R}$ (i.e., dense in the Euclidean topology).*

(b) *For any $A \subseteq \mathbb{R}$, $L^c(A)$ is T_0.*

(c) *$L^c(A)$ is T_1 if and only if A and B are both order dense.*

(d) *$L^c(A)$ is connected if and only if there exists no pair $(a, b) \in A \times B$ with $a < b$. That is, if and only if A is a ray to the right (including $\emptyset$ and $\mathbb{R}$).*

(e) *$L^c(A)$ is never compact.*

Proof. (a) $L^c(A)$ must contain an open set of form $[b, \infty)$ or $(-\infty, a]$, which is not Euclidean open.

Suppose A and B are order dense. Given $x \in \mathbb{R}$ and $\varepsilon > 0$, there exist $a \in A, b \in B$ with $x - \varepsilon < b < x < a < x + \varepsilon$. This shows every Euclidean neighborhood $(x - \varepsilon, x + \varepsilon)$ of x contains a $L^c(A)$ neighborhood $[b, a]$ of x. Suppose every Euclidean neighborhood $(x-\varepsilon, x+\varepsilon)$ of x contains a $L^c(A)$ neighborhood U of x. As a bounded $L^c(A)$-neighborhood, U has form $[b, a]$ for some $b \in B, a \in A$. Thus, $\forall x \in \mathbb{R}, \forall \varepsilon > 0$, there exists $a \in A, b \in B$ with $x - \varepsilon < b \leq x \leq a < x + \varepsilon$, which shows A and B are order dense in $\mathbb{R}$.

(b) Suppose $x < y$ in $\mathbb{R}$. If $x \in A$, then $(-\infty, x]$ is a neighborhood of x excluding y. If $y \in B$, then $[y, \infty)$ separates y from x. If $x \in B, y \in A$, consider $z \in (x, y) = (b, a)$. If $z \in A$, then $(-\infty, z]$ is a neighborhood of $x = b$ which excludes $y = a$. If $z \in B$, then $[z, \infty)$ separates a from b.

(c) Note that B fails to be order dense if and only if A contains an interval of positive length. If A contains an interval $[a, a']$ with $a < a'$, then every neighborhood of a' includes a, so $L^c(A)$ is not T_1. Dually, if A is not order dense, then $L^c(A)$ is not T_1. If A and B are both order dense, (a) shows that $L^c(A)$ is finer than the Euclidean topology and thus is T_2.

(d) Suppose A is a ray to the right. If $A = \emptyset$ or $A = \mathbb{R}$, then there are no two disjoint open sets, so $L^c(A)$ cannot be disconnected. If $A = (b, \infty)$, then every open set contains b. If $A = [a, \infty)$, then every open set contains a. Thus, $L^c(A)$ has no separation. Conversely, suppose there exists $a \in A, b \in B$ with $a < b$. Let $a_+ = \limsup A \cap [a, b)$ and $b_- = \liminf B \cap (a, b]$. Now either $a_+ > a$ or $b_- < b$. The cases are dual, so suppose $a_+ > a$. Then there exists a strictly increasing sequence (a_n) in $[a, a_+) \subseteq A$ converging to a_+. If $a_+ \in B$, then $(-\infty, a_+) = \bigcup\{(-\infty, a_n] : n \in \mathbb{N}\}$ and $[a_+, \infty)$ forms a separation of $L^c(A)$. If $a_+ \in A$, then $a_+ < b$ and there exists a strictly decreasing sequence (b_n) in B converging to a_+, so $(-\infty, a_+]$ and $(a_+, \infty) = \bigcup\{[b_n, \infty) : n \in \mathbb{N}\}$ forms a separation of $L^c(A)$.

(e) If $\sup A = \infty$, then $\{(-\infty, a] : a \in A\}$ is an open cover with no finite subcover. If $\sup A = m < \infty$, then $(m, \infty) \subseteq B$. If $m \in A$, then $\{(-\infty, m]\} \cup \{[b, \infty) : b > m\}$ is an open cover with no finite subcover. If $m \in B$, then there exists a strictly increasing sequence (a_n) in A converging to m, and $\{(-\infty, a_n] : n \in \mathbb{N}\} \cup \{[m, \infty)\}$ is an open cover with no finite subcover. □

6. Hybrids of the Euclidean Topology and the Right-Ray Topology

Consider $P(A)$ having subbase

$$\mathcal{S} = \{(a-\varepsilon, a+\varepsilon) : a \in A, \varepsilon > 0\} \cup \{(b-\varepsilon, \infty) : b \in B, \varepsilon > 0\}.$$

Note that $\mathcal{S}$ is not a basis: If $A = \{0, 2\}$ and $\varepsilon = 1.5$, $(0-\varepsilon, 0+\varepsilon) \cap (2-\varepsilon, 2+\varepsilon) = (.5, 1.5)$ contains no element of $\mathcal{S}$.

$x = a \in A$ or $x = b \in B$	form of the neighborhood base at x ($\varepsilon > 0$)
a	$(a-\varepsilon, a+\varepsilon)$
$\inf A < b$	$(b-\varepsilon, b+\varepsilon)$
$b \leq \inf A$	$(b-\varepsilon, \inf A + \varepsilon)$

Table 2. Neighborhoods of points in $P(A)$.

Example 6.1. A model for $P(A)$. For a real numbers i, consider the set $X = \{(i, y) \in \mathbb{R}^2 : y \geq 0\} \cup \{(x, 0) \in \mathbb{R}^2 : i \leq x\}$ as suggested in Figure 3. Think of the x-axis as being on the ground and the y-axis extending vertically above the ground. It takes energy to travel on the ground (horizontally) and upward, but no added energy to travel downward. This suggests a quasi-metric on X defined by taking the ε-ball around $(r, s) \in X$ to be the set of points accessible from (r, s) if you have enough energy to move a distance of ε units.

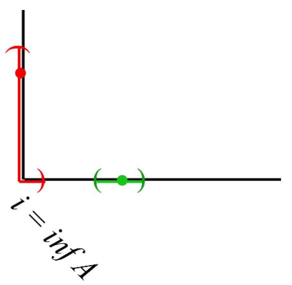

Figure 3. $P(A)$ models distances required to move overland or upward.

Comparing Figures 3 and 2 suggest that the results on $L(A)$ should carry over to $P(A)$ with minor modifications, essentially assuming $\infty \in B$ in the statements of results but not in the defintion of the basis for the topology. In this manner, we see that the results of Theorem 2.2 hold for $P(A)$, interpreting $\sup B = \infty$ in the statements.

Theorem 4.1 is also easily adapted to this situation to give a quasi-metric q for $P(A)$. There is only one case for $q(a, y)$ when $a \in A$, so we may simplify the statement as below. The proof remains valid.

Theorem 6.2. *For $a \in A$, $b \in B = \mathbb{R} - A$, and $y \in \mathbb{R}$, let*

$$q(a, y) = |a - y| \quad \text{if } y \in \mathbb{R} \tag{1}$$

$$q(b, y) = \begin{cases} |b - y| & \text{if } \inf A < b & (5) \\ b - y & \text{if } y < b \leq \inf A & (6) \\ 0 & \text{if } b < y \leq \inf A & (7) \\ y - \inf A & \text{if } b \leq \inf A < y. & (8) \end{cases}$$

Then $q(x, y)$ is a quasi-metric which generates the topology $P(A)$.

The characterization of compact sets given in Theorem 2.4 does not carry over directly from $L(A)$ to $P(A)$. Due to the lack of symmetry, a "dual argument" given in the proof there is no longer valid. Below is a characterization of compact sets in $P(A)$.

Theorem 6.3. *Suppose $A, B \neq \emptyset$. Then a non-empty set $S \subseteq \mathbb{R}$ is not $P(A)$-compact if and only if*

(a) *S does not have a smallest element, or*
(b) *$\inf A \leq \min S$ and S is not Euclidean closed and bounded, or*
(c) *$\min S < \inf A$ and there exists a sequence (s_n) in $S \cap (2\inf A - \min S, \infty)$ converging (in the Euclidean topology) to $x \in (2\inf A - \min S, \infty] - S$.*

Proof. If S does not have a minimum element $\min S \in S$, let (s_n) be a strictly decreasing sequence in S with $\bigcup\{(s_n, \infty) : n \in \mathbb{N}\} = S$. Now $\{(s_n, \infty) : n \in \mathbb{N}\}$ is an open cover of S with no finite subcover.

If $\inf A \leq \min S$, then S inherits the Euclidean topology from $P(A)$, so S is $P(A)$ compact if and only if S is Euclidean closed and bounded.

If $\min S < \inf A$ and there exists a sequence (s_n) in $S \cap (2\inf A - \min S, \infty)$ converging to $x \in (2\inf A - \min S, \infty) - S$, without loss of generality, we may assume (s_n) is strictly monotone.

If (s_n) is strictly decreasing converging to $x \geq 2\inf A - \min S$, we consider two cases: (i) $\inf A \in A$ or (ii) $\inf A \notin A$. In case (i), set $\varepsilon = x - \inf A$ and note that $C_0 = (\inf A - \varepsilon, \inf A + \varepsilon) = (2\inf A - x, x)$ and $x > 2\inf A - \min S$ implies C_0 covers $[\min S, x)$. Now $\{C_0\} \cup \{(s_n, \infty) : n \in \mathbb{N}\}$ is an open cover of S with no finite subcover. In case (ii), let $D = \inf A - \min S$, so $\inf A + D = 2\inf A - \min S$. Now $\delta = x - (2\inf A - \min S) = x - \inf A + D > 0$. Let (a_n) be a strictly decreasing sequence in $A \cap (\inf A, \inf A + \delta)$ converging to $\inf A$. With $\varepsilon = x - a_1$, $(a_1 - \varepsilon, a_1 + \varepsilon) = (a_1 - \varepsilon, x)$ is open. Since $x = \inf A + D + \delta$ and $a_1 < \inf A + \delta$, it follows that $\varepsilon = x - a_1 > D$. Say $\varepsilon = D + \beta$. For $a_n \in A \cap (\inf A, \inf A + \beta)$, we have $a_n - \varepsilon < \inf A - D = \min S < a_n + D < a_1 + D < x$. Thus, $C_0 = (a_n - \varepsilon, a_n + \varepsilon) \cup (a_1 - \varepsilon, a_1 + \varepsilon) = (a_n - \varepsilon, x)$ covers $[\min S, x)$. Now $\{C_0\} \cup \{(s_n, \infty) : n \in \mathbb{N}\}$ is an open cover of S with no finite subcover.

If (s_n) is strictly increasing and $\inf A \in A$, then with $\varepsilon_n = s_n - \inf A$, $\{(\inf A - \varepsilon_n, \inf A + \varepsilon_n) = (\inf A - \varepsilon_n, s_n) : n \in \mathbb{N}\} \cup \{(x, \infty)\}$ is an open cover of S with no finite subcover. If $\inf A \notin A$, pick (a_n) as in the previous paragraph and let $\varepsilon_n = s_n - a_n$. Then $\{(a_n - \varepsilon_n, a_n + \varepsilon_n) = (a_n - \varepsilon_n, s_n) : n \in \mathbb{N}\} \cup \{(x, \infty)\}$ is an open cover of S with no finite subcover.

To show the converse, we must show that S is compact if

(a′) $\min S$ exists, and
(b1′) $\min S < \inf A$ or (b2′) S is Euclidean compact, and
(c1′) $\inf A \leq \min S$ or (c2′) $\min S < \inf A$ and there is no sequence in $S \cap (2\inf A - \min S, \infty)$ converging to $x \in (2\inf A - \min S, \infty] - S$.

Note that (a′, b1′, c1′) cannot occur. If (b2′) occurs, since every $P(A)$-open set is Euclidean open, every $P(A)$-open cover of S has a finite subcover, so S is $P(A)$-compact. In the remaining case (a′, b1′, c2′), suppose $\mathcal{C}$ is an open cover of S by $P(A)$-basic open sets. For $C \in \mathcal{C}$ with $\min S \in C$, if C contains $[\min S, \infty)$, then $\{C\}$ is a finite subcover of S. Otherwise, $C = (\min S - \varepsilon', \infty) \cap (a - \varepsilon, a + \varepsilon)$ for some $\varepsilon', \varepsilon > 0$ and $a \in A$. Since $\min S \in (a - \varepsilon, a + \varepsilon)$, we have $\varepsilon > \inf A - \min S$, so $a + \varepsilon > \inf A + (\inf A - \min S)$ and C covers $[\min S, 2\inf A - \min S + \delta]$ for some $\delta > 0$. Now it only remains to show that $S' = S \cap [2\inf A - \min S + \delta, \infty)$ has a finite subcover from $\mathcal{C}$. The condition (c2′) implies that S' is Euclidean closed and bounded, and since $2\inf A - \min S > \inf A$, $P(A)$ restricted to S' is the Euclidean topology, so S' and thus S are $P(A)$-compact. □

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References

[1] A. Bouziad and E. Sukhacheva, *On Hattori spaces*, Comment. Math. Univ. Carolin. **58** (2), 213–223, 2017.

[2] A. Calderón-Villalobos and I. Sánches, *Hattori topologies on almost topological groups*, Topology Appl. **326**, 108411, 15pp., 2023.

[3] V. A. Chatyrko and Y. Hattori, *A poset of topologies on the set of real numbers*, Comment. Math. Univ. Carolin. **54** (2), 189–196, 2013.

[4] V. Chatyrko and A. Karassev, *Countable dense homogeneity and Hattori Spaces*, Q&A in Gen. Top. **39**, 73–87, 2021.

[5] P. Fletcher and W. F. Lindgren, *Quasi-uniformities with a transitive base*, Pacific J. Math. **43** (3), 619–631, 1972.

[6] R. Fox, *A short proof of the Junnila quasi-metrization theorem*, Proc. AMS **83** (3), 663–664, 1981.

[7] Y. Hattori, *Order and topological structures of posets of the formal balls on metric spaces*, Mem. Fac. Sci. Eng. Shimane Univ. Series B: Mathematical Science **43**, 13–26, 2010.

[8] H. H. Hung, *Quasi-metrizability*, Topology and its Applications **83**, 39–43, 1998.

[9] T. Khmyleva and E. Sukhacheva, *On linear hoemomorphisms of spaces of continuous functions on Hattori spaces*, Topology Appl. **281**, 107209, 8pp., 2020.

[10] R. D. Kopperman, *Which topologies are quasimetrizable?*, Topology and its Applications **52**, 99–107, 1993.

[11] H.-P. A. Künzi and F. Yildiz, *Extensions of T_0-quasi-metrics*, Acta Math. Hungar. **153** (1), 196–215, 2017.

[12] T. Richmond, *Hybrid topologies on the real line*, Applied General Topology **24** (1) 157–168, 2023.

[13] T. Richmond, *General Topology: An Introduction*, De Gruyter, 2020.

[14] L. A. Steen and J. A. Seeback, Jr., *Counterexamples in Topology*, (2nd edition) Springer-Verlag, 1970.