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LINEAR STATIC AND VIBRATION ANALYSIS OF CIRCULAR AND ANNULAR PLATES BY THE HARMONIC DIFFERENTIAL QUADRATURE (HDQ) METHOD

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ABSTRACT : *Numerical solution to static and free vibration analysis of thin circular and annular plates having various supports and load conditions are obtained by the method of Harmonic Differential Quadrature (HDQ). Bending moments, normal stress, and deflections of circular plate are found for uniformly distributed, central concentrated, and non-uniformly loads. Both clamped and simply supported edges are considered as boundary conditions. Results are obtained by using various number of grid points. The obtained results are compared with existing solutions available from other numerical methods and analytical results. The method presented gives efficient accurate results for the deflection and bending analysis of circular plates.*

KEYWORDS : *Harmonic differential quadrature; circular plates; annular plates; free vibration; deflection; bending moment; stress.*

DAİRESEL VE DELİKLİ PLAKLARIN HARMONİK DİFERANSİYEL QUADRATURE (HDQ) YÖNTEMİYLE LİNEER STATİK VE TİTREŞİM HESABI

ÖZET : *Çeşitli yük ve mesnet şartlarına sahip ince dairesel ve delikli plakların statik ve serbest titreşim hesabı için sayısal çözüm Harmonik Diferansiyel Quadratur(HDQ) yöntemiyle elde edilmiştir. Dairesel plağın eğilme momenti, normal gerilmeler ve yer değiştirmeleri düzgün yayılı yük, merkezi tekil yük ve üniform olmayan yük için bulunmuştur. Gerek ankastre mesnet ve gerekse basit mesnet, sınır şartları olarak göz önüne alınmıştır. Farklı düğüm nokta sayıları için sonuçlar elde edilmiştir. Elde edilen sonuçlar mevcut diğer sayısal yöntem sonuçları ve analitik sonuçlar ile karşılaştırılmıştır. Sunulan yöntem dairesel plakların yerdeğiştirme ve eğilme çözümleri için yeter doğrulukta sonuçlar vermiştir.*

ANAHTAR KELİMELER : *Harmonik diferansiyel quadrature; dairesel plak; delikli plak; serbest titreşim; deplasman; eğilme moment; gerilme.*

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I. INTRODUCTION

With the modern computer technology, various numerical methods were well developed and widely used to solve various kinds of engineering and science problems, which are described by the partial differential equations. However, it is not always possible to obtain the analytical mathematical solutions for engineering problems. In fact, rigorous mathematical solutions can be obtained only for certain simplified situations. For problems involving complex material properties, loading, and boundary conditions, the engineer introduces assumptions and idealizations deemed necessary to make the problem mathematically manageable but still capable of providing sufficient approximate solutions and satisfactory results from the point of view of safety and economy. The link between the real physical system and the mathematically feasible solution is provided by the mathematical model which is the symbolic designation for the substitute idealized system including all the assumptions on the physical problem.

Real physical systems or engineering problems are often described by partial differential equations, either linear or nonlinear and in most cases, their closed form solutions are extremely difficult to establish. As a result, approximate numerical methods have been widely used to solve partial differential equations which arise in almost all engineering disciplines. The most commonly used numerical methods for such applications are the finite element, finite difference and boundary element methods, and most engineering problems can be solved by these methods to satisfactory accuracy if a proper and sufficient number of grid points are used. However, in a large number of practical applications where only reasonably accurate solutions at few specified physical coordinates are of interest, the finite element or finite difference method becomes inappropriate since they still require a large number of grid points and so large a computer capacity, especially in the cases of nonlinear problems where iteration becomes inevitable. Consequently, both CPU time and storage requirements are often considerable for the standard methods.

There are many methods available in the literature to study the static and dynamic behavior of thin plates with different boundary and loading conditions. For details, one may refer Timoshenko and Krieger [1] etc. Exact solutions for plate problems are rather difficult to obtain, except for a few simple cases. In many cases, one may have to resort to various approximate namely numerical methods. Each method has its own advantages and disadvantages. Of the various methods proposed in recent times, one can cite the finite differences, finite and boundary element methods as the most efficient and universal methods for solving variant type plate problems [2,3,4,5,6]. In seeking a more efficient numerical method which requires fewer grid points yet achieves acceptable accuracy, the method of differential quadrature (DQ), which is based on the assumptions that the partial derivatives of a function in one direction can be expressed as a linear combination of the function values at all mesh points along that direction, was introduced by Bellman et al. [7]. The method of differential quadrature circumvents the above difficulties by computing a moderately accurate solution from only a few points. Since then, applications of differential quadrature method to various engineering problems have been investigated and their successes have demonstrated the potential of the method as an attractive numerical analysis technique [8,9,10].

In this study, the static and vibration analysis of circular, annular, and sectorial plates subjected to various loads and support conditions are investigated by using harmonic differential quadrature. The accuracy, efficiency and convenience of HDQ are demonstrated throughout the numerical examples. To the authors' knowledge, it is the first time the harmonic differential quadrature method has been successfully applied to thin, isotropic circular plate problems for the analysis of deflection and bending. Free vibration and stability analysis of circular plates have also obtained by using this method [11] before.

II. DIFFERENTIAL QUADRATURE METHOD (DQM)

The basic idea of the differential quadrature method is that the derivative of a function, with respect to a space variable at a given sampling point, is approximated as a weighted linear sum of the sampling points in the domain of that variable. As with the other numerical analysis techniques, such as finite element or finite difference methods, the DQM also transforms the given differential equation into a set of analogous algebraic equations in terms of the unknown function values at the reselected sampling points in the field domain. During recent years, the DQM has been largely promoted by Bert and associates who were the first to introduce the method as a tool for structural analysis [8]. Recent works of Bert and associates, mainly on the vibration analysis of plates, have contributed significantly to the development of the DQM. Various problems in structural mechanics have been solved successfully by this method [12,13,14,15]. All this work has demonstrated that the application of the DQ methods leads to accurate results with less computational effort and that there is a potential that the method may be become an alternative to the conventional methods such as finite differences and finite element. Therefore research on extension and application of the method becomes an important endeavor.

In the differential quadrature method, a partial derivative of a function with respect to a space variable at a discrete point is approximated as a weighted linear sum of the function values at all discrete points in the region of that variable. For simplicity, we consider a one-dimensional function $u(x)$ in the [-1,1] domain , and N discrete points. Then the first derivative at point i, at $x = x_i$ is given by

$$u_{,x}(x_i) = \left.\frac{\partial u}{\partial x}\right|_{x = x_i} = \sum_{j=1}^{N} A_{ij} \; u(x_j) \quad ; \qquad i = 1,2,.....,N \quad (1)$$

where x_j are the discrete points in the variable domain, $u(x_j)$ are the function values at these points and A_{ij} are the weighting coefficients for the first order derivative attached to these function values. Bellman [7] and Bert et al. [8,12] suggested two methods to determine the weighting coefficients. The first one is to let equation (1) be exact for the test functions

$$u_k(x) = x^{k-1}, \qquad k = 1,2,...,N \qquad (2)$$

which leads to a set of linear algebraic equations

$$(k-1)x_i^{k-2} = \sum_{j=1}^{N} A_{ij} x_j^{k-1} \qquad (3)$$

for $i = 1,2,........,N$ and $k = 1,2,........,N$

which represents N sets of N linear algebraic equations. This equation system has a unique solution because its matrix is of Vandermonde form. This equation may be solved for the weighting coefficients analytically using the Hamming's method [16] or numerical method using the certain special algorithms for Vandermonde equations, such as the method of Björck and Pareyra [17]. Equation (3) is also given following matrix form

$$\left\{ \partial u / \partial x \right\}_i = [A_{ij}] \left\{ u(x) \right\}_j \tag{4}$$

In order to reduce the complexity of the derivative approximation formulae and thereby conserve on computational effort, it is advantageous to use quadrature approximation formulae for also the second, third and higher order derivatives. Thus, the weighting coefficients for each formula will be different from those for the first-order derivative. As similar to the first order, the second order derivative can be written as

$$u_{,xx}(x_i) = \left. \frac{\partial^2 u}{\partial x^2} \right|_{x = x_i} = \sum_{j=1}^{N} B_{ij} \, u(x_j) \quad ; \qquad i = 1,2,.....,N \tag{5}$$

where the B_{ij} is the weighting coefficients for the second order derivative. Equation (5) also can be written

$$u_{,xx}(x_i) = \left. \frac{\partial^2 u}{\partial x^2} \right|_{x = x_i} = \sum_{j=1}^{N} A_{ij} \sum_{k=1}^{N} A_{jk} \, u(x_k) \quad ; \quad i = 1,2,........,N \tag{6}$$

Again, the function given by equation (2) is used so that the second order derivative is

$$(k-1)(k-2) x_i^{k-3} = \sum_{j=1}^{N} B_{ij} x_j^{k-1} \tag{7}$$

this can be solved in the same manner as indicated for equation (3) above. Weighting coefficients of the second, third and fourth order derivatives B_{ij}, C_{ij}, D_{ij}, can be obtained by following formulations;

$$D_{ij} = \sum_{k=1}^{N} A_{ik} C_{kj}$$

$$; \ C_{ij} = \sum_{k=1}^{N} A_{ik} B_{kj}; \quad B_{ij} = \sum_{k=1}^{N} A_{ik} A_{kj} \qquad (8,9,10)$$

The second method is to obtain the weighting coefficients is similar to the first one with the exception that a different set of trial or test function is chosen for satisfying equation (1) exactly;

$$u_k(x) = \frac{L_N(x)}{(x - x_k) L_N^{(1)}(x_k)}, \qquad k=1,2,..........,N \qquad (11)$$

where $L_N(x)$ is the Nth order Legendre polynomial and $L_N^{(1)}(x)$ the first order derivative of $L_N(x)$. N is the number of grid points as with the first one. However, it requires that x_k (k=1,2,....,N) have to be chosen to be roots of the shifted Legendre polynomial. This means that once number of grid points N is specified the roots of the shifted Legendre polynomial are given, thus the distribution of the grid points are fixed regardless of the physical problems being considered. By choosing x_k to be roots of the shifted Legendre polynomial and substituting equation (11) into equation (1), we obtained a direct simple algebraic expression for the weighting coefficients A_{ij}

$$A_{ij} = \frac{L'_N(x_i)}{(x_i - x_j) L'_N(x_j)}; \ \text{for} \ \ i \neq j \ \text{ and } \quad A_{ii} = \frac{1 - 2x_i}{2x_i(x_i - 1)}; \ \text{ for } \ i=j \qquad (12,13)$$

i, j= 1,2,.......,N

In this second approach, the weighting coefficients that was defined equation (12) and (13) are easy to obtain without solving algebraic equations or having a singularity problem as with the first one.

III. HARMONIC DIFFERENTIAL QUADRATURE (HDQ)

Despite the increasing application of the DQ method in structural analysis, a draw back regarding its ill conditioning of the weighting coefficients with increasing number of grid points used as well as the increasing order of derivatives was pointed out by

Bellman [7]. One way of overcoming this drawback was recently addressed by by Shu and Richards [14] . They introduced a recurrence relationship, which is used to generate weighting coefficients for any order derivatives from its first-order derivative weighting coefficient. These researchers had been called the generalized differential quadrature of their approximate. A recently approach the original differential quadrature approximation called the Harmonic differential quadrature (HDQ) has been proposed by Striz et all. [18]. Unlike the differential quadrature that uses the polynomial functions, such as Lagrange interpolated, and Legendre polynomials as the test functions, harmonic differential quadrature uses harmonic or trigonometric functions as the test functions. As the name of the test function suggested, this method is called the HDQ method. The harmonic test function $h_k(x)$ used in the HDQ method is defined as [19];

$$h_k(x) = \frac{N(x, x_k)}{P(x_k)} \qquad \text{for } k = 0,1,2,....,N \tag{14}$$

where

$$N(x, x_k) = \prod_{k=0}^{N} \sin\frac{x - x_k}{2}\pi\,; \quad P(x_k) = \prod_{i=0}^{N} \sin\frac{x_k - x_i}{2}\pi \tag{15,16}$$

According to the HDQ, the weighting coefficients of the first-order derivatives A_{ij} for $i \neq j$ can be obtained by using the following formula:

$$A_{ij} = \frac{(\pi/2)P(x_i)}{P(x_j)\sin[(x_i - x_j)/2]\pi}, \qquad i, j = 1,2,3,........,N \tag{17}$$

where

$$P(x_i) = \prod_{j=1, j\neq i}^{N} \sin\left(\frac{x_i - x_j}{2}\pi\right), \quad \text{for } j = 1,2,3,.......,N \tag{18}$$

The weighting coefficients of the second-order derivatives B_{ij} for $i \neq j$ can be obtained using following formula:

$$B_{ij} = A_{ij}\left[2\,A_{ii}^{(1)} - \pi ctg\left(\frac{x_i - x_j}{2}\right)\pi\right], \qquad i\,,j = 1,2,3,\ldots\ldots..,N \tag{19}$$

The weighting coefficients of the first-order and second-order derivatives $A_{ij}^{(p)}$ for $i = j$ are given as

$$A_{ii}^{(p)} = -\sum_{j=1,\, j \neq i}^{N} A_{ij}^{(p)}\ , \qquad p = 1 \text{ or } 2\ ; \text{ and for } i = 1,2,\ldots, N \tag{20}$$

It should be mentioned that in the differential quadrature solutions, the sampling points in the various coordinate directions may be different in number as well as in their type. A natural, an often convenient, choice for sampling points is that of equally spaced point. This type sampling points are given as

$$x_i = \frac{i-1}{N-1}; \qquad i = 1,2,\ldots..,N \tag{21}$$

Some times, the differential quadrature solutions deliver more accurate results with unequally spaced sampling points. A better choice for the positions of the grid points between the first and the last points at the opposite edges is that corresponding to the zeros of orthogonal polynomials such as; the zeros of Chebyshev polynomials [13]. Furthermore, another choice that is found to be even better than the Chebyshev and Legendre polynomials is the set of points proposed by Shu et al. [14,19]. These points are given as

$$x_i = \frac{1}{2}\left[1 - \cos\left(\frac{2i-1}{N-1}\right)\pi\right]; \qquad i = 1,2,\ldots..,N \tag{22}$$

in the related direction.

IV. NUMERICAL APPLICATIONS AND RESULTS

To verify the analytical formulation presented in the previous section, circular plates with two different types boundary and six type load conditions are considered.

Following, the governing differential equations for deflections of plates are presented. The present formulations are based on classical small deflection theory. Then, the harmonic differential quadrature method has been applied to the given differential equations.

IV.I. DEFLECTION ANALYSIS OF CIRCULAR PLATES

Consider a thin circular plate of uniform thickness subject to a general axisymmetric load (Figure 1). The governing differential equation for small deflection is given

$$\frac{d^4u}{dr^4}+\frac{2}{r}\left(\frac{d^3u}{dr^3}\right)-\frac{1}{r^2}\left(\frac{d^2u}{dr^2}\right)+\frac{1}{r^3}\left(\frac{du}{dr}\right)=\frac{q(r)}{D} \tag{23}$$

where D is the flexural rigidity, *q(r)* is the general axisymmetric load on the plate, r is the radial position and u the normal deflection of the circular plate. By normalizing of (23), we have obtain

$$\frac{1}{a^4}\left[\frac{d^4u}{dR^4}+\frac{2}{R}\left(\frac{d^3u}{dR^3}\right)-\frac{1}{R^2}\left(\frac{d^2u}{dR^2}\right)+\frac{1}{R^3}\left(\frac{du}{dR}\right)\right]=\frac{q(Ra)}{D} \tag{24}$$

Where $R = r / a$, and a is known the outside radius of the plate. The moments, stresses, and shear forces in the radial and tangential directions are given [20];

$$M_r=-D\left(\frac{d^2u}{dr^2}+\frac{v}{r}\frac{du}{dr}\right);\ M_\theta=-D\left(\frac{1}{r}\frac{du}{dr}+v\frac{d^2u}{dr^2}\right) \tag{25a,25b}$$

$$\sigma_r=-D\frac{12}{h^3}z\left[\frac{d^2u}{dr^2}+v\left(\frac{1}{r}\frac{du}{dr}+\frac{1}{r^2}\frac{d^2u}{d\theta^2}\right)\right];\sigma_\theta=-D\frac{12}{h^3}z\left[\frac{1}{r}\frac{du}{dr}+\frac{1}{r^2}\frac{d^2u}{d\theta^2}+v\frac{d^2u}{dr^2}\right] \tag{25c,25d}$$

$$Q_r=-D\frac{d}{dr}\left(\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}\right);Q_\theta=-D\frac{1}{r}\frac{d}{d\theta}\left(\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}\right) \tag{26a, 26b}$$

Where ν is the Poisson ratio, *D* is the flexural rigidity, h is the thickness of plate. In case the simply supported outside, the boundary conditions are

$$U = 0 \quad \text{at} \;\; R = 1 \tag{27}$$

$$\left(\frac{d^2 u}{d\,R^2} + \frac{v}{R}\frac{du}{dR} \right) = 0 \quad \text{at} \;\; R = 1 \tag{28}$$

In addition to above boundary conditions the regularity condition must be given for circular plates. This condition is necessary to assure that the plate slope is zero at the origin to avoid a singularity at this location. The regularity condition at the centre of the plate is given by

$$\frac{du}{dR} = 0 \quad \text{at} \;\; R = 0 \tag{29}$$

Applying the differential quadrature approximation to the normalized plate deflection equation, boundary and regularity conditions given by Eq. (24), (27)(28), and (29)

$$\sum_{j=1}^{N} D_{ij} u_j + \frac{2}{R_i} \sum_{j=1}^{N} C_{ij} u_j - \frac{1}{R_i^2} \sum_{j=1}^{N} B_{ij} u_j + \frac{1}{R_i^3} \sum_{j=1}^{N} A_{ij} u_j = \frac{q(R_i a)}{D} a^4 \tag{30}$$

for $\quad i = 2,3,....,(N\text{-}2)$

$$u_N = 0 \tag{31}$$

$$\sum_{j=1}^{N} B_{Nj} u_j + \frac{\upsilon}{R} \sum_{j=1}^{N} A_{Nj} u_j = 0 \tag{32}$$

$$\sum_{j=1}^{N} A_{1j} u_j = 0 \tag{33}$$

Notice that, we only keep the discretized equations for i = 2 to (N-2) in Eq.(30) because there one support condition at R = 0 and there are two boundary conditions at R = 1 point. The boundary conditions for a clamped outside edge are

$$u = 0 \qquad \text{at} \quad R = 1 \tag{34}$$

and

$$du/dR = 1 \quad \text{at} \quad R = 1 \tag{35}$$

Applying the differential quadrature approximation to this boundary conditions at each discrete point on the grid yields

$$u_N = 0 \quad \text{and} \quad \sum_{j=1}^{N} A_{Nj} u_j = 0 \tag{36,37}$$

where the repeated index j means summation from 1 to N.

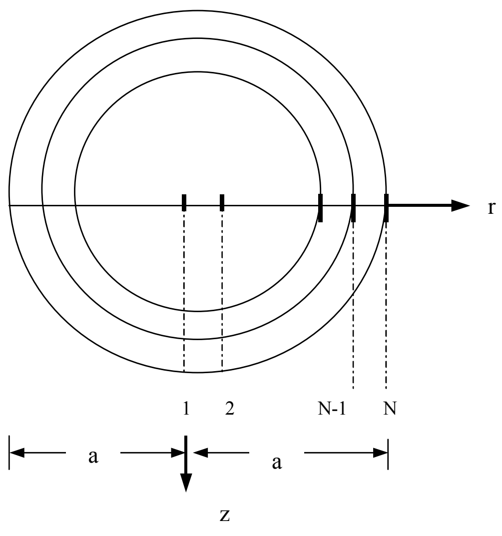

Figure 1. Grid spacing for a thin circular plate

In the numerical applications, different types of loads and support conditions are considered (Figure 2). Results are obtained for each case using various numbers of grid points. It is observed that the convergence of the method is very good. Following, several test examples for different loads and support conditions have been selected to demonstrate the convergence properties, accuracy and the simplicity in numerical implementation of the HDQ procedures. The numerical results for various example circular plate problems are tabulated (Table 1-3), and Figured (Fig. 3-6) and the comparison of the present results with the exact or other numerical values available in the literature, when possible, are made. Table 1 presents the results convergence study of the maximum moments for the simply supported and clamped supported circular plates of various Poisson's ratio, ν, subjected to a uniformly distributed load. The exact solutions of the moments reported by Timoshenko and Woinowsky-Krieger [1] are also given in related Table for the purpose of comparison. For the maximum moments five (N=5) grid points provide acceptable results with a maximum discrepancy of -2.2 % for Case 6. It can be seen that the present results are in excellent agreement with the exact values. The non-dimensionalized deflections obtained using the HDQ method are given in Table 2. The particular solutions of the deflections reported by Berktay [21] are also given in related Table for the purpose of comparison. For the deflections seven (N=7) grid points provide acceptable results with a maximum discrepancy of 2.85 % for Case 2. It can be seen that the present results are in excellent agreement with the exact values. The computational time on a standard PC is less than 5s for these examples.

Table 1. Maximum bending moment for various Poisson ratio (N=5)

ν	Clamped (C) (CASE 5)			Simply Supported (SS) (CASE 6)		
	*Mmax [qa^2] Exact (Ref. 1)	*Mmax[qa^2] HDQ	% Error	*Mmax[qa^2] Exact (Ref. 1)	*Mmax[qa^2] HDQ	% Error
0.0	0.0625	0.0625	0.000	0.1875	0.1875	0
0.2	0.0750	0.0748	0.267	0.2000	0.2008	-0.4
0.4	0.0875	0.0874	0.114	0.2125	0.2100	1.17
0.6	0.1000	0.0989	1.100	0.2250	0.2300	-2.2
0.8	0.1125	0.1126	0.088	0.2375	0.2375	0
1.0	0.1250	0.1250	0.000	0.2500	0.2500	0

*$Mmax= M_r\ max= M_\varphi\ max$

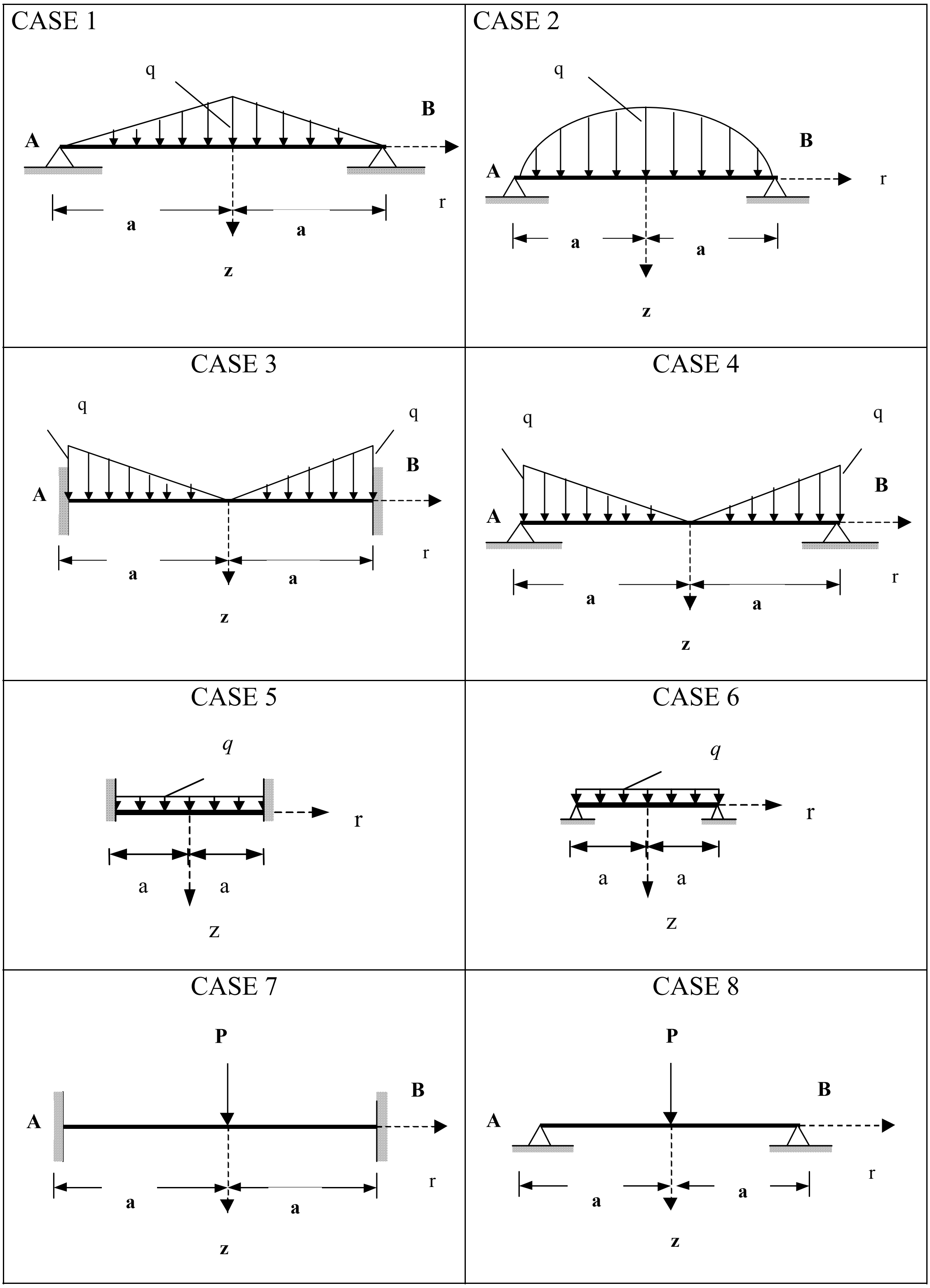

Figure 2. Load and support conditions using numerical applications

Table 2. Non-dimensionalized deflections for non-uniformly loaded plate (N=7)

	CASE 1		CASE 2	
r/a	u [100qa^4/D] Ref. 21	u [100qa^4/D] HDQ	u [100qa^4/D] Ref. 21	u 100qa^4/D] HDQ
0.0	-1.55555	-1.45317	-1.38888	-1.36994
0.1	-1.55535	-1.48356	-1.38732	-1.35206
0.3	-1.54397	-1.59810	-1.37627	-1.38019
0.5	-1.47184	-1.49521	-1.29386	-1.17451
0.7	-1.25509	-1.25802	-1.03414	-1.10948
0.9	-0.79289	-0.80954	-0.45548	-0.50311
1.0	0	0	0	0

Solving the set of combined algebraic equations (30), (31), (32), and (33), the non-dimensional deflections U at various grid points can be found for circular plate in case the simply supported boundary conditions (CASE 8). For clamped edge Eq. (36) and (37) will be used as boundary condition equations (CASE 7). Results are obtained for N=9 grid points using $\nu = 0.3$. Table 3 gives the results together with the exact analytical solutions for comparisons.

Table 3. Deflections for circular plate under central concentrated load (N=9)

	CASE 8			CASE 7		
			($\nu = 0.3$)			
r/a	u [$P/\pi D$] Exact (Ref.20)	u[$P/\pi D$] HDQ	% Error	u [$P/\pi D$] Exact (Ref.20)	u [$P/\pi D$] HDQ	% Error
0.0	0.1586	0.1585	0.123	0.0625	0.0625	0.0
2/10	0.1443	0.1444	-0.14	0.0520	0.0521	-0.33
4/10	0.1149	0.1152	-0.35	0.0342	0.0345	-0.59
6/10	0.0786	0.0783	0.25	0.0170	0.0168	1.17
8/10	0.0393	0.0393	0	0.0047	0.0047	0.0
1.0	0.0000	0.0000	0	0.0000	0.0000	0.0

Figures (3-6) demonstrate the influence of non-dimensional radial coordinate r/a on the bending moments, M_r, M_φ, for two different type support conditions. In this application, linearly varying load having symmetry with respect to the plate center (CASE3 and CASE 4) is taken into consideration. The method presented is shown to give excellent results with a small number of discrete points. It can be observed from these figures all the HDQ results agree with the exact results to within 3%. The exact solution can be found in the literature [20,21].

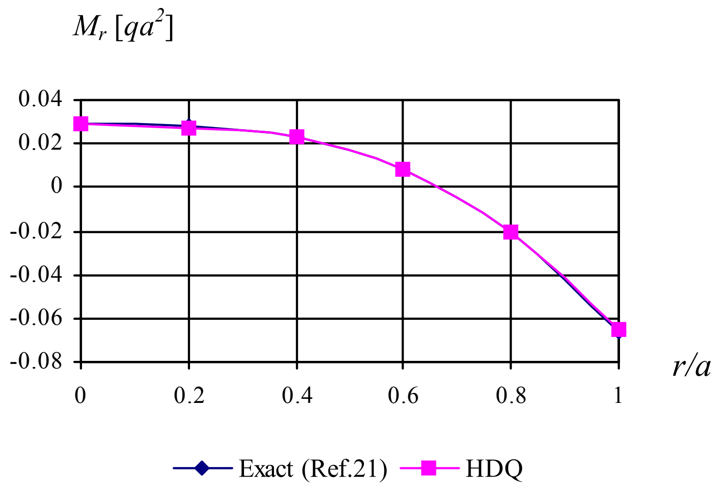

Figure 3. Bending moment in radial direction for clamped edge (CASE 3; ν = 0.3)

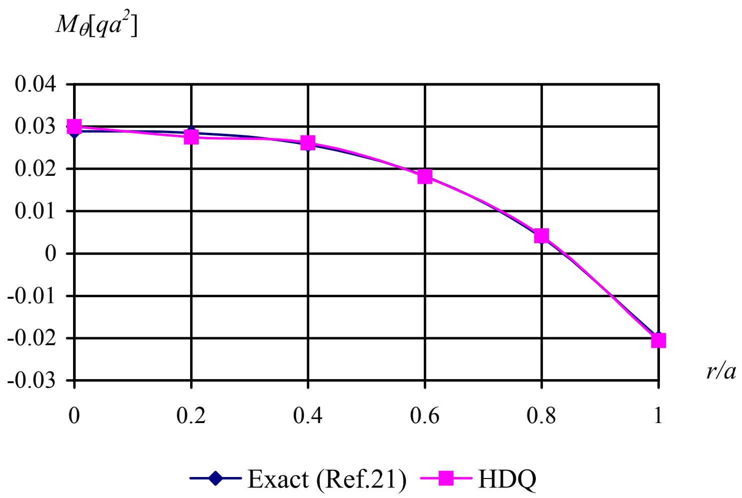

Figure 4. Bending moment in tangential direction for clamped edge (CASE 3; ν = 0.3)

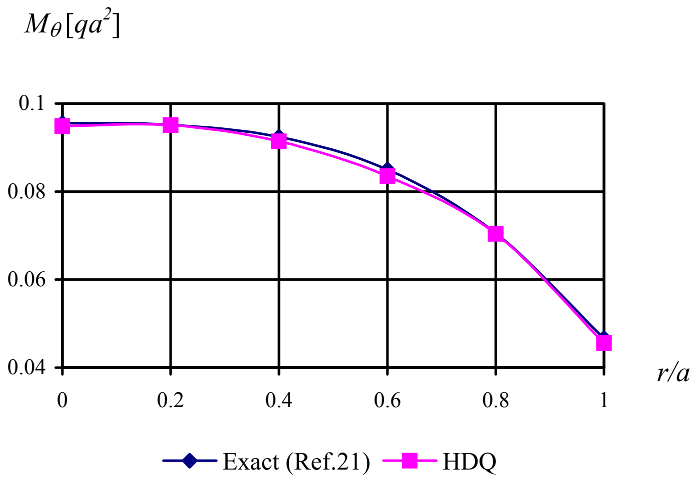

Figure 5. Bending moment in tangential direction for simply supported edge (CASE 4; ν = 0.3)

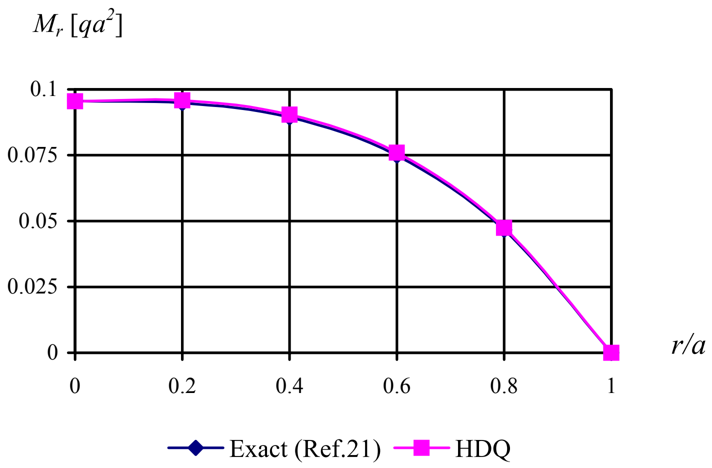

Figure 6. Bending moment in radial direction for simply supported edge (CASE 4; ν = 0.3)

Figures (7-11) demonstrate the influence of non-dimensional radial coordinate *r/a* on the stresses, σ_r, σ_θ for two different type support conditions. In this application, uniformly distributed load (CASE 5 and CASE 6) and central concentrated load (CASE 7 and CASE 8) are taken into consideration. The method presented is shown to give excellent results with a small number of discrete points. It can be observed from these figures all the HDQ results agree with the exact results. Results obtained from polar finite differences method are indicated by FD. For the FD solution 13×13 grid size is used. As can be seen, the HDQ results compare very well with the analytical solutions for only 9×9 grid points. Exact solutions are found in [1,3,6]. The variation of the error with number of grid points was shown in Fig.12 for HDQ and FD method. The percentage error had been reduced as parallel to the increase the grid points. In this figure S-S-S-S support are taken as boundary conditions for uniformly distributed load (CASE-6). The best solution is obtained for 9×9 grid sizes by using HDQ method.

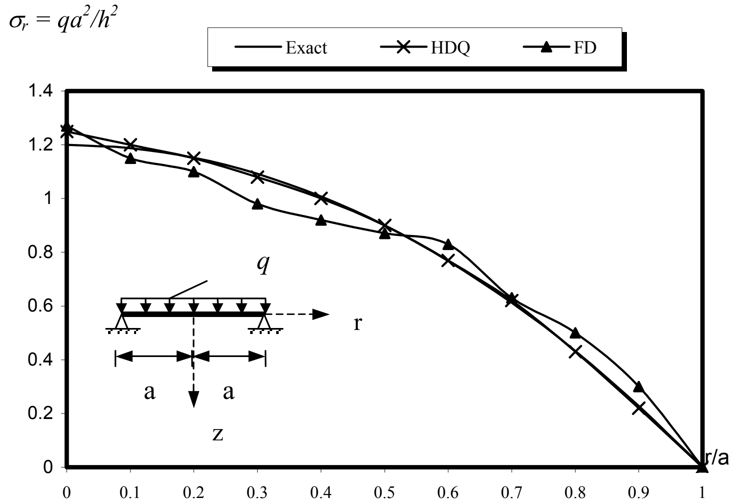

Figure 7. Stress value in radial direction for simply supported plate (ν = 0.2)

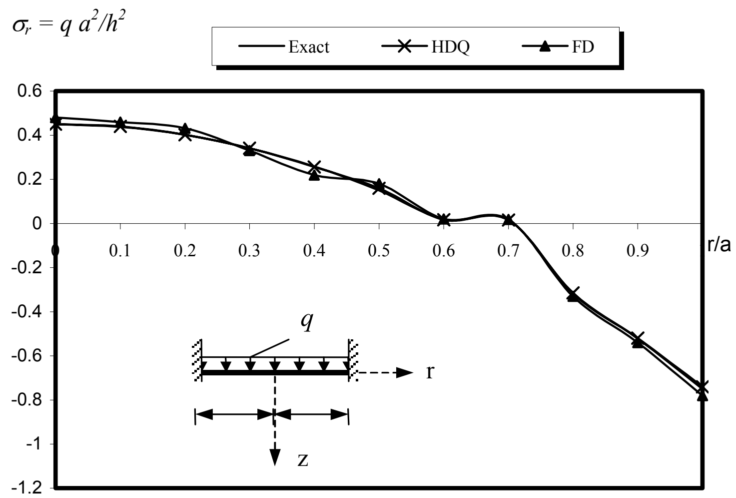

Figure 8. Stress value in radial direction for clamped plate (ν = 0.2)

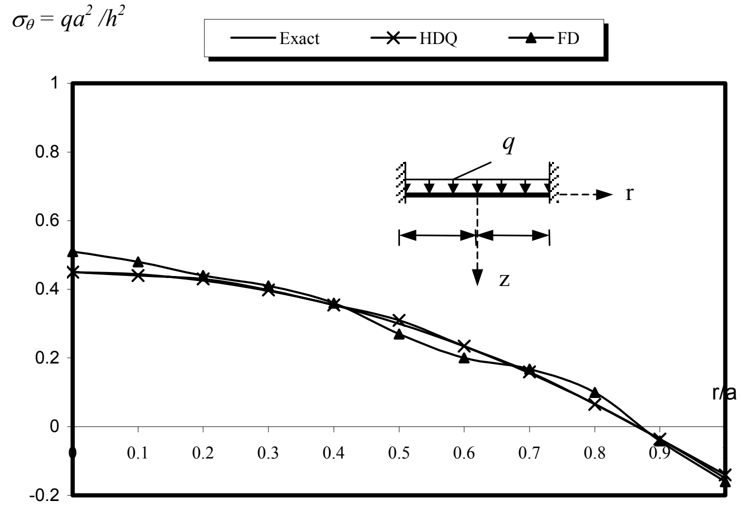

Figure 9. Stress value in tangential direction for clamped plate (ν = 0.2)

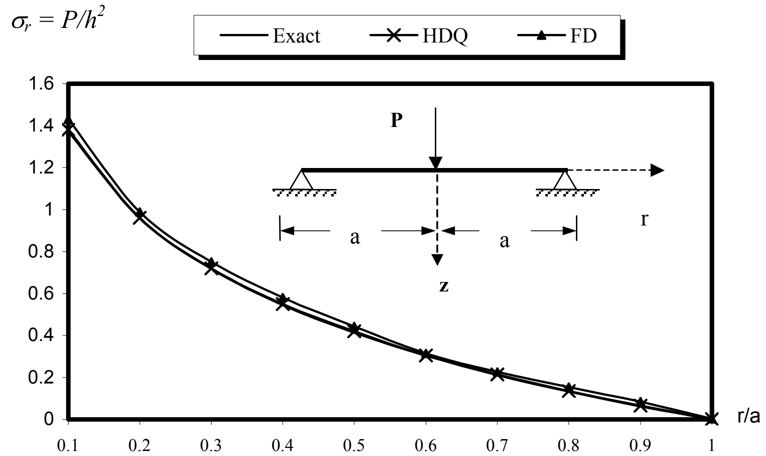

Figure 10. Stress value in radial direction for simply supported plate (ν = 0.25)

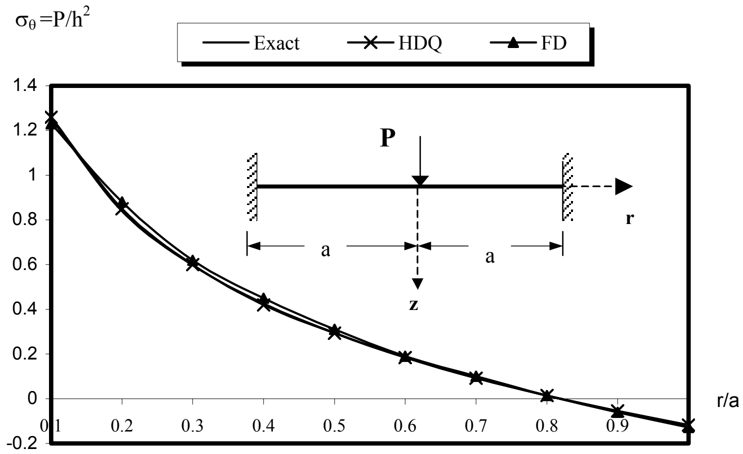

Figure 11. Stress value in tangential direction for clamped plate ($\nu = 0.25$)

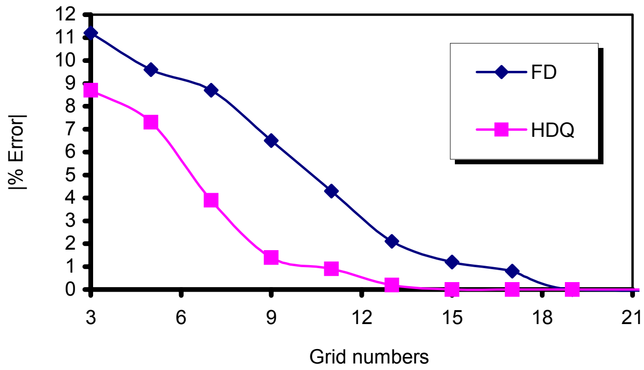

Fig. 12. Percentage error with grid numbers for uniformly loaded simply supported plates (CASE-6; $\nu = 0.2$)

V. DQ METHOD FOR IRREGULAR DOMAINS

Furthermore, let the field of interest be a curvilinear quadrilateral domain in the Cartesian *x-y* plane, as shown in Fig. 13(a). The geometry of this plate can be mapped into a square region in the natural ξ-η plane, as shown in Fig. 13(b).

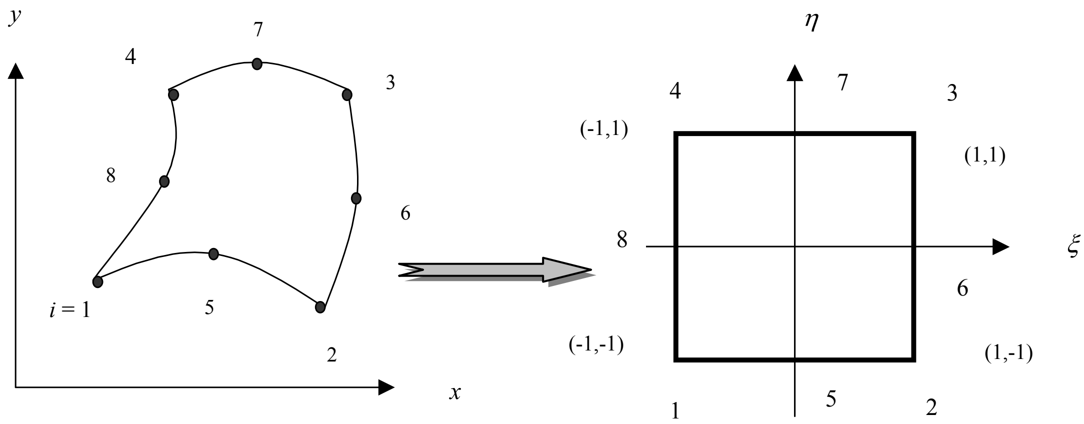

Figure 13(a). Curvilinear quadrilateral domain, (b) standard square domain

In this case, interpolation or shape functions are given for grid i;

$$\Phi_i(\xi,\eta) = \frac{1}{4}(1+\xi\,\xi_i)(1+\eta\,\eta_i)(-1+\xi\,\xi_i+\eta\,\eta_i) \quad \text{for } i = 1,3,5,7 \tag{38a}$$

$$\Phi_i(\xi,\eta) = \frac{1}{2}(1-\eta^2)(1+\xi\,\xi_i) \quad \text{for } i = 4 \text{ and } 8 \tag{38b}$$

$$\Phi_i(\xi,\eta) = \frac{1}{2}(1-\xi^2)(1+\eta\,\eta_i) \quad \text{for } i = 2 \text{ and } 6 \tag{38c}$$

Thus, the following equations are used for the coordinate transformation.

$$x = \sum_{i=1}^{8} x_i \Phi_i(\xi,\eta) \quad \text{and} \quad y = \sum_{i=1}^{8} y_i \Phi_i(\xi,\eta) \tag{39,40}$$

Using the chain rule, the first-order, and second order derivatives of a function are given

$$\begin{Bmatrix} u_x \\ u_y \end{Bmatrix} = J_1^{-1} \begin{Bmatrix} u_\xi \\ u_\eta \end{Bmatrix} \tag{41}$$

$$\begin{Bmatrix} u_{xx} \\ u_{yy} \\ 2u_{yx} \end{Bmatrix} = J_3^{-1} \begin{Bmatrix} u_{\xi\xi} \\ u_{\eta\eta} \\ 2u_{\xi\eta} \end{Bmatrix} - J_3^{-1} J_2 J_1^{-1} \begin{Bmatrix} u_\xi \\ u_\eta \end{Bmatrix} \tag{42}$$

where ξ_i and η_i are the coordinates of Node i in the ξ-η plane, and J_1 , J_2, and J_3 are the Jacobian matrices. These are expressed as follows;

$$J_1 = \begin{bmatrix} x_\xi & y_\xi \\ x_\eta & y_\eta \end{bmatrix};\quad J_2 = \begin{bmatrix} x_{\xi\xi} & y_{\xi\xi} \\ x_{\eta\eta} & y_{\eta\eta} \\ x_{\xi\eta} & y_{\xi\eta} \end{bmatrix};\quad J_3 = \begin{bmatrix} {x_\xi}^2 & {y_\xi}^2 & x_\xi y_\xi \\ {x_\eta}^2 & {y_\eta}^2 & x_\eta y_\eta \\ x_\xi x_\eta & y_\xi y_\eta & \frac{1}{2}(x_\xi y_\eta + x_\eta y_\xi) \end{bmatrix} \tag{43}$$

Bert and Malik give the detailed information about irregular element for DQ [22]. More detailed information can be found in this reference. Details of the transformation process of the DQ method as applied to straight-sided quadrilateral and curvilinear domains are not elaborated here but can be found elsewhere in the literature [22,23,24,25]. In the numerical applications, simply supported (SS) and clamped (C) boundary conditions are considered. At any point of either type of support, the deflection is zero

$$U = 0 \tag{44}$$

For simply supported edge, the normal moment must be zero

$$(\cos^2\alpha + v\sin^2\alpha)\frac{\partial^2 U}{\partial X^2} + (\sin^2\alpha + v\cos^2\alpha)\frac{\partial^2 U}{\partial Y^2} + \left[2(1-v)\cos\alpha\sin\alpha\right]\frac{\partial^2 U}{\partial X\partial Y} = 0 \tag{45}$$

and at a clamped edge, there is no rotation normal to the edge, thus

$$\cos\alpha\frac{\partial U}{\partial X} + \sin\alpha\frac{\partial U}{\partial Y} = 0 \tag{46}$$

where, α is the angle between the normal to the plate boundary and the *x*-axes. In the DQ form, Eqn (44), Eqn (45) and Eqn (46) are given as;

$$U_i = 0 \tag{47}$$

$$(\cos^2\alpha + v\sin^2\alpha)\sum_{j=1}^{N_\xi} B^{\xi}_{mj}U_j + (\sin^2\alpha + v\cos^2\alpha)\sum_{j=1}^{N_\eta} B^{\eta}_{nj}U_j +$$

$$+\left[2(1-v)\cos\alpha\sin\alpha\right]\sum_{j=1}^{N_\xi} A^{\xi}_{mj}\sum_{j=1}^{N_\eta} A^{\eta}_{nj}U_j = 0 \quad (48)$$

$$\cos\alpha\sum_{j=1}^{N_\xi} A^{\xi}_{mj} + \sin\alpha\sum_{j=1}^{N_\eta} A^{\eta}_{nj}U_j = 0 \quad (49)$$

For the straight-sided quadrilateral and curvilinear domains the sampling points are taken as

$$\xi_i = \frac{1}{2}\left[1-\cos\left(\frac{i-1}{N_\xi - 1}\right)\pi\right]; \qquad \eta_j = \frac{1}{2}\left[1-\cos\left(\frac{j-1}{N_\eta - 1}\right)\pi\right]$$

$$i = 1,2,.....,N_\xi \quad \text{and} \quad j = 1,2,.....,N_\eta$$

V.I. FREE VIBRATION ANALYSIS OF ECCENTRIC SECTORIAL PLATE

Consider, free vibration analysis of eccentric sectorial plate as shown in Fig.14.

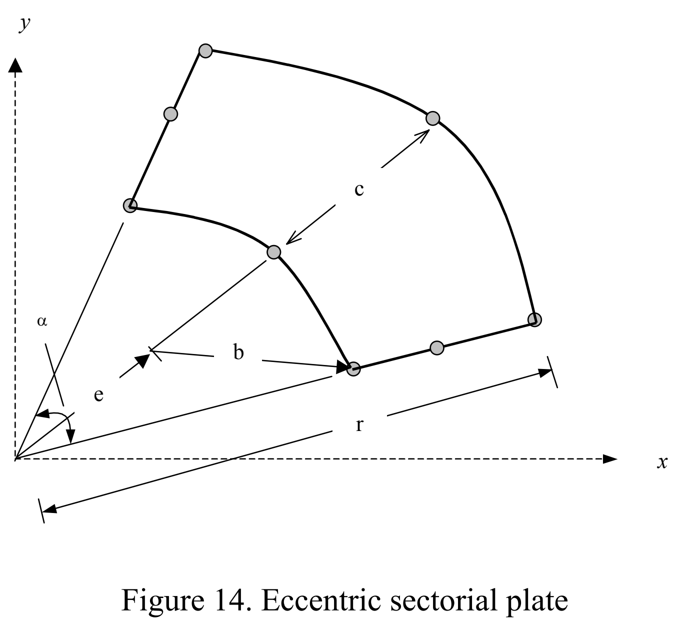

Figure 14. Eccentric sectorial plate

The governing differential equations of free vibration of a thin plate is given in a non-dimensional form as

$$\frac{\partial^4 U}{\partial X^4} + 2k^2 \frac{\partial^4 U}{\partial X^2 \partial Y^2} + k^4 \frac{\partial^4 U}{\partial Y^4} = \Omega^2 U \qquad (50)$$

where U is the dimensionless mode function of the vibration, Ω is the dimensionless frequency and its given $\Omega = \omega^2 a^4 \rho h/D$, $X = x/a$, $Y = y/b$ are the dimensionless coordinates, a and b are the dimensions of the plate as parallel to x-axis and y-axis, $k = a/b$ is ratio of the plate edge length or aspect ratio, ρ the mass density of using material for plate, h the uniform plate thickness, ω the natural frequency, and D denotes the flexural rigidity of plates and its given $D = Eh^3/12(1-\nu^2)$, ν the Poisson' ratio, E the modulus of elasticity of the plate material. Equation (50) can be given by applying the DQM as

$$\sum_{k=1}^{N_\xi} D_{ik}^{\xi} U_{kj} + 2k^2 \sum_{k=1}^{N_\xi} B_{ik}^{\xi} \sum_{m=1}^{N_\eta} B_{jm}^{\eta} U_{km} + k^4 \sum_{k=1}^{N_\eta} D_{jk}^{\eta} U_{ik} = \Omega^2 U_{ij} \qquad (51)$$

for $i = 1,2,\ldots\ldots,N_\xi$ and $j = 1,2,\ldots\ldots,N_\eta$

where D_{ik}^{ξ} D_{jk}^{η} B_{ik}^{ξ} B_{jm}^{η} represent the weighting coefficients of the fourth and second order derivatives along ξ-and η-directions for the differential quadrature approximation. For the simply supported all over the four edges plate 15×15 grid points can produce quite accurate results. The Polar finite differences (FD) results of the vibration for first two frequencies are also given in Table 4 for the purpose of comparison. For the FD solution 19×19 grid size is used. The required time on a Pentium-II with 64 MB RAM is less than 2s for HDQ. But, the required computing time is 5s for the FD solution. It should be noted that the HDQ solution converges at a smaller grid size as compared to the FD solution.

Table 9. Comparison for HDQ and DQ solutions for first three non-dimensional[a] vibration frequencies of sectorial plates (r/b = 2.0; α = 45; S-S-S-S)

	HDQ ($N_\xi = N_\eta = 15$)			Polar Mesh FD (Ref.20)			Ref.26		
	Ω_1	Ω_2	Ω_3	Ω_1	Ω_2	Ω_3	Ω_1	Ω_2	Ω_3
e/b =0	68.380	150.964	189.594	153.851	148.730	192.54	68.379	150.98	189.60

[a] $\Omega = \omega a^2 \sqrt{\rho h D}$

V.II. FREE VIBRATION ANALYSIS OF ANNULAR PLATES

In this example, we solve the free vibration of simply supported circular plates with a concentric circular hole, so-called annular plates (Figure 15). The inner and outer radii of an annular plate are denoted by *b* and *a*, respectively.

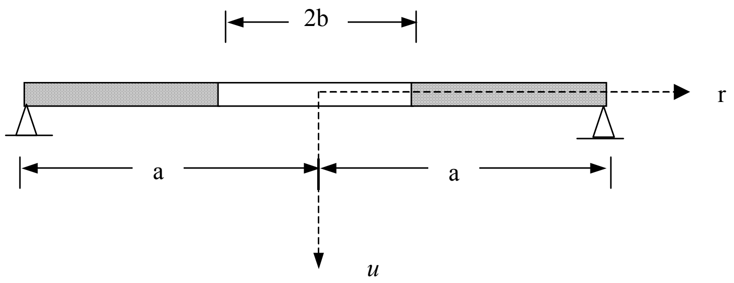

Figure 15. Simply supported annular plates

The governing differential equation of circular annular plates undergoing axisymmetric free vibration and its DQ form are given by

$$U_{RRRR} + \frac{2}{R} U_{RRR} - \frac{1}{R^2} U_{RR} + \frac{1}{R^3} U_R - \Psi^2 U = 0 \tag{52}$$

$$\sum_{j=1}^{N} D_{ij} U_j + \frac{2}{R_i} \sum_{j=1}^{N} C_{ij} U_j - \frac{1}{R_i^2} \sum_{j=1}^{N} B_{ij} U_j + \frac{1}{R_i^3} \sum_{j=1}^{N} A_{ij} U_j - \Psi^2 U_i = 0 \tag{53}$$

i=1,2,.....,(*N-2*)

where, R = r/a, ψ is the non-dimensional frequency and its given by $\Psi = \omega a^2 \sqrt{\rho h/D}$, *D* is the flexural rigidity of the annular plate, *h* is the thickness, ρ is the density of the plate material, and ω is the circular frequency. Considering the simply supported annular plate. That is

$$U = 0 \quad \text{at} \quad R = 1 \tag{54}$$

and

$$\frac{\partial^2 U}{\partial R^2} + \frac{\nu}{R}\frac{\partial U}{\partial R} \quad \text{at} \qquad R = 1 \tag{55}$$

Notice that, we only keep the discretized equations for i = 1 to (N-2) in Eq. (53) because there are two boundary conditions at R = 1 point. The non-dimensional fundamental frequencies of annular plates with simply supported boundary conditions obtained by HDQ shown in Table 10. The non-dimensional fundamental frequency given by Leissa [22] is also presented in this table for comparison. It is shown that, the results compare very well with the solution of Leissa [27]. As observed from Table 10, the obtained fundamental frequencies found by using 13 uniformly spaced grid points are very accurate.

Table10. Non-dimensional* fundamental frequency of annular plates

b/a	Leissa,1969 (Ref.27)	HDQ (13×13)
0.1	14.44	14.44
0.2	17.39	17.4
0.3	21.1	21.12
0.4	28.25	28.26
0.5	40.01	40.04

$^{*}\Psi = \omega a^2 \sqrt{\rho h/D}$

V.III. HYDROSTATICALLY LOADED CIRCULAR PLATES

As the last example, let be consider the simply supported circular plate under a hydrostatic load (Figure 16). Shearing forces are obtained in this example in both the radial and the tangential direction. The obtained results are summarized in Table 11.

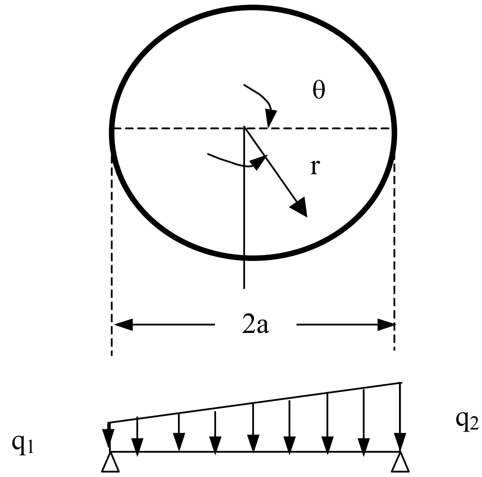

Figure 16. Hydrostatic load

Table 11. Obtained shearing forces in radial and tangential direction (ν = 0.3)

	FEM (N=13) (Ref. 2)		HDQ (N=11)	
r/a	$Q_r = (qa)\cos\theta$	$Q_\theta = (qa)\sin\theta$	$Q_r = (qa)\cos\theta$	$Q_\theta = (qa)\sin\theta$
0	0.136	0.0001	0.132	0
0.1	0.133	-0.0128	0.131	-0.0132
0.2	0.114	-0.0266	0.120	-0.0257
0.3	0.114	-0.0369	0.100	-0.0365
0.4	0.078	-0.0462	0.074	-0.0455
0.5	0.043	-0.0516	0.041	-0.0512

FEM results are also given in Table 11 for comparison. FEM results are obtained using 13 grid points. These results are found in Ref.2 [2]. The exact results can be calculated as 0.040 for r/a =0.5 in the famous book of Timoshenko and Krieger [1] for the shear force in radial direction (Q_r). 0.041 value is obtained by the HDQ method for this point (r/a=0.5) using N=11 grid points. However, 0.043 value has been found by FEM using N=13 grid points. For this value the HDQ method provide acceptable results with a maximum discrepancy of 2.5 % for equally spaced grid points. This discrepancy is being 7 % for FEM. The HDQ results are closer to the results obtained based on the analytical solutions [1], than those calculated based on the finite element analysis of Civalek [2].

The HDQ method was found to require less than three seconds of CPU time for almost all cases on a standard personal computer (Pentium-II processor having 64 MB RAM).

An attractive advantage of the HDQ method is that it can produce the acceptable accuracy of numerical results with very few grid points in the solution domain and therefore can be very useful for rapid evaluation in engineering design.

VI. CONCLUDING REMARKS

A harmonic type differential quadrature method was introduced to study the bending, vibration, and deflections analysis of thin, isotropic circular and annular plates with various support and load conditions. The conventional small deflection theory is used in the study with the governing differential equations transformed into a set of linear algebraic equations by the harmonic differential quadrature formulation. The method of harmonic differential quadrature that was using the paper proposes a very simple algebraic formula to determine the connections weighting coefficients required by differential quadrature approximation without restricting the choice of mesh grids. The known boundary conditions are easily incorporated in the harmonic differential quadrature as well as the other type differential quadrature. The discretizing and programming procedures are straightforward and easy. The authors believe that many researchers will use the harmonic differential quadrature method in their analysis due to its handy accuracy, efficiency and convenience. Because, harmonic differential quadrature method require considerably grid points, much less computer storage and computing time. Several test examples for different support and load conditions have been selected to demonstrate the convergence properties, accuracy and simplicity in numerical implementation of HDQ procedures. This has verified the accuracy and applicability of the HDQ method to the class of problem considered in this study.

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