Maltepe Journal of Mathematics
ISSN:2667-7660, URL:http://dergipark.org.tr/tr/pub/mjm
Volume V Issue 2 (2023), Pages 47-51, Doi: https://doi.org/10.47087/mjm.1314434

ABEL'S CONVOLUTION FORMULAE THROUGH TAYLOR POLYNOMIALS

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Abstract. By making use of the Taylor polynomials, new proofs are presented for three binomial identities including Abel's convolution formula.

§1. **Introduction.** There are numerous identities in mathematical literature. Among them, Newton's binomial theorem is well known

$$\sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}=(x+y)^n.$$

Abel [1] (see [7, §3.1], for example) discovered the following deep generalizations of it with an extra λ-parameter:

$$\sum_{k=0}^{n}\binom{n}{k}x(x+k\lambda)^{k-1}(y-k\lambda)^{n-k}=(x+y)^n. \tag{1}$$

This convolution identity is fundamental in enumerative combinatorics and number theory. The reader can refer to [19] for a historical note. The known proofs can briefly be described as follows:

- Generating function method; see [9] and Chu [3].
- Series rearrangement and finite differences: Chu [4].
- The classical Lagrange expansion formula; see [17, §4.5].
- Lattice path combinatorics; see [15, §4.5] and [16, Appendix].
- The Cauchy residue method of integral representation; see [8, §2.1].
- Gould–Hsu Inverse series relations: Gould–Hsu [12] and Chu–Hsu [6, 2].
- Riordan arrays (which can trace back to Lagrange expansion); see [18].

The aim of this short article is to offer new and simple proofs for (1) and two other binomial identities via Taylor polynomials.

2020 *Mathematics Subject Classification.* Primary 05A10, Secondary 11B65.
Key words and phrases. The binomial theorem; Taylor polynomial; Abel's convolution formulae.

Submitted on June 14th, 2023. Published on December 30th, 2023
Communicated by Hacer Sengul Kandemir.

§2. **Proof of** (1)**.** Denote by $P(y)$ the binomial sum in (1). Its mth derivative at $y=-x$ is determined by

$$P^{(m)}(-x) = x\sum_{k=0}^{n-m}\frac{(n-k)!}{(n-k-m)!}\binom{n}{k}(x+k\lambda)^{k-1}(y-k\lambda)^{n-k-m}\big|_{y=-x}$$
$$= \frac{n!\ x}{(n-m)!}\sum_{k=0}^{n-m}(-1)^{n-m-k}\binom{n-m}{k}(x+k\lambda)^{n-m-1}. \tag{2}$$

To evaluate the last sum, we recall the difference operator Δ, which is defined for a function $f(y)$ at the point y by

$$\Delta f(y) = f(y+1) - f(y).$$

By applying n times of Δ, we have the nth difference

$$\Delta^n f(y) = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}f(y+k).$$

In particular, when $f(y)$ is a polynomial of degree $m \le n$ with the leading coefficient c_m, then by induction, it is not hard to prove the important identity (see [13, Equation 5.42])

$$\Delta^n f(y) = n!\ c_m\ \chi(m=n), \tag{3}$$

where χ is the logical function given by $\chi(\text{true}) = 1$ and $\chi(\text{false}) = 0$.

Therefore, the sum in (2) results in the $(n-m)$th difference of a polynomial of degree $n-m-1$. Consequently, $P^{(m)}(-x)$ vanishes for $0 \le m < n$ and $P^{(n)}(-x) = n!$.

Because $P(y)$ is a polynomial of degree n, we confirm Abel's identity (1) by expressing $P(y)$ in terms of the Taylor polynomial at $y=-x$ as follows:

$$P(y) = \sum_{m=0}^{n}\frac{(x+y)^m}{m!}P^{(m)}(-x) = (x+y)^n. \qquad \square$$

§3. **A binomial transformation.** Gould [11, Equation 1.10] recorded a binomial transformation which can be reproduced equivalently as

$$\sum_{k=0}^{n}\binom{x+1}{n-k}y^k = \sum_{i=0}^{n}\binom{x-i}{n-i}(1+y)^i. \tag{4}$$

Observing that both sides of the above equality are polynomials of degree n in y. Denote by $Q(y)$ the sum on the right–hand side. Its Maclaurin polynomial expression reads as

$$Q(y) = \sum_{k=0}^{n}\frac{y^k}{k!}Q^{(k)}(0).$$

Then we confirm (4) by computing the kth derivative of $Q(y)$ in the following manner

$$\begin{aligned}Q^{(k)}(0) &= k!\sum_{i=k}^{n}\binom{x-i}{n-i}\binom{i}{k}\\ &= k!(-1)^{n-k}\sum_{i=k}^{n}\binom{n-x-1}{n-i}\binom{-k-1}{i-k}\\ &= k!(-1)^{n-k}\binom{n-k-x-2}{n-k} = k!\binom{x+1}{n-k},\end{aligned}$$

where the last step is justified by the Chu–Vandermonde convolution formula. □

§4. **A binomial sum identity.** Let m and n be the two nonnegative integers with $m \le n$. There is an interesting binomial sum (see [20])

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(y+k\lambda)^m}{x+k} = \frac{(y-x\lambda)^m}{x\binom{x+n}{n}}. \tag{5}$$

Clearly, this is an identity between two polynomials of degree m in y. Let $R(y)$ stand for the sum on the left. Then its Taylor polynomial at $y = x\lambda$ is given by

$$R(y) = \sum_{j=0}^{m}\frac{(y-x\lambda)^j}{j!}R^{(j)}(x\lambda).$$

Evaluate the jth derivative by

$$R^{(j)}(x\lambda) = j!\binom{m}{j}\lambda^{m-j}\sum_{k=0}^{n}(-1)^k\binom{n}{k}(x+k)^{m-j-1}.$$

When $0 \le j < m$, the last sum with respect to k is the nth difference of a polynomial of degree $m-j-1 < n$ that equals zero in view of (3). Instead, we have for $j = m$

$$R^{(m)}(x\lambda) = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{m!}{x+k}.$$

Consequently, (5) will be confirmed if we can show that

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1}{x+k} = \frac{n!}{(x)_{n+1}}, \tag{6}$$

where the shifted factorial is defined by

$$(x)_0 = 1 \quad \text{and} \quad (x)_n = x(x+1)\cdots(x+n-1) \quad \text{for} \quad n = 1, 2, \cdots.$$

In fact, it is routine to check that (6) follows from the partial fraction decomposition

$$\frac{n!}{(x)_{n+1}} = \sum_{k=0}^{n}\frac{A_k}{x+k}$$

with the connection coefficients being determined by

$$A_k = \lim_{x\to -k}\frac{n!(x+k)}{(x)_{n+1}} = \binom{n}{k}(-1)^k. \qquad \square$$

§5. **Two companion formulae.** For the formula (1), Abel [1] found also a companion one

$$\sum_{k=0}^{n}\binom{n}{k}x(x+k\lambda)^{k-1}(y-n\lambda)(y-k\lambda)^{n-k-1}=(x+y-n\lambda)(x+y)^{n-1}.$$

Besides, there exists a third one of Jensen type (cf. [14]) found by Gould [10]

$$\sum_{k=0}^{n}\binom{n}{k}(x+k\lambda)^{k}(y-k\lambda)^{n-k}=\sum_{m=0}^{n}\frac{n!}{m!}(x+y)^{m}\lambda^{n-m}.$$

Both of them reduce to the usual binomial theorem when $\lambda=0$. They can be proved by carrying out exactly the same procedure. The interested reader is encouraged to do it as an exercise.

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