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Research Article

On the Generalized Poincaré Distance

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Abstract
In this work, the concept of the generalized Poincaré distance is given and the distance between two points on vertical lines, horizontal lines and semi-ellipses in the upper half-plane are examined. It is also shown that translations parallel to the x-axis and reflections in the vertical lines preserve the generalized Poincaré distance in the upper half-plane.

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1. Introduction

Hyperbolic geometry, also known as Lobachevskian geometry, is a non-Euclidean geometry that has been the subject of intense study and fascination for over a century. One of the most popular models of hyperbolic geometry is the classical Poincaré upper half-plane, which has been widely used to represent various hyperbolic structures in mathematics, physics, and other fields.

In 1854, Riemann proposed a general modification of planar distances, suggesting that mathematicians studied geometries where the length of an arbitrary curve γ in the plane was expressed as

$$L=\int_{\gamma}\sqrt{(Edx^2+2Fdxdy+Gdy^2}.$$

In this equality, the functions E, F and G are subject to the constraints $E>0$, $G>0$ and $EG-F^2>0$. The expression $Edx^2+2Fdxdy+Gdy^2$ is known as the Riemann (or Riemannian) metric. The Euclidean metric, dx^2+dy^2, and the Poincare metric, $\frac{dx^2+dy^2}{y^2}$, are special cases of this general metric, [1]. It is important to note that many metrics are only defined in specific regions of the plane. For example, the Poincare metric is undefined on the x-axis and is typically restricted to the upper half-plane. Riemannian geometry is a branch of mathematics that studies the properties of curved spaces using the concept of metrics. Every metric, which describes the distances and angles in a space, has associated geodesics. Geodesics are the paths that locally minimize distance between points. Riemannian geometry provides a powerful framework for understanding the geometry of curved spaces and has found applications in physics, computer science, and other fields. Its success lies in its ability to capture the rich geometric structures and phenomena that arise in curved spaces, offering deep insights into the nature of space itself.

The upper half-plane refers to the positive ordinate region of the Cartesian coordinate system. By considering half-lines perpendicular to the x-axis and half-circles centered on the x-axis as lines within the upper half-plane, a model for the hyperbolic plane can be constructed. Various models for hyperbolic plane geometries exist, including the one mentioned in this paper and other models described in [1–4]. In this particular model, the hyperbolic length of a curve γ is defined by an integral involving dx, dy and known as the Poincaré metric, which serves as a distance function and metric. The Poincaré half-plane refers to the Euclidean half-plane equipped with the Poincaré metric. Numerous generalized metrics have been developed in recent years, as highlighted in [3–9].

In the Poincaré Half-Plane Model, the geodesics are represented by either vertical lines perpendicular to the x-axis or circular arcs centered on the x-axis. It's important to note that geodesics in the Poincaré Half-Plane Model are different from the Euclidean straight lines.

In 2019, R. Kaya gave some generalizations of the classical Poincaré upper half-plane model by replacing the circular arcs with elliptical arcs with center on the x-axis and foci on the x-axis or on the lines perpendicular to the x-axis at the center, in the upper half-plane. This approach leads to a class of generalized upper half-planes with an infinite number of members, each with its own unique hyperbolic structure. The paper showed that using the Poincaré distance function, the generalized upper half-plane can be used to create a metric geometry known as $\mathbb{H}_k$ [4].

The purpose of this work is to introduce an alternative distance function that differs from the traditional Poincaré distance function and to examine the distance between two points on vertical lines, horizontal lines and semi-ellipses with the center on the x-axis in the upper half-plane. It is also shown that translations parallel to the x-axis and reflections in the vertical lines preserve the generalized Poincaré distance in the upper half-plane.

By investigating this alternative distance function, we aim to unveil new insights into the geometric properties and characteristics of the upper half-plane. The findings presented in this paper contribute to the understanding of non-traditional distance metrics and their implications for determining optimal paths in different geometrical contexts.

2. The generalized Poincaré metric

In the upper half Poincaré model, the hyperbolic length of an arbitrary curve γ is defined by using the Poincaré distance function $\int\limits_{\gamma} \frac{\sqrt{dx^2+dy^2}}{y}$. The Riemann metric with $E=\frac{1}{a^2y^2}$, $G=\frac{1}{b^2y^2}$, $F=0$, where $a>0, b>0$ and $a,b\in\mathbb{R}$ can be expressed as

$$Edx^2+2Fdxdy+Gdy^2=\frac{dx^2}{a^2y^2}+\frac{dy^2}{b^2y^2}.$$

This metric is a generalization of the Poincaré metric in the upper half-plane. In the Poincaré metric, the expressions for the metric coefficients are $E=G=1/y^2$ and $F=0$. By introducing additional parameters a and b, the given metric extends the Poincaré metric and allows for a more generalized representation of distances in the upper half-plane. So both metrics serve to calculate the lengths of curves in this region, but they differ in their constant coefficients.

Definition 1. *In the upper half-plane, the distance between two points P and Q along the γ curve is given by the integral*

$$\int\limits_{\gamma} \frac{\sqrt{\frac{dx^2}{a^2}+\frac{dy^2}{b^2}}}{y}, \qquad a,b\in\mathbb{R}^+.$$

The differentials dx and dy represent infinitesimal changes.

Theorem 2. *Let P and Q be two points with equal x-coordinates in the upper half-plane. The generalized Poincaré length of the Euclidean line segment connecting P and Q is given by $\frac{1}{b}\ln\frac{y_2}{y_1}$, where y_1 and y_2 are the y-coordinates of P and Q, respectively.*

Proof. Let $P(x_1,y_1)$ and $Q(x_2,y_2)$ be on the vertical line $x=h$. The generalized hyperbolic length of the Euclidean vertical line segment $x=h$ joining the points P and Q

$$\int\limits_{y_1}^{y_2} \frac{\sqrt{\frac{dx^2}{a^2}+\frac{dy^2}{b^2}}}{y}.$$

In this case, $dx=0$ and $dy=y_2-y_1$. Considering $y_1\le y_2$, we have the generalized Poincaré length of the line segment joining the points $P(x_1,y_1)$ and $Q(x_2,y_2)$

$$\frac{1}{b}\int_{y_1}^{y_2}\frac{dy}{y}=\frac{1}{b}\ln\frac{y_2}{y_1}.$$

This proves that the distance between two points P and Q on the line $x=h$ in the upper half-plane is given by $\frac{1}{b}\ln\frac{y_2}{y_1}$. ■

Theorem 3. *Let P and Q be two points with equal y-coordinates in the upper half-plane. The generalized Poincaré length of the Euclidean line segment with the equation $y=k$ connecting P and Q is given by $\frac{1}{ak}(x_2-x_1)$, where x_1 and x_2 are the x-coordinates of P and Q, respectively.*

Proof. Let $P(x_1,y_1)$ and $Q(x_2,y_2)$ be on the horizontal line $y=k$. The generalized Poincaré length of the Euclidean horizontal line segment joining the points P and Q is

$$\int_{x_1}^{x_2}\frac{\sqrt{\frac{dx^2}{a^2}+\frac{dy^2}{b^2}}}{y}.$$

In this case, $dy=0$ and $dx=x_2-x_1$. Considering $x_1\leq x_2$, we have the generalized Poincaré length of the line segment joining the points $P(x_1,y_1)$ and $Q(x_2,y_2)$:

$$\frac{1}{a}\int_{x_1}^{x_2}\frac{dx}{k}=\frac{1}{ak}(x_2-x_1).$$

This proves that in the upper half-plane model, the generalized Poincaré distance between two points P and Q on the line $y=k$ is given by $\frac{1}{ak}(x_2-x_1)$. ■

Proposition 4. *Let γ be semi-ellipse having the equation $((x-c)^2/a^2)+y^2/b^2=1$, with the center $C=(c,0)$. If P and Q are points of γ such that the rays CP and CQ make angles α and β ($\alpha<\beta$), respectively, with the positive x-axis, then the generalized Poincaré length of arc PQ is*

$$\frac{1}{b}\ln\frac{\csc\beta-\cot\beta}{\csc\alpha-\cot\alpha}.$$

Proof. Consider the semi-ellipse γ with the equation $((x-c)^2/a^2)+y^2/b^2=1$, centered at $C=(c,0)$, and points P and Q on γ such that the rays CP and CQ make angles α and β ($\alpha<\beta$) with the positive $x-$axis, respectively. If the semi-ellipse γ is parameterized by using t, then

$$x=c+acost,\qquad y=b\sin t.$$

Let t_P and t_Q denote the values of t corresponding to the points P and Q, respectively. Using the parameterization, we can express

$$\frac{dx}{dt}=-a\sin t,\qquad \frac{dy}{dt}=b\cos t.$$

So the generalized Poincaré length of the arc PQ can be written as

$$\int_{\gamma}\frac{\sqrt{\frac{dx^2}{a^2}+\frac{dy^2}{b^2}}}{y}=\frac{1}{b}\int_{t_P}^{t_Q}\frac{dt}{\sin t}.$$

Since $\alpha=t_P$ and $\beta=t_Q$, we can rewrite the formula as

$$\frac{1}{b}\ln\frac{\csc\beta-\cot\beta}{\csc\alpha-\cot\alpha}.$$

■

Theorem 5. *If PQ and $P'Q'$ are the arcs of Euclidean similar ellipses which have the same center C on the x-axis and foci on the x-axis or on the lines perpendicular to the x-axis at the center C and such that each of the triples C,P,P' and C,Q,Q' is collinear in the upper half plane, then two arcs PQ and $P'Q'$ have the same generalized Poincaré length.*

Proof. Let PQ be an arc on the ellipse α with the center $C(c,0)$ and equation

$$\frac{(x-c)^2}{A^2}+\frac{y^2}{B^2}=1, \qquad A,B\in\mathbb{R}^+$$

and let $P'Q'$ be an arc on the ellipse α' with the same center C and equation

$$\frac{(x-c)^2}{A'^2}+\frac{y^2}{B'^2}=1, \qquad A',B'\in\mathbb{R}^+$$

such that C, P, Q and C, P', Q' are collinear points in the upper half plane. Also the angles t_P and t_Q, $t_P<t_Q$ be the angles between the rays CP , CQ and positive x-axis, respectively. Since α and α' are similar ellipses, we get

$$\frac{A}{A'}=\frac{B}{B'}=k.$$

If the semi-ellipse α is parameterized by using t, then

$$x=c+A\cos t, \qquad y=B\sin t.$$

Determine the values of t corresponding two points P and Q, as t_P and t_Q respectively. Using the parametrization, we can express

$$\frac{dx}{dt}=-A\sin t, \qquad \frac{dy}{dt}=B\cos t.$$

So the generalized Poincaré length of the arc PQ is given by

$$\int_{t_P}^{t_Q}\frac{\sqrt{\frac{A^2}{a^2}\sin^2 t+\frac{B^2}{b^2}\cos^2 t}}{B\sin t}dt.$$

Using $A=kA'$ and $B=kB'$,

$$\int_{t_P}^{t_Q}\sqrt{\frac{A'^2\sin^2 t}{a^2}+\frac{B'^2\cos^2 t}{b^2}}dt$$

is obtained. Since the last integral is the generalized Poincaré length of the arc joining P' and Q', two arcs PQ and $P'Q'$ have the same generalized Poincaré length. ■

Theorem 6. *In the upper half-plane, translations parallel to the x-axis preserve the generalized Poincaré distance.*

Proof. Let τ be the translation defined by $\tau(x,y)=(x+h,y)$, where h is a fixed number. Consider a curve γ parametrized as $[u(t),v(t)]$ for $t_1<t<t_2$. The translated curve $\tau(\gamma)$ has the parametrization $[u(t)+h,v(t)]$ for the same interval $t_1<t<t_2$. The differentials dx and dy along both γ and $\tau(\gamma)$ can be expressed as $dx=u'(t)dt, dy=v'(t)dt$. By substituting these differentials into the generalized Poincaré distance formula, we have

$$\int_{t_1}^{t_2}\frac{\sqrt{\frac{u'(t)^2}{a^2}+\frac{v'(t)^2}{b^2}}}{v(t)}dt.$$

Consequently, both γ and $\tau(\gamma)$ have the same the generalized Poincaré length, given by the above integral. Therefore, translations parallel to the x-axis preserve the generalized Poincaré distances. ■

Theorem 7. *Reflections in the lines perpendicular to the x-axis in the upper half-plane preserve the generalized Poincaré distance.*

Proof. Let l be a line perpendicular to the x-axis having equation $x = h$ in the upper half-plane. If γ is any curve parametrized as $[u(t), v(t)]$, $t_1 < t < t_2$, then $\rho_l(\gamma)$ has parametrization $[2h - u(t), v(t)]$, $t_1 < t < t_2$. The differentials dx and dy along γ and $\rho_l(\gamma)$ can be expressed as: $dx = u'(t)dt, dy = v'(t)dt$ and $dx = -u'(t)dt, dy = v'(t)dt$, respectively. By substituting these differentials into the generalized Poincaré distance formula, we have:

$$\int_{t_1}^{t_2} \frac{\sqrt{\frac{u'(t)^2}{a^2} + \frac{v'(t)^2}{b^2}}}{v(t)} dt.$$

Consequently, both γ and $\rho_l(\gamma)$ have the same generalized Poincaré length. ■

3. Conclusions

This paper provides an in-depth exploration of the generalized Poincaré distance in the upper half-plane. It establishes formulas for calculating the generalized Poincaré length in various scenarios, considering equal x-coordinates, equal y-coordinates, and arcs of a semi-ellipse. It is show that the generalized Poincaré arc lengths on the similar semi-ellipses with the same center on the x-axis and foci on the x-axis or on the lines perpendicular to the x-axis at the center subtending the same angle at the center in the upper half-plane are equal. Furthermore, it demonstrates that translations parallel to the x-axis and reflections in the lines perpendicular to the x-axis preserve the generalized Poincaré distance. These findings contribute to the understanding and application of the generalized Poincaré distance in geometric contexts within the upper half-plane.

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