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RESEARCH ARTICLE

Q-HOMOTOPY SHEHU ANALYSIS TRANSFORM METHOD OF TIME-FRACTIONAL COUPLED BURGERS EQUATIONS

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ABSTRACT

In this study, numerical solutions to time-fractional coupled Burgers equations are obtained utilizing the q-homotopy Shehu analysis transform method. The definition of fractional derivatives in the sense of Caputo. q-homotopy Shehu analysis transform method is also used to find the numerical solutions of the time-fractional coupled Burgers equations. In addition, the MAPLE software is utilized to plot the graphs of the solutions. These results demonstrate that the presented method is accurate and simple to implement.

Keywords: q-homotopy Shehu analysis transform method, Coupled Burgers equations, Mittag-Leffler function

1. INTRODUCTION

Fractional differential equations (FDEs) have recently gained popularity in numerous scientific areas. [1-6]. The fractional technique is commonly utilized to simulate a variety of difficulties, and it is widely applied to numerous problems in mechanics, anomalous diffusion, wave propagation, and turbulence, among others [7-9]. Thus, numerous scientists research fractional calculus extensively and work to enhance it. One of the greatest benefits of utilizing FDEs with non-local properties is that they display novel properties for several difficulties. Nonlinear FDEs are challenging to solve for a variety of reasons. Because the majority of FDEs isn't be solved analytically, efficient and potent numerical approaches are devised. But the numbers of the approaches are fairly limited.

However, the fractional order varies based on time and space. The scenario leads to a rapidly expanding field of FPDEs with fractional operators of variable order [10-19]. Several potent numerical approaches were established in the scientific literature, and numerous eminent scholars contributed to this topic. Several of these techniques include the adomian decomposition method (ADM) [20], homotopy perturbation method (HPM) [21-23], homotopy analysis method (HAM) [24-25], the collocation method [26-31], the Sumudu transform method (STM) [32-33], the differential transform method (DTM) [34-38].

For analyzing differential equations in the time domain, the Shehu transform, a generalization of the Laplace and Sumudu integral transforms, is introduced [39]. Iterative Shehu transform method (ISTM) is developed, which leverages the Shehu transform (ST) approach and decomposes the nonlinearity component to create a simple and effective solution for solving FPDEs [40]. A two-dimensional version of the one-dimensional Shehu transform is developed [41]. The fuzzy Shehu transform technique (FSTM) is presented by employing Zadeh's decomposition theorem and fuzzy Riemann integral of real-valued functions on finite intervals [42]. Using the natural transform decomposition technique and the iterative Shehu transform method with a singular kernel derivative, the time-fractional Klein-Gordon

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equation was studied [43]. To solve time-fractional gas dynamics equations, the variational iteration transform method is combined with ST and VIM [44]. By using the established q-Shehu transform, the solutions of the q-fractional kinetic equations are found in terms of the constructed generalized hyper-Bessel function [45]. The notion of ST in q-calculus, q-Shehu transform, is introduced along with its features [46].

The time-fractional coupled Burgers equation is [47]

$$\begin{cases} \dfrac{\partial^\alpha u}{\partial \tau^\alpha} + u\dfrac{\partial u}{\partial \zeta} + w\dfrac{\partial u}{\partial \mu} = \dfrac{1}{Re}\left[\dfrac{\partial^\alpha u}{\partial \zeta^\alpha} + \dfrac{\partial^\alpha u}{\partial \mu^\alpha}\right], \\ \dfrac{\partial^\beta w}{\partial \tau^\beta} + u\dfrac{\partial w}{\partial \zeta} + w\dfrac{\partial w}{\partial \mu} = \dfrac{1}{Re}\left[\dfrac{\partial^\beta w}{\partial \zeta^\beta} + \dfrac{\partial^\beta w}{\partial \mu^\beta}\right], 0 < \zeta, \mu < 1, \quad 0 < \alpha, \beta < 1, \tau > 0. \end{cases} \tag{1}$$

The Burgers model of turbulence is an extremely influential fluid dynamics model. Various scientists have contemplated examining the theory and model of shock waves in order to get theoretical knowledge of a physical flow class and to assess various approximation approaches. Eq. (1) has the advantage over competing numerical formulations of viscous diffusion and nonlinear advection in that it is the simplest. It illustrates the fundamental forms of the dissipation term $w\frac{\partial u}{\partial \mu}$ and the nonlinear advection term $u\frac{\partial u}{\partial \zeta}$, where Re is the Reynolds number that is used to model the naturally occurring phenomena of wave motions and hence determine the behavior of the solution. Cole [48] examined the mathematical characteristics of Equation (1). In physics and applied mathematics, nonlinear phenomena play a significant role. The significance of getting actual or estimated outcomes of PDEs in science in terms of exploring new strategies; this is still a popular topic [49–52]. Utilizing nonlinear PDEs, a variety of strategies for obtaining the real results of the numerous physical models presented have been proposed. Bateman [53] created a notable model and identified its consistent results, that are illustrative of numerous viscous flows. Later, Burgers [48] proposed it as one of the class models describing mathematical turbulence problems. It was described by Hopf [54] in gas dynamics. In addition, they demonstrated separately that the Burgers equation may be solved for any initial condition. Numerical solutions for the one-dimensional Burgers equation was investigated [55]. Nonlinear convection and the viscosity terms unquestionably chasten the Navier–Stokes equation [56-57].

This article aims to introduce a new method, the q-homotopy Shehu analysis transform method, and to use it to get new numerical solutions for coupled Burgers equations.

The remainder of the study is detailed below. In Section 2, fundamental fractional derivative definitions and the Shehu transform of fractional derivatives are provided. In Section 3, q-homotopy Shehu analysis transform method is provided. In Section 4, the numerical solutions of the time-fractional coupled Burgers equations are presented. Section 5 introduces the conclusion.

2. PRELIMINARIES

Several fundamental definitions are provided in the part.

Definition 1 [4, 58-59]. The Riemann-Liouville fractional integral is described as

$$I^a f(x) = \begin{cases} \dfrac{1}{\Gamma(a)}\displaystyle\int_0^x (x-t)^{a-1} f(t)dt, & a > 0, x > 0, \\ I^0 f(x) = f(x), & a = 0. \end{cases} \tag{2}$$

Definition 2 [4, 58-59]. The Caputo fractional derivative (CFD) is given by

$$D^a f(x) = I^{a-n} D^n f(x) = \frac{1}{\Gamma(n-a)} \int_0^x (x-t)^{n-a-1} f^{(n)}(t) dt, \tag{3}$$

where $n-1 < a \leq n,\ n \in N,\ x > 0, f \in C_{-1}^n$.

Definition 3 [12]. The Mittag-Leffler function $\boldsymbol{E_a}$ is defined as

$$E_a(z) = \sum_{n=0}^{\infty} \frac{z^a}{\Gamma(na+1)}, a > 0. \tag{4}$$

Definition 4 [39]. The Shehu transform (ST) of the function $f(t)$ is given by

$$\mathrm{S}[f(t)] = V[s,u] = \int_0^\infty f(t) e^{-\frac{st}{u}} dt\,, s > 0, u > 0. \tag{5}$$

Definition 5 [39]. If $\mathrm{V}(s,u)$ is the ST of the function $f(t)$, then ST of CFD is defined by

$$\mathrm{S}[D^\alpha f(t)] = \left(\frac{\mathrm{s}}{\mathrm{u}}\right)^\alpha \mathrm{V}(s,u) - \sum_{k=0}^{n-1} \left(\frac{\mathrm{s}}{\mathrm{u}}\right)^{\alpha-k-1} f^{(k)}(0), n-1 < \alpha \leq n. \tag{6}$$

3. THE METHODOLOGY OF q-HOMOTOPY SHEHU ANALYSIS TRANSFORM METHOD

In this part, q-HSATM for nonlinear FPDEs is presented. In order to illustrate the technique for the suggested method, the nonlinear FPDEs are written in standard operator form

$$D_t^\alpha u(x,t) + Au(x,t) + Nu(x,t) = g(x,t), t > 0, n-1 < \alpha \leq n, \tag{7}$$

with the initial condition

$$u(x,0) = h(x), \tag{8}$$

where A is a linear operator, N is a nonlinear operator, $g(x,t)$ is a source term and D_t^α is a time-fractional derivative operator of order α.

Now, by performing Shehu transform on Eq. (7) and using the initial condition, it is acquired as

$$S[u(x,t)] - \sum_{k=0}^{n-1} \left(\frac{\mathrm{s}}{\mathrm{u}}\right)^{-k-1} \frac{\partial^k u(x,t)}{\partial t^k}|_{t=0} + \left(\frac{u}{s}\right)^\alpha S[Au(x,t) + Nu(x,t) \tag{9}$$

$$-g(x,t)] = 0. \tag{10}$$

The nonlinear operator by the assist of HAM for real function $\varphi(x,t;q)$ is defined as

$$N[\varphi(x,t;q)\,] = S[\varphi(x,t;q)\,] - \frac{u}{s} \varphi(x,t;q)\,(0^+) + \left(\frac{u}{s}\right)^\alpha \{S[Au(x,t) + Nu(x,t) \tag{11}$$

$$-g(x,t)]\}, \tag{12}$$

where $q\epsilon\left[0,\frac{1}{n}\right]$.
It is established a homotopy as follows

$$(1-nq)S\left[\varphi(x,t;q)-\frac{u}{s}\varphi(x,t;q)\,(0^{+})\right]=hqH(x,t)N[\varphi(x,t;q)], \tag{13}$$

where, $h\neq 0$ is an auxiliary parameter and S represents Shehu transform. For $q=0$ and $q=\frac{1}{n}$, the results in Eq. (13) are respectively provided:

$$\varphi(x,t;0)=u_0(x,t),\varphi\left(x,t;\frac{1}{n}\right)=u(x,t), \tag{14}$$

Therefore, by amplifying q from 0 to $\frac{1}{n}$, then the solution $\varphi(x,t;q)$ converges from $u_0(x,t)$ to the solution $u(x,t)$. Employing the Taylor theorem around q and expanding $\varphi(x,t;q)$ and then, it is obtained as

$$\varphi(x,t;q)=u_0(x,t)+\sum_{i=1}^{\infty}u_m(x,t)q^m, \tag{15}$$

where

$$u_m(x,t)=\frac{1}{m!}\frac{\partial^m\varphi(x,t;q)}{\partial q^m}\Big|_{q=0}. \tag{16}$$

Eq. (15) converges at $q=\frac{1}{n}$ for the appropriate $u_0(x,t)$, n and h. Then, we have one of the solutions of the original nonlinear equation of the form

$$u(x,t)=u_0(x,t)+\sum_{m=1}^{\infty}u_m(x,t)\left(\frac{1}{n}\right)^m. \tag{17}$$

If we differentiate the zeroth order deformation Eq. (13) m −times with respect to q and we divide by $m!$, respectively, then for $q=0$, it is obtained as

$$S[u_m(x,t)-k_m u_{m-1}(x,t)]=hH(x,t)\mathcal{R}_m(\vec{u}_{m-1}), \tag{18}$$

where the vectors are defined by

$$\vec{u}_m=\{u_0(x,t),u_1(x,t),\dots,u_m(x,t)\}. \tag{19}$$

When the inverse Shehu transform to Eq. (18) is applied, then it is obtained as

$$u_m(x,t)=k_m u_{m-1}(x,t)+hS^{-1}[H(x,t)\mathcal{R}_m(\vec{u}_{m-1})], \tag{20}$$

where

$$\mathcal{R}_m(\vec{u}_{m-1}) = \mathrm{S}[u_{m-1}(x,t)] - \left(1 - \frac{k_m}{n}\right)\frac{\mathrm{u}}{s}u_0(x,t), \tag{21}$$

$$+\left(\frac{u}{s}\right)^{\alpha}\mathrm{S}\big(Au_{m-1}(x,t) + H_{m-1}(x,t) - \mathrm{g}(x,t)\big), \tag{22}$$

and

$$k_m = \begin{cases} 0, & m \leq 1, \\ n, & m > 1. \end{cases} \tag{23}$$

Here, H_m is homotopy polynomial and presented as

$$H_{m-1} = \frac{1}{(m-1)!}\frac{\partial^{m-1}\varphi(x,t;q)}{\partial q^{m-1}}|_{q=0}, \tag{24}$$

and

$$\varphi(x,t;q) = \varphi_0 + q\varphi_1 + q^2\varphi_2 + \cdots \tag{25}$$

By using Eqs. (20-22), it is obtained

$$u_m(x,t) = (k_m + h)u_{m-1}(x,t) - \left(1 - \frac{k_m}{n}\right)\frac{\mathrm{u}}{s}u_0(x,t) \tag{26}$$

$$+hS^{-1}\left[\left(\left(\frac{u}{s}\right)^{\alpha}\mathrm{S}\big(Au_{m-1}(x,t) + H_{m-1}(x,t) - \mathrm{g}(x,t)\big)\right)\right]. \tag{27}$$

By utilizing q-HSATM, the series solution is defined by

$$u(x,t) = \sum_{i=0}^{\infty} u_m(x,t). \tag{28}$$

3.1. Convergence Analysis of q-HSATM Solutions

Theorem 1 (Uniqueness theorem) [60] The solution for the Eq. (7) obtained by q-HSATM is unique for every $\boldsymbol{\gamma} \in (\mathbf{0},\mathbf{1})$, where $\boldsymbol{\gamma} = (\boldsymbol{n}+\boldsymbol{h}) + \boldsymbol{h}(\boldsymbol{\epsilon}+\boldsymbol{\alpha})\boldsymbol{T}$.

Theorem 2 (Convergence theorem) [60] Let **X** be a Banach space and $\boldsymbol{F}: \boldsymbol{X} \to \boldsymbol{X}$ be a nonlinear mapping. Assume that

$$\|\boldsymbol{G}(\boldsymbol{a}) - \boldsymbol{G}(\boldsymbol{h})\| \leq \boldsymbol{\gamma}\|\boldsymbol{a} - \boldsymbol{h}\|, \forall \boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{X}, \tag{29}$$

then **G** has a fixed point in view of Banach fixed point theory **[61]**. Moreover, for the arbitrary selection of $\boldsymbol{a_0}, \boldsymbol{b_0} \in \boldsymbol{X}$, the sequence generated by the q-HSATM converges to fixed point of $\boldsymbol{G}$ and

$$\|\boldsymbol{w_m} - \boldsymbol{w_n}\| \leq \frac{\boldsymbol{\gamma^n}}{\mathbf{1}-\boldsymbol{\gamma}}\|\boldsymbol{w_1} - \boldsymbol{w_0}\|, \forall \boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{X}. \tag{30}$$

4. THE NUMERICAL SOLUTIONS OF THE TIME-FRACTIONAL COUPLED BURGERS EQUATIONS

Let us consider the nonlinear time-fractional coupled Burgers equations (TFCBEs) **[47]**

$$\begin{cases} \frac{\partial^\alpha u}{\partial \tau^\alpha} + u\frac{\partial u}{\partial \zeta} + w\frac{\partial u}{\partial \mu} = \frac{1}{Re}\left[\frac{\partial^2 u}{\partial \zeta^2} + \frac{\partial^2 u}{\partial \mu^2}\right], \\ \frac{\partial^\beta w}{\partial \tau^\beta} + u\frac{\partial w}{\partial \zeta} + w\frac{\partial w}{\partial \mu} = \frac{1}{Re}\left[\frac{\partial^2 w}{\partial \zeta^2} + \frac{\partial^2 w}{\partial \mu^2}\right], 0 < \zeta, \mu < 1, \quad 0 < \alpha, \beta < 1, \tau > 0. \end{cases} \tag{31}$$

with initial conditions

$$u(\zeta, \mu, 0) = \frac{3}{4} - \frac{1}{4(1 + \exp((Re/32)(-4\zeta + 4\mu)))}, \tag{32}$$

$$w(\zeta, \mu, 0) = \frac{3}{4} + \frac{1}{4(1 + \exp((Re/32)(-4\zeta + 4\mu)))}, \tag{33}$$

where, Re is Reynolds number.
Now, by applying Shehu transform to Eq. (31) and by using Eqs. (32)-(33), then it is obtained as

$$S[u(\zeta, \mu, \tau)] - \frac{u}{s} u(\zeta, \mu, 0) + \left(\frac{u}{s}\right)^\alpha S\left[u\frac{\partial u}{\partial \zeta} + w\frac{\partial u}{\partial \mu} - \frac{1}{Re}\left[\frac{\partial^2 u}{\partial \zeta^2} + \frac{\partial^2 u}{\partial \mu^2}\right]\right] = 0, \tag{34}$$

$$S[w(\zeta, \mu, \tau)] - \frac{u}{s} w(\zeta, \mu, 0) + \left(\frac{u}{s}\right)^\alpha S\left[u\frac{\partial w}{\partial \zeta} + w\frac{\partial w}{\partial \mu} - \frac{1}{Re}\left[\frac{\partial^2 w}{\partial \zeta^2} + \frac{\partial^2 w}{\partial \mu^2}\right]\right] = 0, \tag{35}$$

The nonlinear operators by using Eqs. (34)-(35) are defined as

$$\begin{aligned} &N^1[\varphi(\zeta, \mu, \tau; q), \psi(\zeta, \mu, \tau; q)] = S[\varphi(\zeta, \mu, \tau; q)] - \frac{u}{s} u(\zeta, \mu, 0) + \left(\frac{u}{s}\right)^\alpha \\ &\times \left\{S\left[\varphi(\zeta, \mu, \tau; q)\frac{\partial \varphi(\zeta,\mu,\tau;q)}{\partial \zeta} + \psi(\zeta, \mu, \tau; q)\frac{\partial \varphi(\zeta,\mu,\tau;q)}{\partial \mu} - \frac{1}{Re}\left[\frac{\partial^2 \varphi(\zeta,\mu,\tau;q)}{\partial \zeta^2} \right.\right.\right. \\ &\left.\left.\left. + \frac{\partial^2 \varphi(\zeta,\mu,\tau;q)}{\partial \mu^2}\right]\right]\right\}, \end{aligned} \tag{36}$$

$$\begin{aligned} &N^2[\varphi(\zeta, \mu, \tau; q), \psi(\zeta, \mu, \tau; q)] = S[\psi(\zeta, \mu, \tau; q)] - \frac{u}{s} w(\zeta, \mu, 0) + \left(\frac{u}{s}\right)^\alpha \\ &\times \left\{S\left[\varphi(\zeta, \mu, \tau; q)\frac{\partial \psi(\zeta,\mu,\tau;q)}{\partial \zeta} + \psi(\zeta, \mu, \tau; q)\frac{\partial \psi(\zeta,\mu,\tau;q)}{\partial \mu} - \frac{1}{Re}\left[\frac{\partial^2 \psi(\zeta,\mu,\tau;q)}{\partial \zeta^2} \right.\right.\right. \\ &\left.\left.\left. + \frac{\partial^2 \psi(\zeta,\mu,\tau;q)}{\partial \mu^2}\right]\right]\right\}. \end{aligned} \tag{37}$$

By applying the proposed algorithm, the $m - th$ order deformation equation is defined by

$$S[u_m(\zeta, \mu, \tau) - k_m u_{m-1}(\zeta, \mu, \tau)] = h\mathcal{R}_{1,m}(\vec{u}_{m-1}), \tag{38}$$

$$S[w_m(\zeta, \mu, \tau) - k_m w_{m-1}(\zeta, \mu, \tau)] = h\mathcal{R}_{2,m}(\vec{w}_{m-1}). \tag{39}$$

where

$$\boldsymbol{\mathcal{R}_{1,m}(\vec{u}_{m-1}) = S[\vec{u}_{m-1}(\zeta,\mu,\tau)] - \frac{u}{s}\mathbf{u}(\zeta,\mu,0) + \left(\frac{u}{s}\right)^{\alpha}}$$
$$\boldsymbol{\times \left\{S\left[\sum_{r=0}^{m-1} u_r \frac{\partial u_{m-1-r}}{\partial \zeta} + \sum_{r=0}^{m-1} w_r \frac{\partial u_{m-1-r}}{\partial \mu} - \frac{1}{Re}\left[\frac{\partial^2 u_{m-1}}{\partial \zeta^2} + \frac{\partial^2 u_{m-1}}{\partial \mu^2}\right]\right\}}, \tag{40}$$
$$\boldsymbol{\mathcal{R}_{2,m}(\vec{w}_{m-1}) = S[\vec{w}_{m-1}(\zeta,\mu,\tau)] - \frac{u}{s}\mathbf{w}(\zeta,\mu,0) + \left(\frac{u}{s}\right)^{\alpha}}$$
$$\boldsymbol{\times \left\{S\left[\sum_{r=0}^{m-1} u_r \frac{\partial w_{m-1-r}}{\partial \zeta} + \sum_{r=0}^{m-1} w_r \frac{\partial w_{m-1-r}}{\partial \mu} - \frac{1}{Re}\left[\frac{\partial^2 w_{m-1}}{\partial \zeta^2} + \frac{\partial^2 w_{m-1}}{\partial \mu^2}\right]\right\}}. \tag{41}$$

On applying inverse Shehu transform to Eqs. (38)-(39), then we have

$$u_m(\zeta,\mu,\tau) = k_m u_{m-1}(\zeta,\mu,\tau) + hS^{-1}\left[\mathcal{R}_{1,m}(\vec{u}_{m-1})\right], \tag{42}$$
$$w_m(\zeta,\mu,\tau) = k_m w_{m-1}(\zeta,\mu,\tau) + hS^{-1}\left[\mathcal{R}_{2,m}(\vec{w}_{m-1})\right]. \tag{43}$$

By the use of initial conditions, then it is obtained as

$$u_0(\zeta,\mu,\tau) = \frac{3}{4} - \frac{1}{4(1+\exp((\mathrm{Re}/32)(-4\zeta+4\mu)))}, \tag{44}$$
$$w_0(\zeta,\mu,\tau) = \frac{3}{4} + \frac{1}{4(1+\exp((\mathrm{Re}/32)(-4\zeta+4\mu)))}. \tag{45}$$

To find the values of $u_1(\zeta,\mu,\tau)$ and $w_1(\zeta,\mu,\tau)$, putting $m=1$ in Eqs. (42)-(43), then it is obtained as

$$u_1(\zeta,\mu,\tau) = \mathrm{hRe}\frac{\exp(-\mathrm{Re}(\zeta-\mu)/8)t^{\alpha}}{128(1+\exp(-\mathrm{Re}(\zeta-\mu)/8))^2\Gamma(\alpha+1)}, \tag{46}$$
$$w_1(\zeta,\mu,\tau) = -\mathrm{hRe}\frac{\exp(-\mathrm{Re}(\zeta-\mu)/8)t^{\beta}}{128(1+\exp(-\mathrm{Re}(\zeta-\mu)/8))^2\Gamma(\beta+1)}. \tag{47}$$

Similarly, to find values of $u_2(\zeta,\mu,\tau)$ and $w_2(\zeta,\mu,\tau)$, putting $m=2$ in Eqs. (42)-(43), then it is found as

$$u_2(\zeta,\mu,\tau) = (\mathrm{n}+\mathrm{h})\mathrm{hRe}\frac{\exp(-\mathrm{Re}(\zeta-\mu)/8)t^{\alpha}}{128(1+\exp(-\mathrm{Re}(\zeta-\mu)/8))^2\Gamma(\alpha+1)},$$
$$-\mathrm{h}^2\mathrm{Re}^2\frac{\exp(-\mathrm{Re}(\zeta-\mu)/8)}{4096(1+\exp(-\mathrm{Re}(\zeta-\mu)/8))^4\Gamma(\alpha+1)\Gamma(\beta+1)}$$
$$\times\left[\left(\frac{-\Gamma(\alpha+1)\Gamma(\beta+1)t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{\Gamma(\alpha+1)\Gamma(\beta+1)t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\right)\exp(-\mathrm{Re}(\zeta-\mu)/8)\right.$$
$$+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)(\exp(-\mathrm{Re}(\zeta-\mu)/4)-1)t^{2\alpha}}{\Gamma(2\alpha+1)}. \tag{48}$$
$$w_2(\zeta,\mu,\tau) = -(\mathrm{n}+\mathrm{h})\mathrm{hRe}\frac{\exp(-\mathrm{Re}(\zeta-\mu)/8)t^{\beta}}{128(1+\exp(-\mathrm{Re}(\zeta-\mu)/8))^2\Gamma(\beta+1)},$$
$$+\mathrm{h}^2(\mathrm{Re})^2\frac{\exp(-\mathrm{Re}(\zeta-\mu)/8)}{4096(1+\exp(-\mathrm{Re}(\zeta-\mu)/8))^4\Gamma(\alpha+1)\Gamma(\beta+1)}$$

$$\times\left[\left(\frac{-\Gamma(\alpha+1)\Gamma(\beta+1)t^{2\beta}}{\Gamma(2\beta+1)}+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\right)\exp(-\mathrm{Re}(\zeta-\mu)/8)\right.$$
$$\left.+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)(\exp(-\mathrm{Re}(\zeta-\mu)/4)-1)t^{2\beta}}{\Gamma(2\beta+1)}\right]. \tag{49}$$

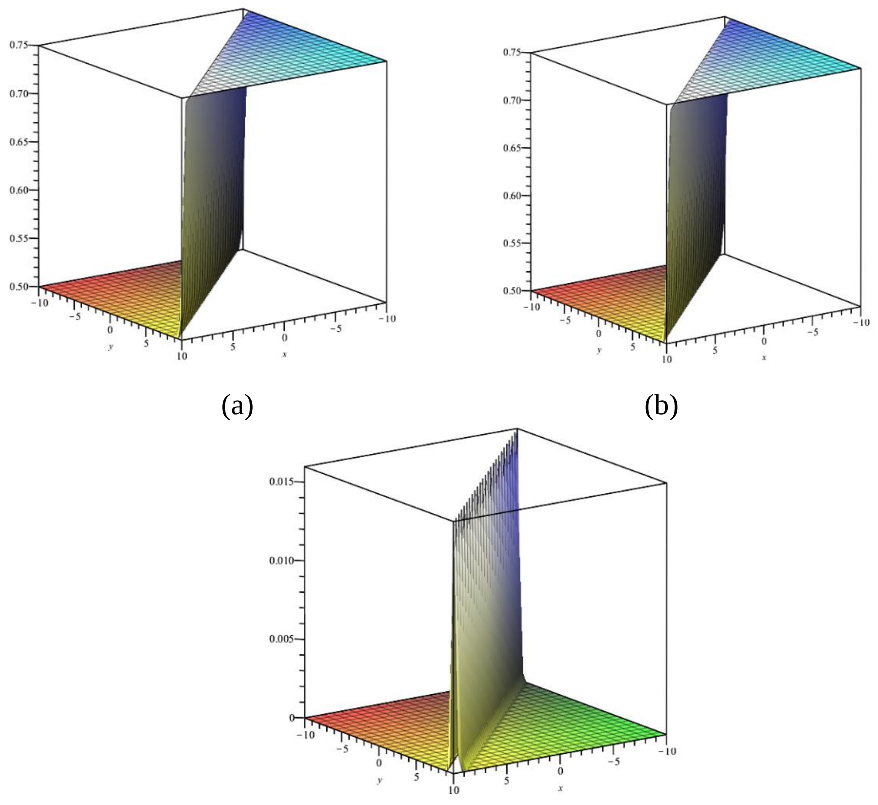
In this way, the other terms can be obtained. Thus, the q-HSATM solution of Eq. (31) is given by

$$\mathrm{u}(\zeta,\mu,\tau)=u_0(\zeta,\mu,\tau)+\sum_{m=1}^{\infty}u_m(\zeta,\mu,\tau)\left(\frac{1}{n}\right)^m, \tag{50}$$

$$\mathrm{w}(\zeta,\mu,\tau)=w_0(\zeta,\mu,\tau)+\sum_{m=1}^{\infty}w_m(\zeta,\mu,\tau)\left(\frac{1}{n}\right)^m. \tag{51}$$

Substituting the values of $\alpha=1,\beta=1,n=1,\ h=-1$ in Eqs. (50)-(51), then the obtained results $\sum_{m=1}^{M}u_m(\zeta,\mu,\tau)\left(\frac{1}{n}\right)^m$ and $\sum_{m=1}^{M}w_m(\zeta,\mu,\tau)\left(\frac{1}{n}\right)^m$ converge to the analytical solutions $\mathrm{u}(\zeta,\mu,\tau)=\frac{3}{4}-\frac{1}{4(1+\exp((\mathrm{Re}/32)(-4\zeta+4\mu-\tau)))}$ and $\mathrm{w}(\zeta,\mu,\tau)=\frac{3}{4}+\frac{1}{4(1+\exp((\mathrm{Re}/32)(-4\zeta+4\mu-\tau)))}$ of TFCBEs when $M\to\infty$.

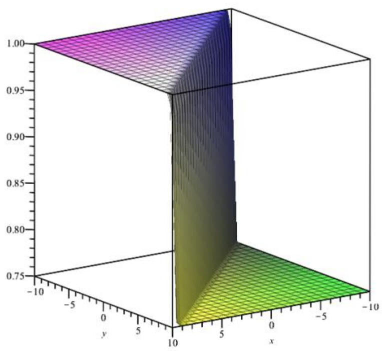
Figure 1 shows 3D graphs for the $\mathrm{u}(\zeta,\mu,\tau)$ solution of q-HSATM, the exact solution, and the absolute error for $\alpha=1,\ \beta=1$.



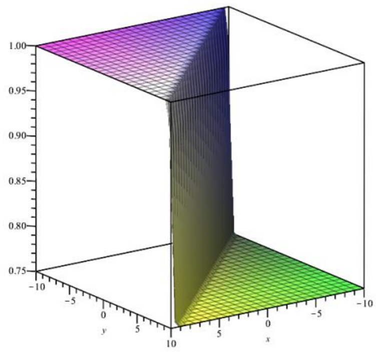
(a) (b) (c)

Figure 1. (a) $\mathrm{u}(\zeta,\mu,\tau)$ solution of q-HSATM **(b)** Exact solution of $\mathrm{u}(\zeta,\mu,\tau)$ **(c)** Nature of absolute error=$|u_{exact}-u_{q-HSATM}|$ at $Re=100,\mathrm{h}=-1,\mathrm{n}=1,\tau=0.5,\alpha=1,\beta=1$.

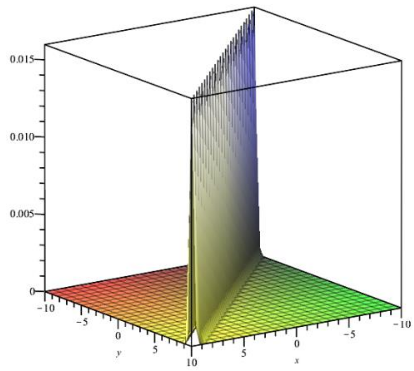
Figure 2 shows 3D graphs for the $\mathrm{w}(\zeta,\mu,\tau)$ solution of q-HSATM, the exact solution, and the absolute error for $\alpha = 1,\ \beta = 1$.



(a)



(b)



(c)

Figure 2. (a) $\mathrm{w}(\zeta,\mu,\tau)$ solution of q-HSATM **(b)** Exact solution of $\mathrm{w}(\zeta,\mu,\tau)$**(c)** Nature of absolute error=$\left|w_{exact} - w_{q-HSATM}\right|$ at $Re = 100, \mathrm{h} = -1, \mathrm{n} = 1, \tau = 0.5, \alpha = 1, \beta = 1$.

The 2D graphs for q-HSATM solution by varying α and β values are depicted in Figure 3.

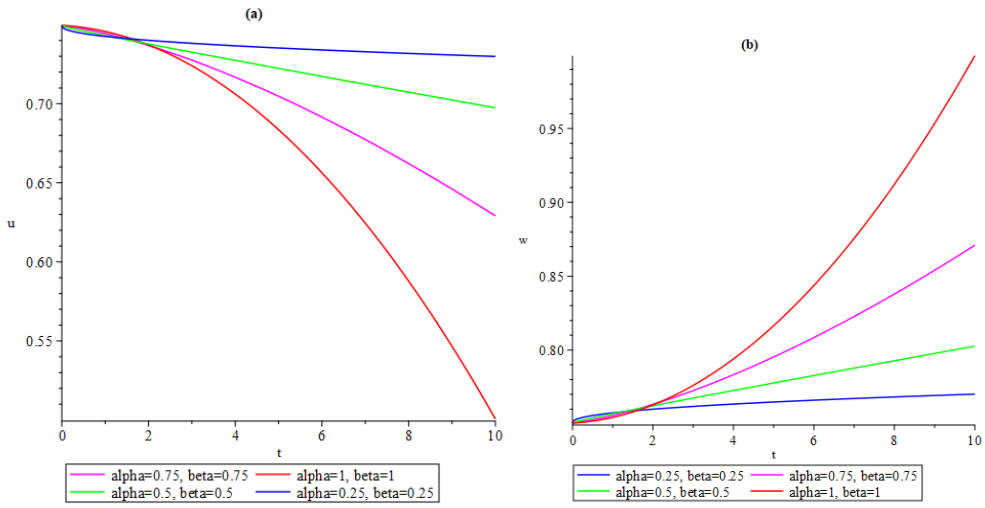


Figure 3. (a) $\mathrm{u}(\zeta,\mu,\tau)$ solution of q-HSATM with respect to t when $\mathrm{Re} = 100, \mathrm{x} = 0.5, \mathrm{y} = 1$ with distinct α and β. **(b)** $\mathrm{w}(\zeta,\mu,\tau)$ solution of q-HSATM with respect to t when $\mathrm{Re} = 100, \mathrm{x} = 0.5, \mathrm{y} = 1$ with distinct α and β.

q-HSATM is observed to be more efficient than FNTDM for $u(\zeta,\mu,\tau)$ numerical solution Table 1.

Table 1. Comparative study between FNTDM **[47]** and q-HSATM for the numierical solutions $u(\zeta,\mu,\tau)$ at $Re = 100,\ y = 1, \alpha = 1$ and $\beta = 1$.

ζ	τ	$\lvert u_{exact} - u_{FNTDM}\rvert$	$\lvert u_{exact} - u_{q-HSATM}\rvert$
0.1	0.1	8.4×10^{-5}	1.7×10^{-8}
	0.2	8.2×10^{-5}	1.5×10^{-7}
	0.3	7.8×10^{-5}	5.7×10^{-7}
	0.4	7.4×10^{-5}	1.4×10^{-6}
	0.5	6.9×10^{-5}	3.2×10^{-6}
0.2	0.1	9.9×10^{-4}	6.2×10^{-8}
	0.2	9.9×10^{-4}	5.4×10^{-7}
	0.3	9.7×10^{-4}	2.0×10^{-6}
	0.4	9.6×10^{-4}	5.2×10^{-6}
	0.5	9.4×10^{-4}	1.1×10^{-5}
0.3	0.1	1.1×10^{-2}	2.1×10^{-7}
	0.2	1.1×10^{-2}	1.8×10^{-6}
	0.3	1.1×10^{-2}	6.9×10^{-6}
	0.4	1.1×10^{-2}	1.8×10^{-5}
	0.5	1.1×10^{-2}	3.9×10^{-5}
0.4	0.1	1.3×10^{-1}	7.5×10^{-7}
	0.2	1.3×10^{-1}	6.5×10^{-6}
	0.3	1.3×10^{-1}	2.4×10^{-5}
	0.4	1.3×10^{-1}	6.3×10^{-5}
	0.5	1.3×10^{-1}	1.3×10^{-4}
0.5	0.1	1.5×10^{0}	2.6×10^{-6}
	0.2	1.5×10^{0}	2.2×10^{-5}
	0.3	1.5×10^{0}	8.3×10^{-5}
	0.4	1.5×10^{0}	2.1×10^{-4}
	0.5	1.5×10^{0}	4.6×10^{-4}

In Table 2, q-HSATM is observed to be more efficient than FNTDM for $w(\zeta,\mu,\tau)$ numerical solution.

Table 2. Comparative study between FNTDM **[47]** and q-HSATM for the numerical solutions $w(\zeta,\mu,\tau)$ at $Re = 100,\ y = 1, \alpha = 1$ and $\beta = 1$.

ζ	τ	$\lvert w_{exact} - w_{FNTDM}\rvert$	$\lvert w_{exact} - w_{q-HSATM}\rvert$
0.1	0.1	3.4×10^{-5}	1.7×10^{-8}
	0.2	1.3×10^{-5}	1.5×10^{-7}
	0.3	4.2×10^{-6}	5.7×10^{-7}
	0.4	2.8×10^{-6}	1.4×10^{-6}
	0.5	9.9×10^{-6}	3.2×10^{-6}
0.2	0.1	2.9×10^{-5}	6.2×10^{-8}
	0.2	1.5×10^{-6}	5.4×10^{-7}
	0.3	1.5×10^{-5}	2.0×10^{-6}
	0.4	3.3×10^{-5}	5.2×10^{-6}
	0.5	5.3×10^{-5}	1.1×10^{-5}
0.3	0.1	9.1×10^{-6}	2.1×10^{-7}
	0.2	4.1×10^{-5}	1.8×10^{-6}
	0.3	8.6×10^{-5}	6.9×10^{-6}
	0.4	1.3×10^{-4}	1.8×10^{-5}
	0.5	2.0×10^{-4}	3.9×10^{-5}
0.4	0.1	6.0×10^{-5}	7.5×10^{-7}
	0.2	1.8×10^{-4}	6.5×10^{-6}
	0.3	3.3×10^{-4}	2.4×10^{-5}
	0.4	5.0×10^{-4}	6.3×10^{-5}
	0.5	7.2×10^{-4}	1.3×10^{-4}
0.5	0.1	3.0×10^{-4}	2.6×10^{-6}
	0.2	7.0×10^{-4}	2.2×10^{-5}
	0.3	1.1×10^{-3}	8.3×10^{-5}
	0.4	1.7×10^{-3}	2.1×10^{-4}
	0.5	2.5×10^{-3}	4.6×10^{-4}

5. CONCLUSION

q-HSATM is used to analyze the coupled Burgers equations in this paper. In addition, MAPLE software was used to obtain the graphs of the numerical solutions of these equations for the various alpha and beta values. For coupled Burgers equations, it is observed that the general structure of surface graphs plotted in Maple software differs. Coupled Burgers equations for which numerical solutions have been quickly and successfully obtained. Therefore, it may be extrapolated that q-HSATM is overly effective and robust for obtaining numerical solutions for various fractional nonlinear partial differential equations.

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CONFLICT OF INTEREST

The authors stated that there are no conflicts of interest regarding the publication of this article.

AUTHORSHIP CONTRIBUTIONS

Autors' contributions are equal.

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