Hacettepe Journal of
Mathematics & Statistics

Hacet. J. Math. Stat.
Volume XX (x) (XXXX), 1 – 18
DOI : 10.15672/hujms.xx

Research Article

A numerical approach for a dynamical system of fractional infectious disease problem

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Abstract

In this investigation, we study for a dynamical system aimed at elucidating a disease model under the influence of environmental stress from a broad perspective. The model is articulated through both standard differential equations and their Caputo fractional form. Our methodology involves a numerical approach using the Adams-Bashforth-Moulton technique to solve the system of differential equations, including the initial conditions. The existence, uniqueness and convergence of the technique are also briefly discussed. This study aims not only to improve the current technique, but also to introduce a novel design for obtaining numerical solutions to issues discussed in the existing literature, thus paving the way for further research. We also perform a stability analysis focusing on the coexistence equilibrium. In addition, we present visualisations of the results to elucidate the behaviour of the system, time evolution and phase plane plots with respect to specific parameters.

Mathematics Subject Classification (2020). 35A01, 65L10, 65L12, 65L20, 65L70

Keywords. disease modelling, Adams-Bashforth-Moulton method, Caputo fractional derivative, stability analysis

1. Introduction

Over the past couple of decades, many people lost their lives due to infectious diseases, some of which gave rise to large scale epidemic outbreaks. These include swine flu, SARS, Ebola and Coronavirus which is still ongoing and needs critical treatment strategies. In order to understand the disease mechanism and develop new ideas to control and prevent these diseases, mathematical models associated with the procedure of the disease, transmission rate as well as the infection related factors are needed.

Although hundreds of pioneering study for infectious diseases exist in the literature, mathematical modelling of the epidemic diseases traces its roots back to influential work

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Received: 14.06.2023; Accepted: 18.11.2023

performed by [37], who theoretically investigated the transmission and control of infectious diseases. Then, because of continuously emerging epidemic outbreaks, great deal of specific and general realistic formulation have been developed, see [44, 45, 57, 59, 60] for some studies for the mathematical applications of infectious diseases occurred in various conditions.

Since the transmission of infectious diseases is affected by many direct and indirect factors, one of the primary topics in mathematical epidemiology is to investigate the consequences of different environmental and disease related reasons. Most of the available data on infectious disease modelling is based on direct transmission, where the infection is transmitted depending on the direct contact rate with infectious individuals. However, disease route and transmission rate are influenced by various factors including environmental conditions such as pollution (in air, soil and water), that damages the immune system of healthy individuals and leave them more prone to infection. This environmental fluctuation may give rise to increase in the infection transmission among individuals. Although mathematical modelling for analysing the impact of environmental pollution for disease transmission has a great importance, the existing studies in the literature are quite rare [42]. Besides some other works that take the role of environmental fluctuations and climate into account for some specific diseases such as vector-born diseases have been performed. For instance, bifurcation Analysis of a SIRS Epidemic Model has been investigated by Alexander and Moghadas [2]. It has been demonstrated that the basic reproductive number remains independent of incidence functional form. Two kinds of incidence are investigated, namely, unbounded and saturated contact rates and detailed theoretical results are provided, which are then numerically illustrated. The results determine the ranges for the periodicity behavior of the model based on two crucial parameters: the basic reproductive number and the rate of loss of natural immunity. A weather-driven model of malaria transmission is presented in [34]. The paper presents a mathematical-biological model of parasite dynamics, which includes both weather-dependent stages within the vector and weather-independent stages within the host. Numerical assessments of the model concerning both time and space demonstrate its ability to qualitatively reconstruct the incidence of infection. Yang *et. al.* [61] created a mathematical model to evaluate how temperature affects the probability of a dengue outbreak. Lafferty [46] studied the ecology of climate change and infectious diseases. The author states that climate can influence species distributions through the impact on physiology caused by temperature and precipitation. Dobson [20] investigated climate variability, global change, immunity, and the dynamics of infectious diseases. For a comprehensive review on the topic we refer the reader to the above references.

Fractional calculus and its applications play an important role in engineering, physics, chemistry, biology, and other fields [52]. In this context, fractional derivatives bring a comprehensive concept for describing miscellaneous process and materials by hereditary and memory properties. Correspondingly, the fractional differential equations open further discussions and their applications provide a useful mathematical tool for dynamical systems from the modelling, numerical, and controlling aspects. The outcomes of such investigation give us a better understanding of the underlying mechanism of the systems. Besides, data-fitting, non-locality, and nonlinear modelling research studies benefit from fractional calculus. On the other hand, non well-behaved, singularities of the solutions, and some limitations of the models which consist of the fractional differential equations have difficulties to obtain the solutions of the systems.

From another point of view, numerical approximation of the dynamical systems comprising fractional differential equations with initial and boundary values can be obtained by several techniques. A good example of one of these techniques is a well-known algorithm called the predictor-corrector method of Adams-Bashforth-Moulton which is applied on some biological models [12], [21], [54]. Besides, collocation approach by block-pulse functions and Bernoulli polynomials is used for the solution nonlinear fractional integro-differential equations [49], the solutions of a differential equations system including fractional order in the sense of Caputo have been obtained by the sinc collocation numerical method and introduced by Hatipoğlu *et al.* [33], Frobenius method for solving fractional differential equations [40], and RungeKutta method for solving dynamic systems in the application of neural network for nonlinear dusty plasma [11] are different approaches used in the field. The sextic B-spline collocation scheme was employed in [51] to derive the approximate solution for the generalized equal width wave equation.The modelling aspect of the fractional differential equations is widely used in many research areas. Particularly, these equations have received a great attention in medical applications. This also relates the infectious diseases for being defined by dynamical systems including parameters and the non-integer order differential equations. Here, the non-integer order differential equations may help to explain the dynamical systems in a comprehensive aspect. Therefore, we apply the modelling techniques on the construction of the systems which are explained clearly by analyses and simulations. In fact, the current studies show us the crucial results on the subject such as numerical investigation for the Q fever disease by the fractal-fractional operators [4], the novel coronavirus expressed by the Caputo fractional epidemic model [38], the general fractional model for the COVID-19 with isolation as a comparative study [9], and so on.

In the literature of ecological modelling, the imprecision of the environment would cause complexities in the fractional order model of interaction of species, for example see [39]. In this framework, differential models for environmental pollution based on fractional order derivatives gain the attraction of several researchers due to the advantages of the fractional derivatives mentioned above. In the following articles focused on environmental stress, authors introduce that fractional order models perform better than integer order models. In [26], authors have modelled the atmospheric distribution of pollutants in the sense of Caputo fractional derivative. Here, the solutions of the fractional models have been compared with real data. It has been seen that the fractional derivative models perform better than the ordinary models existing in the literature. In [27], a model on the distribution of contaminants has been constructed for the planetary boundary layer. According to the findings, it has been found that there should be a relation between the physical structure of the turbulent flow and the order of the fractional derivative. Another work on the dispersion of pollutants in the planetary boundary layer that have better results for non-integer order model than integer order model is [50]. The [50]. The authors have showed that the best results are obtained for fractional order $\alpha = 0.95$ by simulation results from the fractional order model. Also in [1] authors have extended a fractional integer order model for waterborne diseases in case of environmental stressors.

Recently there are several studies in the literature which focus on the fractional order models of infectious diseases. A relatively new non-integer order epidemiological model for chickenpox virus is proposed in [53]. Here, authors have compared the efficiency of the various fractional order derivative definitions. In [47] authors have the impact of environmental pollution and the fractional parameters (memory effect) of the economic variables on economic growth. A relatively new model to investigate the transmission of transmission of infectious diseases in a predator and prey system has been proposed in [25]. Regarding the findings of the study the model based on fractional-order operators real-world phenomena better than the model based on integer-order differential equations because of their memory-related properties. Another fractional compartment model that involves involves maternally-derived immunity (M), susceptible (S), exposed (E), infected (I) and recovered/Immune (R) compartments has been presented in [3], based on the fractional derivative. Here, authors have found that the fractional model fits the real data better than the classic MSEIR model. The paper [5] examines the effect of memory on gonorrhoea transmission in a structured population using Caputo's fractional derivative.

For this purpose the dynamics of gonorrhea spread between females and males is studied. In addition to numerical results the stability and sensitivity analysis are given. Authors showed the importance of memory in a structured population in comparison to previous studies. The study [6] focuses on the fractional order model of the heartwater disease spread between domestic ruminants and amblyomma ticks involving Caputo derivatives. The numerical simulations demonstrate that a modification of memory impacts the basic reproduction number. Additionally, vaccinated domestic ruminants exhibit a smooth trajectory as the fractional order diminishes. This implies that vaccinating domestic ruminants gradually promotes the quantity of healthy domestic ruminants.

By better representing complex dynamics, Caputo derivative models help to improve biological and epidemiological systems. Caputo derivative models store previous states and inputs, in particular when the initial conditions are not equal to zero. This makes it possible to model phenomena with memory and to exhibit certain relaxing behaviors observed in real processes. Caputo-derivative dynamics can model long-term memory effects and describe biological processes more precisely by incorporating non-integer-order derivatives. This is because memory affects biological systems' cell signals, immune reactions, neuronal function, and disease susceptibility. Moreover, ecological and epidemiological parameters can be estimated and identified using Caputo derivative systems. Control strategies, such as vaccination programs to prevent the spread of disease, can then be designed using these parameters. To the best of our knowledge the model presented by [42] has been studied in various perspectives such as modelling of COVID-19 [36, 43], an agent based modeling of disaster response and recovery phases [56], modeling of Zika outbreak [41] and a delayed SIS modeling to examine the effects of environmental pollution [29]. However the effect of the memory on the proposed model has not been studied so far. Therefore this research builds on the improvement of [42] with memory effect by the motivation of successful results of fractional derivatives in models [4], [38], [9], and [1]. By taking into account the memory effect via Caputo fractional-order derivative, we developed a fractional model to show the role of environmental pollution in the development of disease transmission dynamics using a new application for the successful fractional method studied by Garrappa *et al.* [22].

The organisation of the paper is as follows. In Section 2, firstly the classical infectious disease model comprising the impact of pollution is taken into consideration. Here, the equilibria and stability analysis are revisited in Section 2.1. Then the corresponding system of fractional differential equations is described in Section 2.2. Section 3 deals with the numerical procedure. Particularly, the predictor-corrector or Predict, Evaluate, Correct, Evaluate (PECE) method of Adams-Bashforth-Moulton approach. In Section 4, single parameter bifurcation diagram of the model is performed and time evolutions of the fractional model with various order are compared. Lastly, in Section 5 summary of the work is presented and possible future directions are discussed.

2. Mathematical preliminaries

2.1. Kumari and Sharma classical model

Here, we analyse the role of environmental pollution on the development of disease dynamics regarding a compartmental model which is proposed by Kumari and Sharma [42]. In this context, a total population of N size is divided into three class (i) susceptible class where individuals are not affected by environmental pollution (X) (ii) susceptible class where individuals are exposed to pollution (P) and (iii) infected class (Y). It is assumed that environmental pollution does not have a direct role on the contact rate and hence susceptible can only get infection through direct contact with infected individuals. However, environmental pollution may weaken the immune system of individuals and make them more prone to infection.

The classical model proposed by Kumari and Sharma [42] is given by

$$\frac{\mathrm{d}X(t)}{\mathrm{d}t} = mA - \theta X(t) - \lambda X(t)Y(t) - n\xi Y(t) - \mu X(t) = f(X,P,Y), \tag{2.1}$$

$$\begin{aligned}\frac{\mathrm{d}P(t)}{\mathrm{d}t} &= (1-m)A + \theta X(t) - \lambda(1+\delta\lambda')P(t)Y(t) + (1-n)\xi Y(t) - \mu P(t)\\ &= g(X,P,Y),\end{aligned} \tag{2.2}$$

$$\frac{\mathrm{d}Y(t)}{\mathrm{d}t} = \lambda X(t)Y(t) + \lambda(1+\delta\lambda')P(t)Y(t) - (\xi+\varphi+\mu)Y(t) = h(X,P,Y), \tag{2.3}$$

subject to non-negative initial conditions

$$X(0) \geq 0, \quad P(0) \geq 0 \;\; Y(0) \geq 0.$$

All parameters in the system are considered positive and their description are presented in Table 1. In the system given by Equations (2.1)-(2.3), where $\lambda(1+\delta\lambda')$ stands for the transmission ratio of individuals affected by pollution. Besides $(1-n)$ of the recovered population is transferred into pollution affected stressed class. Individuals from class X enter the class P at a constant rate represented by the parameter θ. The natural death rate for all classes is denoted by μ, and the recovery rate is presented by ξ. For more detailed information on model description, we refer the reader to [42].

Table 1. Parameters and variables used in the model (2.1)-(2.3).

parameter/variables	biological meaning
A	recruitment rate of newborns
λ	transmission rate of the disease
δ	the strength of environmental pollution that affects transmission
λ'	gauge impact of pollution
θ	the ratio of susceptible class who are transmitted into stressed class
μ	natural death rate for all classes (X, P, Y)
ξ	the recovery ratio of infected individuals
ϕ	disease related death ratio
n	the ratio of recovery individuals moving to susceptible class

2.2. Fractional differential equations

Fractional differential equations provide some useful properties for a further analysis of the models used for real world phenomena. In this section an effective numerical method, based on [22], will be considered to solve the fractional order problem corresponding to the model given in Eqs. (2.1)-(2.3).

In order to better understand the effect of the memory on the model (2.1)-(2.3), we consider the improved version with fractional order derivatives of the model given by [42].The fractional mathematical model corresponding to the system (2.1)-(2.3) is given by

$$\mathcal{D}^{\alpha}_{0,t}X(t) = mA - \theta X(t) - \lambda X(t)Y(t) - n\xi Y(t) - \mu X(t) = f^{\alpha}(X,P,Y), \tag{2.4}$$

$$\begin{aligned}\mathcal{D}^{\alpha}_{0,t}P(t) &= (1-m)A + \theta X(t) - \lambda(1+\delta\lambda')P(t)Y(t) + (1-n)\xi Y(t) - \mu P(t)\\ &= g^{\alpha}(X,P,Y),\end{aligned} \tag{2.5}$$

$$\mathcal{D}^{\alpha}_{0,t}Y(t) = \lambda X(t)Y(t) + \lambda(1+\delta\lambda')P(t)Y(t) - (\xi+\varphi+\mu)Y(t) = h^{\alpha}(X,P,Y), \tag{2.6}$$

where $\mathcal{D}_*^\alpha$ is the differential operator in the sense of Caputo and subject to initial conditions, i.e.

$$\mathcal{D}_*^\alpha u(t) = J^{n-\alpha} D^n u(t), \tag{2.7}$$
$$X(0) \geq 0, \quad P(0), \geq 0 \;\; Y(0) \geq 0,$$

where $n := \lceil \alpha \rceil$ is the nearest integer to α around 1. Besides, we have $X(t) = u_1(t)$, $P(t) = u_2(t)$, $Y(t) = u_3(t)$ for any $u_s(t)$, $(s = 1, 2, 3)$. We also describe the RiemannLiouville integral operator J^β for the order of $\beta > 0$.

$$J^\beta u(t) = \frac{1}{\Gamma(\beta)} \int_t^{t_0} (t-v)^{\beta-1} u(v) dv. \tag{2.8}$$

For $\alpha = 1$, we obtain a specific state for the RiemannLiouville integral operator. This is called as the generalisation of the standard integral [22]. On the other hand, we consider the formulation in (2.8) and by using the integral of (2.7)

$$u(t) := \sum_{\mu=0}^{n-1} u^\mu(0) \frac{t^\mu}{\mu!} + \frac{1}{\Gamma[\alpha]} \int_{t_0}^t [(t-v)^{\alpha-1} f(v, u(v))] dv. \tag{2.9}$$

Then we obtain a Volterra integral equation.

2.3. Steady states and stability

Here we express the steady states of the model given by (2.1)-(2.3). As stated in [42] and in many other infectious disease models in the literature, the model has a trivial disease free steady state $S_1 = (X_*, P_*, 0)$ and coexisting state $S_2 = (X^*, P^*, Y^*)$, which can be explicitly found using $\frac{\mathrm{d}X}{\mathrm{d}t} = \frac{\mathrm{d}P}{\mathrm{d}t} = \frac{\mathrm{d}Y}{\mathrm{d}t} = 0$. Thus the disease free state ($Y_* = 0$) is

$$X_* = \frac{mA}{\theta + \mu}, \quad P_* = \frac{1}{\mu}\left((1-m)A + \frac{mA\theta}{\theta+\mu}\right), \quad Y_* = 0,$$

and coexisting state is

$$\begin{aligned} X^* &= \frac{\xi + \varphi + \mu}{\lambda} - (1 + \delta\lambda')P^*, \\ Y^* &= \frac{(1-m)\lambda A + \theta(\xi+\varphi+\mu) - \lambda\left[\theta(1+\delta\lambda') + \mu\right] P^*}{\lambda\left(\lambda(1+\delta\lambda')P^* - (1-n)\xi\right)}. \end{aligned}$$

Here the variable P is solved from $\Psi_3 P^2 + \Psi_2 P + \Psi_1 = 0$ leading to a positive root

$$P^* = \frac{-\Psi_2 - \sqrt{\Psi^2 - 4\Psi_1\Psi_3}}{2\Psi_3}, \tag{2.10}$$

where

$$\begin{aligned} \Psi_1 &= a_1 a_6 a_9 + a_1 a_3 a_{10} + a_6 a_7 + a_3 a_8, \\ \Psi_2 &= a_9(a_1 a_5 + a_2 a_6) + a_{10}(a_2 a_3 + a_1 a_4) + a_5 a_7 + a_4 a_8, \\ \Psi_3 &= a_2 a_5 a_9 + a_2 a_4 a_{10}, \end{aligned}$$

with

$$\begin{aligned} & a_1 = \frac{\xi+\varphi+\mu}{\lambda}, \quad a_2 = -(1+\delta\lambda'), \quad a_3 = (1-m)\lambda A + \theta(\xi+\varphi+\mu), \\ & a_4 = -\lambda\left[\theta(1+\delta\lambda') + \mu\right], \quad a_5 = \lambda^2(1+\delta\lambda'), \quad a_6 = -\lambda(1-n)\xi, \\ & a_7 = -mA, \quad a_8 = -n\xi, \quad a_9 = \theta + \mu, \quad a_{10} = \lambda. \end{aligned}$$

Linearisation around the steady state, where $S_i = S_i + \tilde{S}_2$ $(i = 1, 2)$ gives the Jacobian matrix which can be written in a general form

$$\mathcal{J} = \begin{pmatrix} f_X & f_P & f_Y \\ g_X & g_P & g_Y \\ h_X & h_P & h_Y \end{pmatrix}\Bigg|_{S_i}, \quad i = 1, 2,$$

where

$$\begin{aligned} &f_X = -\theta - \lambda Y - \mu, \quad f_P = 0, \quad f_Y = -\lambda X + n\xi, \\ &g_X = \theta \quad g_P = -\lambda(1 + \delta\lambda')Y - \mu, \quad g_Y = -\lambda(1 + \delta\lambda')P + (1 - n)\xi, \\ &h_X = \lambda Y, \quad h_P = \lambda(1 + \delta\lambda')Y, \quad h_Y = \lambda X + \lambda(1 + \delta\lambda')P - (\xi + \varphi + \mu). \end{aligned}$$

Here the accents $(\tilde{\cdot})$ are omitted for simplicity.

3. Numerical procedure

In the 20th century, the importance of numerical scheme to investigate the approximate solutions of the initial value problems is well-known and the research direction of the field focused on such techniques. J.C. Butcher introduced a comprehensive study regarding the historical evolution of such methods [10]. The essential proposal for the extension of these techniques is applied to the system of differential equations. Therefore, we obtain beneficial results to understand approximations to the dynamical systems which support getting a comprehensive insight into the outcomes of behavioural approach of the dynamical systems. Early research by Bashforth and Adams (1883) and Runge (1895) describes fundamental aspects of step-by-step methods which let us have a methodological approximation to the solutions of the problems. A basic idea of step-by-step methods was applied to the computer-based algorithms which then gave an efficient utilization to enrich the techniques. Therefore, we acquire a better understanding to improve these approaches for the dynamical system requirements.

The elemental aspect of the Adams-Bashforth methods is mainly to establish a stepwise approach to the solution. Thus, we consider mesh points also denote the step sizes. Then the differential equation is represented by the integral on the given interval. This step continues with approximation by the interpolation polynomial. Afterward, substitution by using Newton's interpolation formula, we obtain an essential formulation. Besides, a series is applied to the interpolation and we get an approximation of the solution. On the other hand, the variable step sizes are also applied to Adams-Bashforth methods to obtain an approximation [48].

Here, we consider the numerical approximation for the solution of the dynamical system in (2.4)-(2.6) including the initial values in (2.7). As we described at the Introduction, we consider the schematic algorithms for the explicit multistep methods, particularly present technique. In a general scheme, Adams-Bashforth method is known as one of the explicit multistep methods. A special class of the method is used to provide numerical solutions of the problems which are given in the form (2.4)-(2.7). Particularly, we describe a multistep method for such investigation, more specifically, the PECE method of Adams-Bashforth-Moulton approach. We consider the solution of the problem (2.4)-(2.7) is represented by (2.9). Here, the derivation of Eq. (2.9) gives us a correspondence to the integral defined by the initial conditions in (2.7). Therefore, we introduce the solution of (2.9) instead of the initial value problem.

First, we introduce the equations in (2.4)-(2.6) [13], [14], [15], [31], [23].

$$\mathcal{D}_*^\alpha[X(t)] = f(X, P, Y), \tag{3.1}$$

$$\mathcal{D}_*^\alpha[P(t)] = g(X, P, Y), \tag{3.2}$$

$$\mathcal{D}_*^\alpha[Y(t)] = h(X, P, Y). \tag{3.3}$$

For $0 < \alpha < 1$ the Volterra equation (2.9) is considered as weakly singular while we acknowledge the general solution of the right-hand side equations [15]. For the simplicity, we have

$$\mathcal{D}_*^\alpha[u(t)] = f(X, P, Y). \tag{3.4}$$

Now, we focus on the solution of the weakly singular Volterra equation (2.9). Thus, we consider the nodes t_j for all $j = 0, 1, ..., n+1$ and for the step size $h > 0$ which is defined as $h = b/n + 1$ for $n + 1$ steps [8]. Therefore, we obtain the approximation

$$u(t_{n+1}) = u(t_n) + \int_{t_n}^{t_{n+1}} f(v, u(v))dv, \tag{3.5}$$

which represents the integration of (3.4). We follow the steps in the implicit one-step AdamsMoulton algorithm by using the trapezoidal quadrature formula for two-points [23], [8], [24], [16]. Subsequently, we consider the approximation for $h > 0$ and name it $u_h(t_n)$. Then we provide

$$u_h(t_{n+1}) = u_h(t_n) + \frac{h}{2}\left[f(t_n, u_h(t_n)) + f(t_{n+1}, u_h(t_{n+1}))\right]. \tag{3.6}$$

Now, we consider the prior approximation by defining $u_h^P(t_{n+1})$ which is called predictor in AdamsMoulton method [16]. We replace it in the trapezoidal quadrature formula by using the rectangle rule and we obtain the following explicit formulation

$$u_h^P(t_{n+1}) = u_h(t_n) + h\, f(t_n, u_h(t_n)). \tag{3.7}$$

This method formulation is called as the forward Euler (also known as Adams-Bashforth) technique which is described in detail in [32]. Afterwards, we obtain the following equation by using (3.6)

$$u_h^P(t_{n+1}) = u_h(t_n) + \frac{h}{2}\left[f(t_n, u_h(t_n)) + f(t_{n+1}, u_h^P(t_{n+1}))\right]. \tag{3.8}$$

Furthermore, we construct the scheme by using the product trapezoidal quadrature formula and the nodes. We achieve the next step by a product integration and consider the nodes together with the weight function $(t_{n+1} - .)^{\beta-1}$ [13]. Afterwards, we replace the integral in (2.9). The approximation is applied to

$$\int_{t_0}^{t_{n+1}} (t_{n+1} - v)^{\beta-1} g(v)dv \approx \int_{t_0}^{t_{n+1}} (t_{n+1} - v)^{\beta-1} g_{n+1}(v)dv, \tag{3.9}$$

where g_{n+1} is the interpolant function of g also it has the piecewise linearity [24]. It is applied on the nodes t_j for all $j = 0, 1, ..., n+1$. Then we continue with the explicit calculation [16]

$$\int_{t_0}^{t_{n+1}} (t_{n+1} - v)^{\beta-1} g_{n+1}(v)dv = \sum_{j=0}^{n+1} a_{j,n+1} g(t_j), \tag{3.10}$$

where we define $a_{j,n+1}$

$$a_{j,n+1} = \int_{t_0}^{t_{n+1}} (t_n + 1 - v)^{\beta-1} \varphi_{j,n+1}(v)dv, \tag{3.11}$$

together with

$$\varphi_{j,n+1}(v) = \begin{cases} \frac{v - t_{j-1}}{t_j - t_{j-1}}, & \text{if } t_{j-1} < v < t_j, \\ \frac{t_{j+1} - v}{t_{j+1} - t_j}, & \text{if } t_j < v < t_{j+1}, \\ 0, & \text{otherwise.} \end{cases} \tag{3.12}$$

If we consider the nodes as equispaced for $t_j = t_0 + jh$, (h is a fixed integer), we compile (3.11) with (3.12) and we obtain a reduction of (3.11) [13]

$$a_{j,n+1} = \begin{cases} \frac{h^\beta}{\beta(\beta+1)}(n^{\beta+1} - (n-\beta)(n+1)^\beta), & \text{if } j = 0, \\ \frac{h^\beta}{\beta(\beta+1)}, & \text{if } j = n+1, \end{cases} \tag{3.13}$$

where we examine the case $1 \geq j \geq 1$ for our formulation and we get

$$a_{j,n+1} = \frac{h^\beta}{\beta(\beta+1)}((n-j+2)^{\beta+1} - 2(n-j+1)^{\beta+1} + (n-j)^{\beta+1}). \tag{3.14}$$

This is called as corrector formula and it supports us to have the fractional form of the one-step Adams-Moulton technique. Accordingly, we obtain

$$u_{n+1}(t_{n+1}) = u_0(t_{n+1}) + \frac{1}{\Gamma(\beta)}\left(\sum_{j=0}^{n} a_{j,n+1} f(t_j, u_j) + a_{n+1,n+1} f(t_{n+1}, u^P_{n+1})\right). \tag{3.15}$$

Now, we compute u^P_{n+1} by using the determined predictor formulation. We follow the steps from the Adams-Bashforth tecnique in generalised case. Then we replace it to the integral in (2.9) [13].

$$\int_{t_0}^{t_{n+1}} (t_{n+1} - v)^{\beta-1} g(v) dv \approx \sum_{j=0}^{n} \ell_{j,n+1} g(t_j), \tag{3.16}$$

where $\ell_{j,n+1}$ is defined as

$$\ell_{j,n+1} = \int_{t_j}^{t_{j+1}} (t_{n+1} - v)^{\beta-1} dv = \frac{1}{\beta}\left((t_{n+1} - t_j)^\beta - (t_{n+1} - t_{j+1})^\beta\right). \tag{3.17}$$

Subsequently, we describe (3.17) for the equispaced nodes as

$$\ell_{j,n+1} = \frac{h^\beta}{\beta}\left((n+1-j)^\beta - (n-j)^\beta\right). \tag{3.18}$$

Furthermore, we determine the predictor as

$$u^P_{n+1}(t_{n+1} = u_0(t_{n+1}) + \frac{1}{\Gamma(\beta)} \sum_{j=0}^{n} \ell_{j,n+1} f(t_j, u_j). \tag{3.19}$$

Accordingly, we calculate the function $f(t_{n+1}, u^P_{n+1})$ after we provide the predictor u^P_{n+1} in Eq. (3.19) by using the corrector in (3.15). Aftermath, we deal with the integration by the evaluation of $f(t_{n+1}, u_{n+1})$. Consequently, we accomplish the fractional form of the Adams-Bashforth-Moulton method. The technique is known as the step-by-step implicit method by using the analogue version of the classical Adams-Moulton method with the combination of the Adams-Bashforth method. Due to the algorithm steps, the method is simply known as the predictor-corrector technique which described in [24], [13] as the PECE method. This algorithm provides us guarantee for the numerical solutions by means of its properties which are introduced in the following part [13], [16], [55], [17].

3.1. Convergence

Here, the existence and uniqueness of the solutions and the convergent of the method are proposed below [17]:

Theorem 3.1. *Let be $\mathfrak{C} := [0, b] \times [u(0) - \gamma, u(0) + \gamma]$ for any $b > 0$ and $\gamma > 0$. Besides, let us consider a continuous function $f : \mathfrak{C} \to \mathbb{R}$. Now, we define $b := \min\{b, (\gamma\Gamma(\alpha + 1)/\|f\|_\infty)^{1/\alpha}\}$. Thus, the initial value problem (2.4)-(2.7) is solved by the existence of a function $u : [0, b] \to \mathbb{R}$ [17].*

Proof. The proof is obtained by using the integral operator with the order α which was defined in (2.8). Therefore, the transmission for the fractional calculus is acquired (see [55], Sec. 2, [19], Sec. 3, and [17], p. 235). Here, we apply (2.8) to both sides of the system in (3.1)-(3.3). A convex, closed, and nonempty to itself set is defined $\mathfrak{V} = \{u \in \mathfrak{C}[0,b] : \|u(t) - u(0)\|_\infty \leq \gamma\}$ which is mapped from J. Then we consider continuity property of the operator J in (2.8). Besides, Lipschitz property of the function f is used. By the help of continuity of f on a compact set, we say that there it is uniformly continuous and we use the ε definition and prove the continuity property of J. After that we can easily see the $J(\mathfrak{V})$ is pointwise bounded. Then we can show that

$$
\begin{aligned}
|Ju(t_1) - Ju(t_2)| &= \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_1} (t_1 - v)^{\alpha-1} f(v, u(v))dv - \int_0^{t_2} (t_2 - v)^{\alpha-1} f(v, u(v))dv \right| \\
&= \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_1} ((t_1 - v)^{\alpha-1} - (t_2 - v)^{\alpha-1}) f(v, u(v))dv + \int_{t_1}^{t_2} (t_2 - v)^{\alpha-1} f(v, u(v))dv \right| \\
&\leq \frac{\|f\|_\infty}{\Gamma(\alpha)} \left| \int_0^{t_1} ((t_1 - v)^{\alpha-1} - (t_2 - v)^{\alpha-1}) f(v, u(v))dv + \int_{t_1}^{t_2} (t_2 - v)^{\alpha-1} f(v, u(v))dv \right| \\
&= \frac{\|f\|_\infty}{\Gamma(\alpha+1)} (2(t_2 - t_1)^\alpha + t_1^\alpha - t_2^\alpha) \qquad (3.20) \\
&\leq 2 \frac{\|f\|_\infty}{\Gamma(\alpha+1)} (t_2 - t_1)^\alpha \qquad (3.21)
\end{aligned}
$$

where $0 \leq t_1 \leq t_2 \leq b$. For an arbitrary $\delta > 0$, we have $|t_2 - t_1| < \delta$ and obtain

$$
|Ju(t_1) Ju(t_2)| \leq 2 \frac{\|f\|_\infty}{\Gamma(\alpha+1)} \delta^\alpha.
$$

Then we use the Arzelà-Ascoli and the Shauder's fixed point theorems, with the idea of $J(\mathfrak{V})$ is equicontinuous, to show every sequence of is of uniformly convergent and $J(\mathfrak{V})$ is obtained as relatively compact and the solution of (2.4)-(2.7) is found by a fixed point in J regarding $J(\mathfrak{V})$. □

Theorem 3.2. *Let be $\mathfrak{C} := [0,b] \times [u(0) - \gamma, u(0) + \gamma]$ for any $b > 0$ and $\gamma > 0$. Besides, let us consider a continuous function $f : \mathfrak{C} \to \mathbb{R}$ bounded by b and satisfies the Lipschitz condition:*

$$
\|f(x,y) - f(x,z)\| \leq L|y - z|.
$$

There exists a function $u : [0,b] \to \mathbb{R}$ for b in (3.1) and $L > 0$ is a constant [17].

We know that the problem in (2.4)-(2.7) is equivalent to Eq. (2.9) when f is continuous. The uniqueness theorem is proven by using the generalised version of Banach's fixed point theorem [58].

Theorem 3.3. *Let us consider B is a Banach space, have its closed and nonempty subset J, and $\gamma_n \geq 0$ where $n \in \mathbb{N}$. We also have $\sum_{n=0}^{\infty} \gamma_n$ is convergent, then*

$$
\|J^n v - J^n y\| \leq \gamma_n \|v - y\|, \qquad (3.22)
$$

where $n \in \mathbb{N}$ and $v, y \in J$ for $J : \mathfrak{V} \to \mathfrak{V}$. Therefore, we have has a unique fixed point v^* in J where $(J^n v_0)_{n=1}^{\infty}$ converges to v^* for $v_0 \in J$ [17].

Proof. *(Theorem 3.2)* Let us consider a reduced equation in (2.9) and define a nonempty set $\mathfrak{V} = \{u \in \mathfrak{C}[0,b] : \|u(t) - u(0)\|_\infty \leq \gamma\}$ in Banach space for the functions which are continuous and in the interval $[0,b]$. We already know that the operator J is continuous and now, see that is of fixed point. In (3.20), we have that Ju is also continuous. Therefore, $Ju \in \mathfrak{C}$ when $u \in \mathfrak{C}$ and $J : \mathfrak{C} \to \mathfrak{C}$. Then we obtain

$$
\|J^n u(t) - J^n \tilde{u}(t)\|_{L_\infty[0,b]} \leq \frac{(Lt^\alpha)^n}{\Gamma(1+\alpha n)} \|u(t) - \tilde{u}(t)\|_{L_\infty[0,b]} \qquad (3.23)
$$

For $n-1 \longrightarrow n$:

$$\begin{aligned}\|J^n u(t) - J^n \tilde{u}(t)\|_{L_\infty[0,b]} &= \left\| J(J^{n-1}u(t)) - J(J^{n-1}\tilde{u}(t)) \right\|_{L_\infty[0,b]} \\ &= \frac{1}{\Gamma(\alpha)} \sup_{0\le\omega\le t} \left| \int_0^\omega (\omega - v)^{\alpha-1} [f(v, J^{n-1}u(v)) - f(v, J^{n-1}\tilde{u}(v))] dv \right|\end{aligned}$$

On the other hand, it is trivial to see the result for $n = 0$. By using the Lipschitz assumption and Chebyschev norm in Eq. (3.22), we can prove (3.23) (see [55], Sec. 2 and [17], p. 232). Then we can easily see the convergence of the γ_n and we define the Mittag-Leffler with the order of α:

$$M_\alpha(Lb^\alpha) := \sum_{n=0}^{\infty} \frac{(Lb^\alpha)^n}{\Gamma(1+\alpha n)}$$

Then we obtain the uniqueness result by using the fixed point theorem ([18], Chap. 18). □

Now, we introduce the convergence of the technique as follows:

Theorem 3.4. *Let us consider an error bound which is described in the form*

$$\max_{0\le i\le N} |t_{i+1} - t_i| = \begin{cases} \mathcal{O}(h^2), & \text{if } \alpha \ge 1, \\ \mathcal{O}(h^p), & \text{if } \alpha < 1, \end{cases} \tag{3.24}$$

where $D_*^\alpha u \in U^2[0T]$, $h = \max_i(t_{i+1} - t_i)$. Besides, we have it for the fixed $t^* > 0$, $t_j \in [t_0, t_0 + t^*]$ for $\alpha > 0$. Then it is convergent of order p [16].

Proof. First we assume that u is the solution of (2.4)-(2.7). Then we have

$$\begin{aligned}\left| \int_0^{t_{k+1}} (t_{k+1} - t)^{\alpha-1} D_*^\alpha u(t) dt - \sum_{i=0}^{k} d_{i,k+1} D_*^\alpha u(t_i) \right| &\ge U_1 t_{k+1}^{\nu_1} h^{\mu_1} \\ \left| \int_0^{t_{k+1}} (t_{k+1} - t)^{\alpha-1} D_*^\alpha u(t) dt - \sum_{i=0}^{k} c_{i,k+1} D_*^\alpha u(t_i) \right| &\ge U_2 t_{k+1}^{\nu_2} h^{\mu_2} \end{aligned} \tag{3.25}$$

where $\nu_1, \nu_2 \le 0$ and $\mu_1, \mu_2 > 0$. Therefore, for a suitable T, $N = [T/h]$ and $q = \min\{\mu_1 + \alpha, \mu_2\}$, we have $\max_{0\le i\le N} |u(t_i) - u_i| = \mathcal{O}(h^p)$. Secondly, we assume $\nu_1 = \nu_2 = \alpha > 0$, $\mu_1 = 1$, and $\mu_2 = 2$. Then the error bound is defined as follows:

$$q = \min\{1+\alpha, 2\} = \begin{cases} 2, & \text{if } \alpha \ge 1, \\ 1+\alpha, & \text{if} \alpha < 1, \end{cases}$$

Finally, we obtain (3.24) for the error bound $\mathcal{O}(h^p)$ [14], pp. 40-43 and [32], Sec. 5. □

Therefore, we guarantee the existence, uniqueness and convergence of the technique and the numerical solutions of such a system including the initial conditions. A numerical model is convergent if and only if a sequence of model solutions with increasingly refined solution domains approaches a fixed value. Therefore, we obtain consistency for the numerical solutions by the present technique.
Now we apply the algorithm as the next step.

3.2. The algorithm

Here, we consider the algorithmic description for the implementation of the numerical approach in Section (3) [24]. The algorithmic steps show us practical information regarding the solution path of the present method. On the other hand, we have a programming based settlement which lead future directions of the work. Therefore, we present the steps in (1).

Algorithm 1 An algorithm to introduce a solution for the problem in (2.4)-(2.6) [24]

Require: Reach approximations of $X(t)$, $P(t)$, and $Y(t)$.
Ensure: Integrate the IVP for the FDEs system, of order $\alpha > 0$.
$D^\alpha X(t) := FDE\_FUN(T, X(t))$ ▷ FDE_FUN is the corresponding vector field of the system of FDEs
$D^\alpha P(t) := FDE\_FUN(T, P(t))$ ▷ D^α is the Caputo fractional derivative
$D^\alpha Y(t) := FDE\_FUN(T, Y(t))$
$X(T\_0) = P(T\_0) = Y(T\_0) \geq 0$
$[T, Y] := FDE\_12(\alpha, FDE\_FUN, T\_0, T\_FINAL, Y\_0, H, PARA)$ ▷ $H > 0$ is the step-size
while $[T, Y] := FDE\_12(\alpha, FDE\_FUN, T\_0, T\_FINAL, Y\_\_\_0, H, PARA, MU)$ **do**
  if $MU = 0$ **then**
    find the solution by the predictor method. No corrector evaluation.
  end if
  if $MU > 0$ **then**
    find the solution by the predictor method. Evaluate the corrector.
  end if
  if $M = 1$ **then**
    use the classical PECE method. Evaluate the corrector.
    if $MU = Inf$ **then**
      test the convergence for two consecutive iterates. Evaluate the corrector.
    end if
  end if
end while
$[T, Y] := FDE\_12(\alpha, FDE\_FUN, T\_0, T\_FINAL, Y\_0, H, PARA, MU, MU\_TOL)$
▷ test the convergence.

4. Numerical simulations

In this section stability of equilibria and the behavior of the system (2.4)-(2.6) is illustrated. All computations and graphs are computed and plotted by using MATLAB software. Unless stated otherwise the parameters of the model are fixed to $A = 200$, $\theta = 0.004$, $\mu = 0.035$, $m = 0.8$, $\delta = 0.3$, $\lambda = 0.00002, \lambda' = 0.1$, $\phi = 0.01$, $\xi = 0.012$, $n = 0.7$.

In Figure 1, single parameter bifurcation diagrams for susceptible population (X) in regard to parameters A, δ, λ', n and μ respectively. Numerical continuation of variables P and Y can be performed as a function of various system parameters and similar stability behavior is obtained (not shown here for simplicity). The solid and dotted line represent the stable branch and dashed line stand for the unstable branch. Here the number of eigenvalues with a positive real part is 1 for the dashed line. As seen two equilibria, one is stable and the other is unstable, intersect in a transcritical bifurcation. In Figure 1(b), it is observed that the continuation in terms of parameters δ and λ' demonstrate very similar behaviour for which the dotted line is the stable branch for parameter λ' and solid line is the stable branch for parameter δ.

Time evolution of the fractional system (2.4)-(2.6) for two different initial conditions is given in Figure 2. Firstly, the initial value $(X_0, P_0, Y_0) = (200, 100, 300)$ is considered then the behavior of the variables X, P and Y in Figure 2 (a,c,e) respectively for various orders of α such as $\alpha = 0.7, \alpha = 0.8, \alpha = 0.9$ and $\alpha = 1$ are presented in order to see the effect of the memory. Secondly, the initial value $(X_0, P_0, Y_0) = (4000, 2000, 4000)$ is considered and

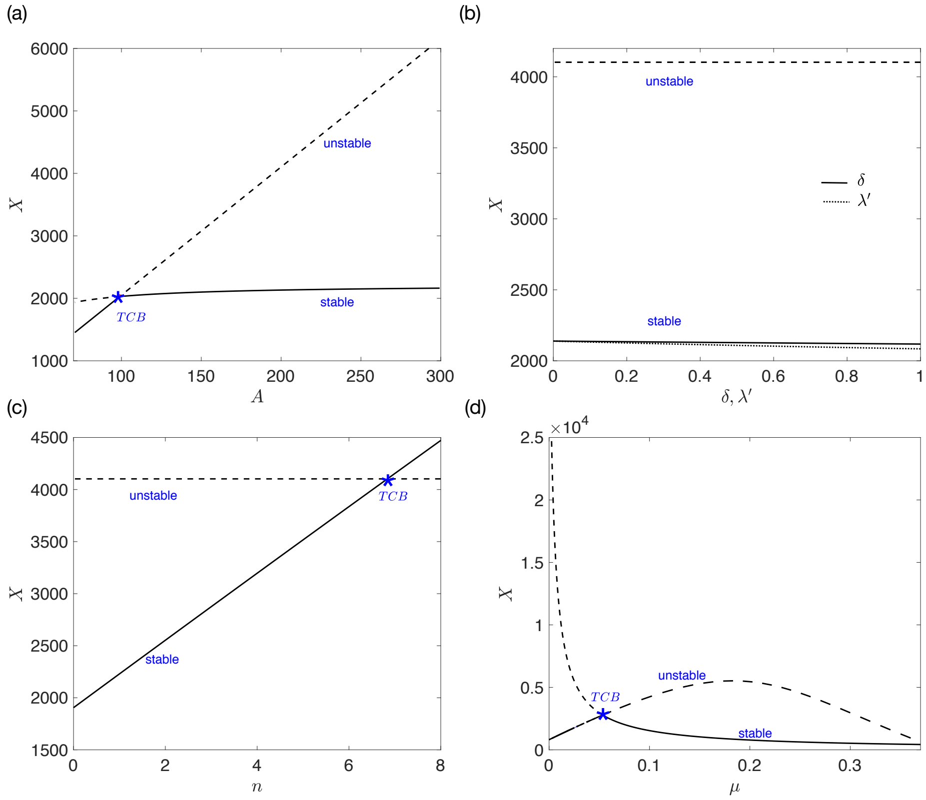

Figure 1. Single parameter numerical continuation of variable X with $\alpha = 1$ in terms of various parameters.

the graph obtained for various orders of α such as $\alpha = 0.7, \alpha = 0.8, \alpha = 0.9$ and $\alpha = 1$ are presented in order to see the effect of the memoryare demonstrated in Figure 1 (b,d,f).

The results presented in 2 are confirmed with phase plane diagrams of the system (2.4)-(2.6) in 3-D plane, which is illustrated in Figure 3 for $\alpha = 0.7, 0.8, 0.9$ and 1. In Figure 3 (a) the initial value $(X_0, P_0, Y_0) = (200, 100, 300)$ is considered and the 3-D graph of the fractional system with initial points $(X_0, P_0, Y_0) = (4000, 2000, 4000)$ are presented in Figure 3 (b).

5. Conclusions

Dynamical behavior of infectious diseases helps us to understand disease dynamic which is mainly affected by environmental pollution. Limited investigation has been performed to understand the effect of environmental conditions on the spread of diseases. In this paper, we analysed the dynamics of an epidemic model with environmental stress through both standard differential equations and its Caputo fractional version. Classic models of ordinary differential equations do not have memory, as their solutions are independent of previous instances. One approach to incorporate memory effects into a mathematical model is by altering the derivative order of a classical model to a non-integer value. Therefore fractional calculus is a valuable mathematical tool for describing biological systems characterized by memory effects [7]. In this context, fractional derivatives exhibit non-local characteristics, effectively capturing memory effects, while time delays convey information about earlier states [35]. In this study,firstly the classical system given by (2.1)-(2.3) is revisited and stability analysis around the coexistence equilibrium is determined [28]. Then the reformulation of the model allows us to analyse the system with

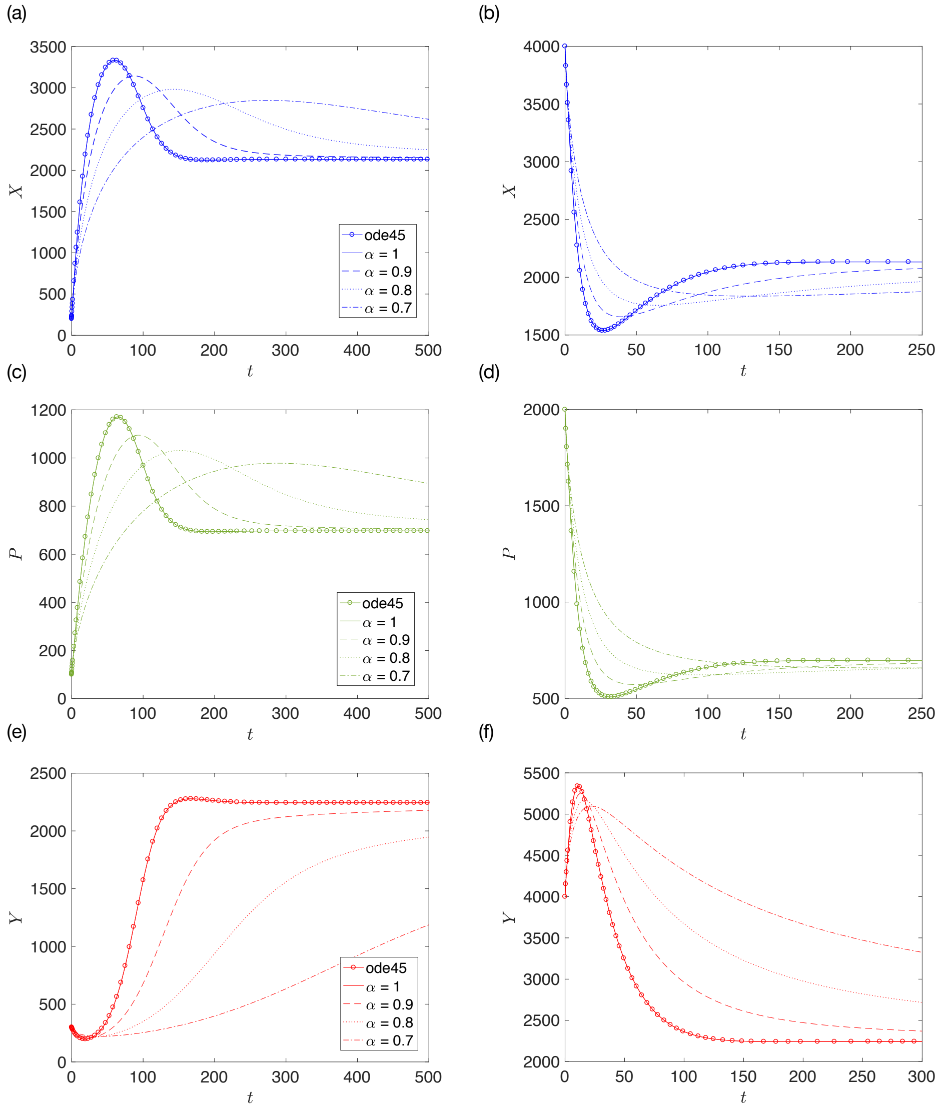

Figure 2. Time evolution of the system given by (2.4)-(2.6) with parameters given in Section 4 for two different initial conditions (i) $(X_0, P_0, Y_0) = (200, 100, 300)$ (a,c,e) and (ii) $(X_0, P_0, Y_0) = (4000, 2000, 4000)$ (b,d,f).

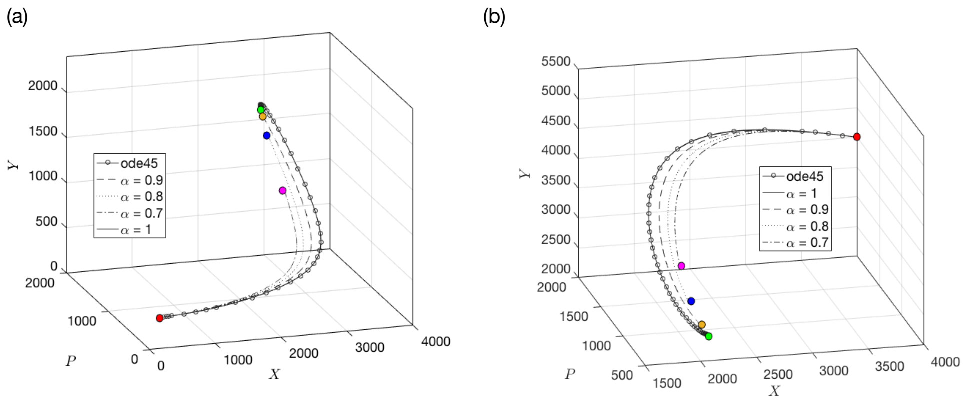

Figure 3. Phase plane plots corresponding to each column in Figure 2 with $(X_0, P_0, Y_0) = (200, 100, 300)$ (a) and $(X_0, P_0, Y_0) = (4000, 2000, 4000)$ (b).

fractional order operators in (2.4)-(2.6). We show that the behavior of the classes non-affected (X), susceptible (P) and infected (Y) by environmental pollution with fractional order derivatives converge to the ordinary differential model while α converges to 1. Therefore, our results allow flexibility for modelling the dynamics of an epidemic model with environmental stress. Besides, we consider the Adams-Bashforth-Moulton method for the solution of the systems including the initial conditions. The technique is guaranteed the existence, uniqueness and the convergence of the solutions. An appropriate composition of the numerical method to the fractional problems attracted us to adapt the method to our model problem. Our study gives an adapted approach for the model problem which goes toward a novel contribution to the field.

The illustrative results are shown by the figures which give us practical understanding for the dynamical system analyses and the approximation to the solutions. The model consists of susceptible populations affected/not affected by pollution and infected individuals, and it can be expanded to include further refinements such as incorporating the increasing level of pollution in the environment.For future studies, we can implement the modified versions of the method by adding the fade memory, additional corrector iterations, different spacing of the mesh, reduction of the arithmetic complexity, and so on. This provides us to follow the path for investigating further open problems in the field. By incorporating both fractional derivatives and delays, physical and biological problems can be more accurately and comprehensively modeled. Thus, fractional tools used in this paper can be expanded to incorporate time delays, leading to fractional delay differential equations. This approach is also reserved for the future. Furthermore, especially for parameter estimation ones, our model could provide more realistic modelling features for the effects of the environmental pollution with various values of α. Another straightforward direction would be to extend the model to delay differential equation system and analyse the dynamics of the interactions among different compartments [30]. Finally, from the point of numerical solution technique view, various numerical techniques such as those given in the introduction section can be extended to the proposed model.

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