Hacettepe Journal of
Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 53 (6) (2024), 1497 – 1514
DOI : 10.15672/hujms.1318033

Research Article

On approximation of functions, conjugate to the functions of several variables belonging to weighted Lipschitz and Zygmund classes

Shailesh Kumar Srivastava*, Sachin Devaiya

Department of Mathematics, Sardar Vallabhbhai National Institute of Technology, Surat, 395007, Gujarat, India

Abstract

In the present paper, we introduce a new weighted Lipschitz class $W(L^p(T^N), \xi_1(s_1), \dots, \xi_N(s_N))$ and Zygmund class $Z(L^p(T^N), \xi_1(s_1), \dots, \xi_N(s_N))$ for $N \in \mathbb{N}$, which generalizes the classes given in [12, 16]. We prove two theorems about the degree (error) of approximation of functions, conjugate to the N-variable functions (2π-periodic in each variable) belonging to these classes using the N-multiple matrix means of their N-multiple conjugate Fourier series. We improve the results of Móricz and Rhoades [11] and Móricz and Shi [12], which are given in the form of corollaries.

Mathematics Subject Classification (2020). 26A16, 40C05, 41A25

Keywords. Degree (error) of approximation, weighted Lipschitz class, weighted Zygmund class, N-multiple matrix means

1. Introduction

Many investigators [1,3–11,16] have investigated the approximation of 2π-periodic functions of two variables belonging to various Lipschitz and Zygmund classes using different summability techniques. In 1987, Móricz and Rhoades [10] obtained some results on error of approximation for 2π-periodic functions of two variables belonging to $Lip(\alpha)$, for $\alpha \in (0, 1]$, using Nörlund means. They also used the same summability method in [11] to approximate continuous functions $g(x, y)$ belonging to function classes $Lip(\alpha, \beta)$ and $Z(\alpha, \beta)$. Recently, Rathore and Singh [16] generalized the classes $Lip(\alpha, \beta)$ and $Z(\alpha, \beta)$ by introducing the classes $Lip(\alpha, \beta; p)$ and $Z(\alpha, \beta; p)$, respectively. They approximated functions of two variables belonging to these classes using almost Euler means of their double Fourier series. It seems that less research work has been conducted in the direction of approximating the conjugate functions of two variables using the conjugate double Fourier series, as evidenced by the limited body of work published in [12–15]. In 1987, Móricz and Shi [12] approximated conjugate functions of two variables belonging to $Lip(\alpha, \beta)$ and $Z(\alpha, \beta)$ classes using Cesàro means of conjugate double Fourier series. In 2012, Nigam and Sharma [14] approximated the conjugate of functions of two variables using double

*Corresponding Author.
Email addresses: shaileshiitr2010@gmail.com; shailesh@amhd.svnit.ac.in (S.K. Srivastava), sbdevaiya18695@gmail.com (S. Devaiya)
Received: 21.06.2023; Accepted: 11.12.2023

matrix means of its double conjugate Fourier series. In 2022, Nigam and Saha [13] approximated the conjugate of functions of two variables belonging to generalized Hölder space by double Hausdorff means of its double conjugate Fourier series. Recently, Patel and Vyas [15] approximated the functions, conjugate to the functions of two variables in mixed Lebegue space using double Karmata means of their double conjugate Fourier series.

2. Results for functions of two variables

Let $g(x,y)$ be a complex valued function, 2π-periodic in each variable, integrable over the two-dimensional torus $T^2 = (-\pi,\pi)\times(-\pi,\pi)$.
The double Fourier series of a function $g(x,y)$ is given by [12]

$$g(x,y) = \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} c_{kl}\, e^{i(kx+ly)}, \tag{2.1}$$

where

$$c_{kl} \;=\; c_{kl}(g) = \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} g(s,t)\, e^{-i(ks+lt)} ds\, dt,\;\; k,l\in\mathbb{Z}.$$

The conjugate series of (2.1) is given by [12]:

$$\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} (-i\,\mathrm{sign}\, k)(-i\,\mathrm{sign}\, l) c_{kl}\, e^{i(kx+ly)}, \tag{2.2}$$

(conjugate with respect to x and y).
The conjugate function of $g(x,y)$, denoted by $\widetilde{g}(x,y)$, is defined as [12]

$$\begin{aligned}\widetilde{g}(x,y) = \frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi} \frac{1}{\tan\frac{s}{2}\tan\frac{t}{2}} \{g(x+s,y+t) - g(x-s,y+t) \\ -g(x+s,y-t) + g(x-s,y-t)\}\, ds\, dt.\end{aligned} \tag{2.3}$$

Let $\{\widetilde{s}_{k,l}\}$ be the sequence of $(k+1,l+1)^{\text{th}}$ partial sums corresponding to the conjugate Fourier series (2.2), which is defined as

$$\widetilde{s}_{k,l} = \sum_{m=0}^{k}\sum_{n=0}^{l}(-i\,\mathrm{sign}\, m)(-i\,\mathrm{sign}\, n)\, c_{mn}\, e^{i(mx+ny)}.$$

Define

$$\widetilde{t}_{k,l}^{A,B} = \sum_{i=0}^{k}\sum_{j=0}^{l} a_{k,i}\, b_{l,j}\; \widetilde{s}_{i,j}, \qquad k,l\in\mathbb{N}\cup\{0\},$$

where $A\equiv(a_{k,i})$ and $B\equiv(b_{l,j})$ are two lower triangular matrices with non-decreasing and non-negative entries with respect to i and j such that $a_{k,-1} = b_{l,-1} = 0, A_{k,\sigma} = \sum_{i=0}^{\sigma} a_{k,i}, B_{l,\tau} = \sum_{j=0}^{\tau} b_{l,j}$, and $A_{k,k} = B_{l,l} = 1$. If $\widetilde{t}_{k,l}^{A,B}\to s_1$ as $k\to\infty$ and $l\to\infty$, then the conjugate series (2.2) is said to be summable to s_1 by the double matrix means.
The regularity conditions of the double matrix means are same as given in [14].
Some particular cases of double matrix means are as follows:

1. (1) If we take $a_{k,i} = \frac{k!\,\gamma!}{(k+\gamma)!}\binom{k+\gamma-i-1}{\gamma-1}$ and $b_{l,j} = \frac{l!\,\delta!}{(l+\delta)!}\binom{l+\delta-j-1}{\delta-1}$, where $\gamma,\delta>-1$, then double matrix means reduces to double Cesàro means of order γ and δ, denoted by (C,γ,δ)-means.
2. (2) If we take $\gamma=\delta=1$ in the above matrix, then (C,γ,δ)-means reduces to $(C,1,1)$-means.
3. (3) If we take $a_{k,i} = \frac{1}{(k-i+1)\log(k+1)}$ and $b_{l,j} = \frac{1}{(l-j+1)\log(l+1)}$, then double matrix means reduces to double Harmonic means, denoted by $(H,1,1)$-means.

(4) If we take $a_{k,i} = \frac{1}{(1+q_1)^k}\binom{k}{i}q_1^{k-i}$ and $b_{l,j} = \frac{1}{(1+q_2)^l}\binom{l}{j}q_2^{l-j}$, where $q_1, q_2 > 0$, then double matrix means reduces to double Euler summability of order q_1 and q_2, denoted by (E, q_1, q_2)-means.

(5) If we take $q_1 = q_2 = 1$ in the above matrix, then (E, q_1, q_2)-means reduces to $(E, 1, 1)$-means.

(6) If we take $a_{k,i} = \frac{p_{k-i}}{P_k}$ and $b_{l,j} = \frac{q_{l-j}}{Q_l}$, where $P_k = \sum_{i=0}^{k} p_i \neq 0$ and $Q_l = \sum_{j=0}^{l} q_j \neq 0$, then double matrix means reduces to double Nörlund summability, denoted by (N, p_k, q_l)-means.

The degree (error) of approximation, $E_{k,l}$, of a function $\widetilde{g}(x,y) \in L^p(T^2)$ by trigonometric polynomial $\widetilde{T}_{k,l}(x,y)$ of degree $(k+l)$ is given by

$$E_{k,l}(\widetilde{g}) = \min_{\widetilde{T}_{k,l}} \|\widetilde{g}(x,y) - \widetilde{T}_{k,l}(x,y)\|_p.$$

The $\widetilde{T}_{k,l}(x,y)$ is called approximant of $\widetilde{g}(x,y)$ and this method of approximation is called the trigonometric Fourier approximation.

The space of Lebesgue functions on T^2 is denoted by $L^p(T^2), p \geq 1$ and the norm on it is defined by

$$\|g\|_p = \left\{\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |g(x,y)|^p dx\, dy\right\}^{\frac{1}{p}} \quad (1 \leq p < \infty) \text{ and}$$
$$\|g\|_\infty = \sup_{-\pi \leq x,y \leq \pi} |g(x,y)|.$$

Let $L_1^{\mu,\nu}(g,s,t)$ be a weighted integral modulus of continuity of function g, which is defined as

$$L_1^{\mu,\nu}(g,s,t) = \sup_{|\eta|\leq s, |\theta|\leq t} \left\{\|\{g(x+\eta, y+\theta) - g(x,y)\}w(s,t)\|_p\right\},$$

where $w(s,t)$ is a weight function, defined by

$$w(s,t) = \sqrt{\frac{s^2\,\sin^{2\mu}(\frac{x}{2}) + t^2\,\sin^{2\nu}(\frac{y}{2})}{s^2+t^2}} \text{ for } \mu,\nu \geq 0, s^2+t^2 \neq 0. \tag{2.4}$$

The new weighted Lipschitz class $W(L^p(T^2), \xi_1(s), \xi_2(t))$ is introduced as follows:

Definition 2.1 ($W(L^p(T^2), \xi_1(s), \xi_2(t))$-class)**.** For positive increasing functions $\xi_1(s)$ and $\xi_2(t)$, the weighted Lipschitz class, denoted by $W(L^p(T^2), \xi_1(s), \xi_2(t))$, is defined as

$$W(L^p(T^2), \xi_1(s), \xi_2(t)) = \Big\{g \in L^p(T^2) : L_{1,x}^{\mu}(g,s) = O(\xi_1(s)) \text{ and } L_{1,y}^{\nu}(g,t) = O(\xi_2(t))\Big\},$$

where $L_{1,x}^{\mu}(g,s)$ and $L_{1,y}^{\nu}(g,t)$ are partial weighted integral moduli of continuity of function g, which are defined as

$$L_{1,x}^{\mu}(g,s) = L_1^{\mu,\nu}(g,s,0) = \sup_{|\eta|\leq s}\{\|\{g(x+\eta,y) - g(x,y)\}\sin^{\mu}(\tfrac{x}{2})\|_p\},$$

and

$$L_{1,y}^{\nu}(g,t) = L_1^{\mu,\nu}(g,0,t) = \sup_{|\theta|\leq t}\{\|\{g(x,y+\theta) - g(x,y)\}\sin^{\nu}(\tfrac{y}{2})\|_p\}.$$

Definition 2.2 ($Lip(\alpha,\beta;p)$-class [16])**.** For $\alpha, \beta \in (0,1]$, the $Lip(\alpha,\beta;p)$-class is defined as

$$Lip(\alpha,\beta;p) = \Big\{g \in L^p(T^2) : L_{1,x}^{p}(g,s) = O(s^\alpha) \text{ and } L_{1,y}^{p}(g,t) = O(t^\beta)\Big\},$$

where $L_{1,x}^{p}(g,s)$ and $L_{1,y}^{p}(g,t)$ are given by

$$L_{1,x}^{p}(g,s) = \sup_{|\eta|\leq s}\{\|g(x+\eta,y) - g(x,y)\|_p\},$$

and

$$L^p_{1,y}(g,t) = \sup_{|\theta|\le t}\{\|g(x,y+\theta) - g(x,y)\|_p\}.$$

Definition 2.3 ($Lip(\alpha,\beta)$-class [12]). For $\alpha,\beta \in (0,1]$, the $Lip(\alpha,\beta)$-class is defined as

$$Lip(\alpha,\beta) = \left\{g : T^2 \to \mathbb{R} : L_{1,x}(g,s) = O(s^\alpha) \text{ and } L_{1,y}(g,t) = O(t^\beta)\right\},$$

where $L_{1,x}(g,s)$ and $L_{1,y}(g,t)$ are given by

$$L_{1,x}(g,s) = \sup_{x,y}\sup_{|\eta|\le s}\{|g(x+\eta,y) - g(x,y)|\},$$

and

$$L_{1,y}(g,t) = \sup_{x,y}\sup_{|\theta|\le t}\{|g(x,y+\theta) - g(x,y)|\}.$$

Remark 2.4. If we take $\mu = 0, \nu = 0, \xi_1(s) = s^\alpha$, and $\xi_2(t) = t^\beta$, for $\alpha,\beta \in (0,1]$ in Definition 2.1, then $W(L^p(T^2),\xi_1(s),\xi_2(t))$ reduces to $Lip(\alpha,\beta;p)$. If we take $p \to \infty$ in Definition 2.2, then $Lip(\alpha,\beta;p)$ reduces to $Lip(\alpha,\beta)$. Then, we can write

$$Lip(\alpha,\beta) \subseteq Lip(\alpha,\beta;p) \subseteq W(L^p(T^2),\xi_1(s),\xi_2(t)).$$

Here, we define the total weighted integral modulus of symmetric smoothness of a function g by

$$Z_2^{\mu,\nu}(g,s,t) = \sup_{|\eta|\le s, |\theta|\le t}\left\{\|\psi(\eta,\theta)w(s,t)\|_p\right\},$$

where $w(s,t)$ is weight function, defined as (2.4), and

$$\psi(\eta,\theta) = g(x+\eta,y+\theta) + g(x-\eta,y+\theta) + g(x+\eta,y-\theta) + g(x-\eta,y-\theta) - 4g(x,y).$$

Now, the new weighted Zygmund class $Z(L^p(T^2),\xi_1(s),\xi_2(t))$ is introduced as follows:

Definition 2.5 ($Z(L^p(T^2),\xi_1(s),\xi_2(t))$-class). For positive increasing functions $\xi_1(s)$ and $\xi_2(t)$, the weighted Zygmund class, denoted by $Z(L^p(T^2),\xi_1(s),\xi_2(t))$, is defined as

$$\begin{aligned} Z(L^p(T^2),\xi_1(s),\xi_2(t)) = \quad & \Big\{g \in L^p(T^2) : Z^\mu_{2,x}(g,s) = O(\xi_1(s)) \text{ and} \\ & \qquad Z^\nu_{2,y}(g,t) = O(\xi_2(t))\Big\}, \end{aligned}$$

where $Z^\mu_{2,x}(g,s)$ and $Z^\nu_{2,y}(g,t)$ are partial weighted integral moduli of smoothness of function g, which are defined as

$$Z^\mu_{2,x}(g,s) = \frac{Z_2^{\mu,\nu}(g,s,0)}{2} = \sup_{|\eta|\le s}\{\|\{g(x+\eta,y) + g(x-\eta,y) - 2g(x,y)\}\sin^\mu(\tfrac{x}{2})\|_p\},$$

and

$$Z^\nu_{2,y}(g,t) = \frac{Z_2^{\mu,\nu}(g,0,t)}{2} = \sup_{|\theta|\le t}\{\|\{g(x,y+\theta) + g(x,y-\theta) - 2g(x,y)\}\sin^\nu(\tfrac{y}{2})\|_p\}.$$

Definition 2.6 ($Z(\alpha,\beta;p)$-class [16]). For $\alpha,\beta \in (0,2]$, the $Z(\alpha,\beta;p)$-class is defined as

$$Z(\alpha,\beta;p) = \left\{g \in L^p(T^2) : Z^p_{2,x}(g,s) = O(s^\alpha) \text{ and } Z^p_{2,y}(g,t) = O(t^\beta)\right\},$$

where $Z^p_{2,x}(g,s)$ and $Z^p_{2,y}(g,t)$ are given by

$$Z^p_{2,x}(g,s) = \sup_{|\eta|\le s}\{\|g(x+\eta,y) + g(x-\eta,y) - 2g(x,y)\|_p\},$$

and

$$Z^p_{2,y}(g,t) = \sup_{|\theta|\le t}\{\|g(x,y+\theta) + g(x,y-\theta) - 2g(x,y)\|_p\}.$$

Definition 2.7 ($Z(\alpha,\beta)$-class [12])**.** For $\alpha,\beta\in(0,2]$, the $Z(\alpha,\beta)$-class is defined as

$$Z(\alpha,\beta)=\left\{g:T^2\to\mathbb{R}:Z_{2,x}(g,s)=O(s^{\alpha})\,\text{and}\,Z_{2,y}(g,t)=O(t^{\beta})\right\},$$

where $Z_{2,x}(g,s)$ and $Z_{2,y}(g,t)$ are given by

$$Z_{2,x}(g,s)=\sup_{x,y}\sup_{|\eta|\le s}\{|g(x+\eta,y)+g(x-\eta,y)-2g(x,y)|\},$$

and

$$Z_{2,y}(g,t)=\sup_{x,y}\sup_{|\theta|\le t}\{|g(x,y+\theta)+g(x,y-\theta)-2g(x,y)|\}.$$

Remark 2.8. If we take $\mu=0,\nu=0,\xi_1(s)=s^{\alpha}$, and $\xi_2(t)=t^{\beta}$, for $\alpha,\beta\in(0,2]$ in Definition 2.5, then $Z(L^p(T^2),\xi_1(s),\xi_2(t))$ reduces to $Z(\alpha,\beta;p)$. If we take $p\to\infty$ in Definition 2.6, then $Z(\alpha,\beta;p)$ reduces to $Z(\alpha,\beta)$. Then, we can write

$$Z(\alpha,\beta)\subseteq Z(\alpha,\beta;p)\subseteq Z(L^p(T^2),\xi_1(s),\xi_2(t)).$$

It is clear that $Z_2^{\mu,\nu}(g,s,t), Z_{2,x}^{\mu}(g,s)$, and $Z_{2,y}^{\nu}(g,t)$ are increasing functions of s and t, and they satisfies the following inequalities

$$2\max\{Z_{2,x}^{\mu}(g,s),Z_{2,y}^{\nu}(g,t)\}\le Z_2^{\mu,\nu}(g,s,t)\le 2\{Z_{2,x}^{\mu}(g,s)+Z_{2,y}^{\nu}(g,t)\},$$

and

$$Z_{2,x}^{\mu}(g,s)\le 2L_{1,x}^{\mu}(g,s),\;\; Z_{2,y}^{\nu}(g,t)\le 2L_{1,y}^{\nu}(g,t). \tag{2.5}$$

From (2.5), it is clear that

$$W(L^p(T^2),\xi_1(s),\xi_2(t))\subseteq Z(L^p(T^2),\xi_1(s),\xi_2(t)).$$

We also write

$$\widetilde{R}_k(s)=\sum_{i=0}^{k}\frac{a_{k,k-i}\,\cos(2k-2i+1)\frac{s}{2}}{2\pi\,\sin\frac{s}{2}}\;,\;\widetilde{S}_l(t)=\sum_{j=0}^{l}\frac{b_{l,l-j}\,\cos(2l-2j+1)\frac{t}{2}}{2\pi\,\sin\frac{t}{2}},$$

$\sigma:=[\frac{1}{s}]$, and $\tau:=[\frac{1}{t}]$ are the integral part of $\frac{1}{s}$ and $\frac{1}{t}$, respectively and

$$\varphi(s,t)=g(x+s,y+t)-g(x+s,y-t)-g(x-s,y+t)+g(x-s,y-t).$$

Note 2.9. We can prove the following inequalities:

$$|\varphi(s,t)|\le 2(Z_{2,x}(g,s)+Z_{2,y}(g,t)), \tag{2.6}$$

and

$$\|\varphi(s,t)\|_p\le 2\left(\frac{Z_{2,x}^{\mu}(g,s)}{s^{\mu}}+\frac{Z_{2,y}^{\nu}(g,t)}{t^{\nu}}\right),\text{ for } s,t\neq 0. \tag{2.7}$$

Here we need some lemmas to prove our theorems:

Lemma 2.10. $|\widetilde{R}_k(s)|=O\left(\frac{1}{s}\right)$*, for* $0<s\le\frac{\pi}{k+1}$ *and* $|\widetilde{S}_l(t)|=O\left(\frac{1}{t}\right)$*, for* $0<t\le\frac{\pi}{l+1}$.
The proof is given in [17, Lemma 1].

Lemma 2.11. $|\widetilde{R}_k(s)|=O\left(\frac{A_{k,k-\sigma}}{s}\right)$*, for* $\frac{\pi}{k+1}<s\le\pi$ *and* $|\widetilde{S}_l(t)|=O\left(\frac{B_{l,l-\tau}}{t}\right)$*, for* $\frac{\pi}{l+1}<t\le\pi$.
The proof is given in [17, Lemma 2].

Theorem 2.12. *Let* $\widetilde{g}(x,y)$ *be the conjugate function of* 2π*-periodic function* $g(x,y)$ *belonging to* $Z(L^p(T^2),\xi_1(s),\xi_2(t))$*. Then the degree (error) of approximation of function* $\widetilde{g}(x,y)$ *through the double matrix means of its double conjugate Fourier series is given by:*

$$\|\widetilde{t}_{k,l}^{A,B}-\widetilde{g}\|_p=O\left((k+1)^{\mu}\,\xi_1\left(\frac{\pi}{k+1}\right)\right)+O\left((l+1)^{\nu}\,\xi_2\left(\frac{\pi}{l+1}\right)\right),$$

provided the positive increasing functions $\xi_1(s)$ and $\xi_2(t)$ satisfy following conditions:

$$\left(\int_{\frac{\pi}{k+1}}^{\pi}\left(\frac{\xi_1(s)}{s^{\mu+1}}\right)^p ds\right)^{\frac{1}{p}} = O\left((k+1)^{\mu+1-\frac{1}{p}}\,\xi_1\left(\frac{\pi}{k+1}\right)\right), \tag{2.8}$$

and

$$\left(\int_{\frac{\pi}{l+1}}^{\pi}\left(\frac{\xi_2(t)}{t^{\nu+1}}\right)^q dt\right)^{1/q} = O\left((l+1)^{\nu+1-\frac{1}{q}}\,\xi_2\left(\frac{\pi}{l+1}\right)\right). \tag{2.9}$$

Proof. We have

$$\widetilde{s}_{k,l}-\widetilde{g} = \tfrac{1}{4\pi^2}\int_0^\pi\int_0^\pi \varphi(s,t)\times\left\{\frac{\cos(k+\frac12)s\,\cos(l+\frac12)t}{\sin\frac{t}{2}\,\sin\frac{s}{2}} - \frac{\cos(k+\frac12)s\,\cos\frac{t}{2}}{\sin\frac{t}{2}\,\sin\frac{s}{2}} - \frac{\cos\frac{s}{2}\,\cos(l+\frac12)t}{\sin\frac{t}{2}\,\sin\frac{s}{2}}\right\}ds\,dt.$$

Then

$$\begin{aligned}
\widetilde{t}^{A,B}_{k,l}-\widetilde{g} &= \sum_{i=0}^{k}\sum_{j=0}^{l} a_{k,i}\,b_{l,j}\{\,\widetilde{s}_{i,j}-\widetilde{g}\,\}\\
&= \int_0^\pi\int_0^\pi \frac{\varphi(s,t)}{4\pi^2}\sum_{i=0}^{k}\sum_{j=0}^{l} a_{k,i}\,b_{l,j}\left\{\frac{\cos(2i+1)\frac{s}{2}\,\cos(2j+1)\frac{t}{2}}{\sin\frac{s}{2}\,\sin\frac{t}{2}} - \frac{\cos(2i+1)\frac{s}{2}\,\cos\frac{t}{2}}{\sin\frac{s}{2}\,\sin\frac{t}{2}} - \frac{\cos(2j+1)\frac{t}{2}\,\cos\frac{s}{2}}{\sin\frac{s}{2}\,\sin\frac{t}{2}}\right\}ds\,dt\\
&= \int_0^\pi\int_0^\pi \frac{\varphi(s,t)}{2\pi}\{2\pi\,\widetilde{R}_k(s)\,\widetilde{S}_l(t) - \cot\tfrac{s}{2}\,\widetilde{S}_l(t) - \cot\tfrac{t}{2}\,\widetilde{R}_k(s)\}ds\,dt.
\end{aligned}$$

Using inequality (2.7), we have

$$\begin{aligned}
\|\widetilde{t}^{A,B}_{k,l}-\widetilde{g}\|_p &= \int_0^\pi\int_0^\pi \|\varphi(s,t)\|_p\{2\pi\,\widetilde{R}_k(s)\,\widetilde{S}_l(t) - \cot\tfrac{s}{2}\,\widetilde{S}_l(t) - \cot\tfrac{t}{2}\,\widetilde{R}_k(s)\}ds\,dt\\
&= \left(\int_0^{\frac{\pi}{k+1}}\int_0^{\frac{\pi}{l+1}} + \int_{\frac{\pi}{k+1}}^{\pi}\int_0^{\frac{\pi}{l+1}} + \int_0^{\frac{\pi}{k+1}}\int_{\frac{\pi}{l+1}}^{\pi} + \int_{\frac{\pi}{k+1}}^{\pi}\int_{\frac{\pi}{l+1}}^{\pi}\right)\|\varphi(s,t)\|_p\times\\
&\quad \{2\pi\,\widetilde{R}_k(s)\,\widetilde{S}_l(t) - \cot\tfrac{s}{2}\,\widetilde{S}_l(t) - \cot\tfrac{t}{2}\,\widetilde{R}_k(s)\}ds\,dt\\
&\le \left(\int_0^{\frac{\pi}{k+1}}\int_0^{\frac{\pi}{l+1}} + \int_{\frac{\pi}{k+1}}^{\pi}\int_0^{\frac{\pi}{l+1}} + \int_0^{\frac{\pi}{k+1}}\int_{\frac{\pi}{l+1}}^{\pi} + \int_{\frac{\pi}{k+1}}^{\pi}\int_{\frac{\pi}{l+1}}^{\pi}\right)\left(\frac{Z^{\mu}_{2,x}(g,s)}{s^\mu} + \right.\\
&\quad \left.\frac{Z^{\nu}_{2,y}(g,t)}{t^\nu}\right)\{2\pi\,\widetilde{R}_k(s)\,\widetilde{S}_l(t) - \cot\tfrac{s}{2}\,\widetilde{S}_l(t) - \cot\tfrac{t}{2}\,\widetilde{R}_k(s)\}ds\,dt\\
&= \left(\int_0^{\frac{\pi}{k+1}}\int_0^{\frac{\pi}{l+1}} + \int_{\frac{\pi}{k+1}}^{\pi}\int_0^{\frac{\pi}{l+1}} + \int_0^{\frac{\pi}{k+1}}\int_{\frac{\pi}{l+1}}^{\pi} + \int_{\frac{\pi}{k+1}}^{\pi}\int_{\frac{\pi}{l+1}}^{\pi}\right)\times\\
&\quad \left\{\left(\frac{\xi_1(s)}{s^\mu}\,|\widetilde{R}_k(s)|\,|\widetilde{S}_l(t)|\,ds\,dt + \frac{\xi_2(t)}{t^\nu}\,|\widetilde{R}_k(s)|\,|\widetilde{S}_l(t)|\,ds\,dt\right)\right.\\
&\quad + \left(\frac{\xi_1(s)}{s^\mu}\,|\cot\tfrac{s}{2}|\,|\widetilde{S}_l(t)|\,ds\,dt + \frac{\xi_2(t)}{t^\nu}\,|\cot\tfrac{s}{2}|\,|\widetilde{S}_l(t)|\,ds\,dt\right)\\
&\quad \left.+ \left(\frac{\xi_1(s)}{s^\mu}\,|\widetilde{R}_k(s)|\,|\cot\tfrac{t}{2}|\,ds\,dt + \frac{\xi_2(t)}{t^\nu}\,|\widetilde{R}_k(s)|\,|\cot\tfrac{t}{2}|\,ds\,dt\right)\right\}\\
&= \left(\widetilde{I}_1+\widetilde{I}_2+\widetilde{I}_3+\widetilde{I}_4\right)\times\left\{\widetilde{J}_1+\widetilde{J}_2+\widetilde{J}_3\right\},\ \text{(say)}.
\end{aligned} \tag{2.10}$$

Using Lemma 2.10, second mean value theorem for integrals, and Hölder's inequality, we have

$$
\begin{aligned}
|\widetilde{I}_1\widetilde{J}_1| &= \left(\int_0^{\frac{\pi}{k+1}}\int_0^{\frac{\pi}{l+1}}\left\{\frac{\xi_1(s)\,|\widetilde{R}_k(s)|}{s^{\mu}}\right\}^p ds\,dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{\pi}{k+1}}\int_0^{\frac{\pi}{l+1}}\left\{|\widetilde{S}_l(t)|\right\}^q ds\,dt\right)^{\frac{1}{q}}\\
&\quad+\left(\int_0^{\frac{\pi}{k+1}}\int_0^{\frac{\pi}{l+1}}\left\{|\widetilde{R}_k(s)|\right\}^p ds\,dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{\pi}{k+1}}\int_0^{\frac{\pi}{l+1}}\left\{\frac{\xi_2(t)\,|\widetilde{S}_l(t)|}{t^{\nu}}\right\}^q ds\,dt\right)^{\frac{1}{q}}\\
&= O\left(\xi_1\left(\frac{\pi}{k+1}\right)(k+1)\left(\frac{\pi}{k+1}\right)^{\frac{1}{p}-\mu+\frac{1}{q}}(l+1)\left(\frac{\pi}{l+1}\right)^{\frac{1}{q}+\frac{1}{p}}\right)\\
&\quad+O\left((k+1)\left(\frac{\pi}{k+1}\right)^{\frac{1}{q}+\frac{1}{p}}\xi_2\left(\frac{\pi}{l+1}\right)(l+1)\left(\frac{\pi}{l+1}\right)^{\frac{1}{q}-\nu+\frac{1}{p}}\right)\\
&= O\left((k+1)^{\mu}\,\xi_1\left(\frac{\pi}{k+1}\right)\right)+O\left((l+1)^{\nu}\,\xi_2\left(\frac{\pi}{l+1}\right)\right). \qquad (2.11)
\end{aligned}
$$

Using Lemma 2.10, $\sin(\frac{s}{2})\geq\frac{s}{\pi}$, $\cos(\frac{s}{2})\leq 1$, second mean value theorem for integrals, and Hölder's inequality, we have

$$
\begin{aligned}
|\widetilde{I}_1\widetilde{J}_2| &= \left(\int_0^{\frac{\pi}{k+1}}\int_0^{\frac{\pi}{l+1}}\left\{\frac{\xi_1(s)}{s^{\mu+1}}\right\}^p ds\,dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{\pi}{k+1}}\int_0^{\frac{\pi}{l+1}}\left\{|\widetilde{S}_l(t)|\right\}^q ds\,dt\right)^{\frac{1}{q}}\\
&\quad+\left(\int_0^{\frac{\pi}{k+1}}\int_0^{\frac{\pi}{l+1}}\left\{\frac{1}{s}\right\}^p ds\,dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{\pi}{k+1}}\int_0^{\frac{\pi}{l+1}}\left\{\frac{\xi_2(t)\,|\widetilde{S}_l(t)|}{t^{\nu}}\right\}^q ds\,dt\right)^{\frac{1}{q}}\\
&= O\left(\xi_1\left(\frac{\pi}{k+1}\right)(k+1)\left(\frac{\pi}{k+1}\right)^{\frac{1}{p}-\mu+\frac{1}{q}}(l+1)\left(\frac{\pi}{l+1}\right)^{\frac{1}{q}+\frac{1}{p}}\right)\\
&\quad+O\left((k+1)\left(\frac{\pi}{k+1}\right)^{\frac{1}{q}+\frac{1}{p}}\xi_2\left(\frac{\pi}{l+1}\right)(l+1)\left(\frac{\pi}{l+1}\right)^{\frac{1}{q}-\nu+\frac{1}{p}}\right)\\
&= O\left((k+1)^{\mu}\,\xi_1\left(\frac{\pi}{k+1}\right)\right)+O\left((l+1)^{\nu}\,\xi_2\left(\frac{\pi}{l+1}\right)\right). \qquad (2.12)
\end{aligned}
$$

Similarly, we can prove

$$
|\widetilde{I}_1\widetilde{J}_3| = O\left((k+1)^{\mu}\,\xi_1\left(\frac{\pi}{k+1}\right)\right)+O\left((l+1)^{\nu}\,\xi_2\left(\frac{\pi}{l+1}\right)\right). \qquad (2.13)
$$

Using (2.8), second mean value theorem for integrals, Lemma 2.10, 2.11, and Hölder's inequality, we have

$$
\begin{aligned}
|\widetilde{I}_2\widetilde{J}_1| &= \left(\int_{\frac{\pi}{k+1}}^{\pi}\int_0^{\frac{\pi}{l+1}}\left\{\frac{\xi_1(s)}{s^{\mu+1}}\right\}^p ds\,dt\right)^{\frac{1}{p}}\left(\int_{\frac{\pi}{k+1}}^{\pi}\int_0^{\frac{\pi}{l+1}}\left\{s\,|\widetilde{R}_k(s)|\,|\widetilde{S}_l(t)|\right\}^q ds\,dt\right)^{\frac{1}{q}}\\
&\quad+\left(\int_{\frac{\pi}{k+1}}^{\pi}\int_0^{\frac{\pi}{l+1}}\left|\frac{A_{k,k-\sigma}}{s}\right|^p ds\,dt\right)^{\frac{1}{p}}\left(\int_{\frac{\pi}{k+1}}^{\pi}\int_0^{\frac{\pi}{l+1}}\left\{\frac{\xi_2(t)\,(l+1)}{t^{\nu}}\right\}^q ds\,dt\right)^{\frac{1}{q}}\\
&= O\left(\xi_1\left(\frac{\pi}{k+1}\right)\left(\frac{\pi}{l+1}\right)^{\frac{1}{p}}(k+1)^{\mu+1-\frac{1}{p}}(k+1)^{1-\frac{1}{q}}\frac{(l+1)}{(k+1)}\left(\frac{\pi}{l+1}\right)^{\frac{1}{q}}\right)\\
&\quad+O\left(\left(\frac{\pi}{l+1}\right)^{\frac{1}{p}}\frac{\pi}{(k+1)}(k+1)^{2-\frac{1}{p}}\left(\frac{\pi}{k+1}\right)^{\frac{1}{q}}\xi_2\left(\frac{\pi}{l+1}\right)(l+1)\,(l+1)^{\nu-\frac{1}{q}}\right)\\
&= O\left((k+1)^{\mu}\,\xi_1\left(\frac{\pi}{k+1}\right)\right)+O\left((l+1)^{\nu}\,\xi_2\left(\frac{\pi}{l+1}\right)\right), \qquad (2.14)
\end{aligned}
$$

as $A_{k,k-\sigma} = O\left(\frac{\pi}{(k+1)s}\right)$ (with the help of regularity condition of $a_{k,i}$).
Using (2.8), $\sin(\frac{s}{2}) \geq \frac{s}{\pi}$, $\cos(\frac{s}{2}) \leq 1$, second mean value theorem for integrals, Lemma 2.10, 2.11, and Hölder's inequality, we have

$$\begin{aligned}
|\widetilde{I}_2\widetilde{J}_2| &= \left(\int_{\frac{\pi}{k+1}}^{\pi}\int_0^{\frac{\pi}{l+1}}\left\{\frac{\xi_1(s)}{s^{\mu+1}}\right\}^p ds\,dt\right)^{\frac{1}{p}}\left(\int_{\frac{\pi}{k+1}}^{\pi}\int_0^{\frac{\pi}{l+1}}\left\{|\widetilde{S}_l(t)|\right\}^q ds\,dt\right)^{\frac{1}{q}} \\
&\quad + \left(\int_{\frac{\pi}{k+1}}^{\pi}\int_0^{\frac{\pi}{l+1}}\left\{\frac{1}{s}\right\}^p ds\,dt\right)^{\frac{1}{p}}\left(\int_{\frac{\pi}{k+1}}^{\pi}\int_0^{\frac{\pi}{l+1}}\left\{\frac{\xi_2(t)\,(l+1)}{t^{\nu}}\right\}^q ds\,dt\right)^{\frac{1}{q}} \\
&= \left(\frac{\pi}{l+1}\right)^{\frac{1}{p}}\left(\int_{\frac{\pi}{k+1}}^{\pi}\left\{\frac{\xi_1(s)}{s^{\mu+1}}\right\}^p ds\right)^{\frac{1}{p}}(l+1)\left(\frac{\pi}{l+1}\right)^{\frac{1}{q}}\left(\int_{\frac{\pi}{k+1}}^{\pi} ds\right)^{\frac{1}{q}} \\
&\quad + \left(\frac{\pi}{l+1}\right)^{\frac{1}{p}}\left(\int_{\frac{\pi}{k+1}}^{\pi}\frac{ds}{s^p}\right)^{\frac{1}{p}}\left(\int_{\frac{\pi}{k+1}}^{\pi} ds\right)^{\frac{1}{q}}\xi_2\left(\frac{\pi}{l+1}\right)(l+1)\left(\int_0^{\frac{\pi}{l+1}}\frac{dt}{t^{q\nu}}\right)^{\frac{1}{q}} \\
&= O\left((k+1)^{\mu}\,\xi_1\left(\frac{\pi}{k+1}\right)\right) + O\left((l+1)^{\nu}\,\xi_2\left(\frac{\pi}{l+1}\right)\right). \qquad (2.15)
\end{aligned}$$

Similarly, we can prove

$$|\widetilde{I}_2\widetilde{J}_3| = O\left((k+1)^{\mu}\,\xi_1\left(\frac{\pi}{k+1}\right)\right) + O\left((l+1)^{\nu}\,\xi_2\left(\frac{\pi}{l+1}\right)\right). \qquad (2.16)$$

We can easily calculate $\widetilde{I}_3\widetilde{J}_1$, $\widetilde{I}_3\widetilde{J}_2$, and $\widetilde{I}_3\widetilde{J}_3$ using the similar steps of $\widetilde{I}_2\widetilde{J}_1$, $\widetilde{I}_2\widetilde{J}_2$, and $\widetilde{I}_2\widetilde{J}_3$, respectively.
Using (2.8), (2.9), Lemma 2.11, second mean value theorem for integrals, and Hölder's inequality, we have

$$\begin{aligned}
|\widetilde{I}_4\widetilde{J}_1| &= \left(\int_{\frac{\pi}{k+1}}^{\pi}\int_{\frac{\pi}{l+1}}^{\pi}\left\{\frac{\xi_1(s)\,B_{l,l-\tau}}{s^{\mu+1}\,t}\right\}^p ds\,dt\right)^{\frac{1}{p}}\left(\int_{\frac{\pi}{k+1}}^{\pi}\int_{\frac{\pi}{l+1}}^{\pi}\{A_{k,k-\sigma}\}^q ds\,dt\right)^{\frac{1}{q}} \\
&\quad + \left(\int_{\frac{\pi}{k+1}}^{\pi}\int_{\frac{\pi}{l+1}}^{\pi}\{B_{l,l-\tau}\}^p ds\,dt\right)^{\frac{1}{p}}\left(\int_{\frac{\pi}{k+1}}^{\pi}\int_{\frac{\pi}{l+1}}^{\pi}\left\{\frac{\xi_2(t)\,A_{k,k-\sigma}}{t^{\nu+1}\,s}\right\}^q ds\,dt\right)^{\frac{1}{q}} \\
&= O\left(\frac{(l+1)^{2-\frac{1}{p}}}{(l+1)(k+1)}\right)\left(\int_{\frac{\pi}{k+1}}^{\pi}\left\{\frac{\xi_1(s)}{s^{\mu+1}}\right\}^p ds\right)^{\frac{1}{p}}\left(\frac{\pi}{l+1}\right)^{\frac{1}{q}}\left(\int_{\frac{\pi}{k+1}}^{\pi}\frac{ds}{s^q}\right)^{\frac{1}{q}} \\
&\quad + O\left(\frac{(k+1)^{2-\frac{1}{q}}}{(l+1)(k+1)}\right)\left(\frac{\pi}{k+1}\right)^{\frac{1}{p}}\left(\int_{\frac{\pi}{l+1}}^{\pi}\frac{ds}{t^p}\right)^{\frac{1}{p}}\left(\int_{\frac{\pi}{l+1}}^{\pi}\left\{\frac{\xi_2(t)}{t^{\nu+1}}\right\}^q dt\right)^{\frac{1}{q}} \\
&= O\left((k+1)^{\mu}\,\xi_1\left(\frac{\pi}{k+1}\right)\right) + O\left((l+1)^{\nu}\,\xi_2\left(\frac{\pi}{l+1}\right)\right), \qquad (2.17)
\end{aligned}$$

as $A_{k,k-\sigma} = O\left(\frac{\pi}{(k+1)s}\right)$ and $B_{l,l-\tau} = O\left(\frac{\pi}{(l+1)t}\right)$ (with the help of regularity conditions of $a_{k,i}$ and $b_{l,j}$).
Using (2.8), (2.9), Lemma 2.11, $\sin(\frac{s}{2}) \geq \frac{s}{\pi}$, $\cos(\frac{s}{2}) \leq 1$, second mean value theorem for

integrals, and Hölder's inequality, we have

$$\begin{aligned}
|\widetilde{I}_4\widetilde{J}_2| &= \left(\int_{\frac{\pi}{k+1}}^{\pi}\int_{\frac{\pi}{l+1}}^{\pi}\left\{\frac{\xi_1(s)\,B_{l,l-\tau}}{s^{\mu+1}\,t}\right\}^p ds\,dt\right)^{\frac{1}{p}}\left(\int_{\frac{\pi}{k+1}}^{\pi}\int_{\frac{\pi}{l+1}}^{\pi}\left\{\frac{1}{s}\right\}^q ds\,dt\right)^{\frac{1}{q}}\\
&\quad+\left(\int_{\frac{\pi}{k+1}}^{\pi}\int_{\frac{\pi}{l+1}}^{\pi}\{B_{l,l-\tau}\}^p\,ds\,dt\right)^{\frac{1}{p}}\left(\int_{\frac{\pi}{k+1}}^{\pi}\int_{\frac{\pi}{l+1}}^{\pi}\left\{\frac{\xi_2(t)}{t^{\nu+1}\,s^2}\right\}^q ds\,dt\right)^{\frac{1}{q}}\\
&= O\left(\frac{(l+1)^{2-\frac{1}{p}}}{(l+1)(k+1)}\right)\left(\int_{\frac{\pi}{k+1}}^{\pi}\left\{\frac{\xi_1(s)}{s^{\mu+1}}\right\}^p ds\right)^{\frac{1}{p}}\left(\frac{\pi}{l+1}\right)^{\frac{1}{q}}\left(\int_{\frac{\pi}{k+1}}^{\pi}\frac{ds}{s^q}\right)^{\frac{1}{q}}\\
&\quad+O\left(\frac{(k+1)^{2-\frac{1}{q}}}{(l+1)(k+1)}\right)\left(\frac{\pi}{k+1}\right)^{\frac{1}{p}}\left(\int_{\frac{\pi}{l+1}}^{\pi}\frac{ds}{t^p}\right)^{\frac{1}{p}}\left(\int_{\frac{\pi}{l+1}}^{\pi}\left\{\frac{\xi_2(t)}{t^{\nu+1}}\right\}^q dt\right)^{\frac{1}{q}}\\
&= O\left((k+1)^{\mu}\,\xi_1\left(\frac{\pi}{k+1}\right)\right)+O\left((l+1)^{\nu}\,\xi_2\left(\frac{\pi}{l+1}\right)\right). \qquad (2.18)
\end{aligned}$$

Similarly, we can prove

$$|\widetilde{I}_4\widetilde{J}_3| = O\left((k+1)^{\mu}\,\xi_1\left(\frac{\pi}{k+1}\right)\right)+O\left((l+1)^{\nu}\,\xi_2\left(\frac{\pi}{l+1}\right)\right). \qquad (2.19)$$

Combining $(2.10-2.19)$, we have

$$\|\widetilde{t}_{k,l}^{A,B}-\widetilde{g}\|_p = O\left((k+1)^{\mu}\,\xi_1\left(\frac{\pi}{k+1}\right)\right)+O\left((l+1)^{\nu}\,\xi_2\left(\frac{\pi}{l+1}\right)\right).$$

This completes the proof of Theorem 2.12. □

Corollary 2.13. *For $\widetilde{g}\in Z(\alpha,\beta;p)$, using double Nörlund summability method, we get*

$$\|\widetilde{t}_{k,l}^{A,B}-\widetilde{g}\|_p = \begin{cases} O\left(p_k^{\alpha}+q_l^{\beta}\right), & \alpha,\beta\in(0,1),\\ O\left(p_k\log\frac{\pi}{p_k}+q_l^{\beta}\right), & \beta\in(0,1),\alpha=1,\\ O\left(p_k^{\alpha}+q_l\log\frac{\pi}{q_l}\right), & \alpha\in(0,1),\beta=1,\\ O\left(p_k\log\frac{\pi}{p_k}+q_l\log\frac{\pi}{q_l}\right), & \alpha=\beta=1,\end{cases}$$

in view of $(k+1)p_k\geq 1$ and $(l+1)q_l\geq 1$.

Corollary 2.14. *For $\widetilde{g}\in Z(\alpha,\beta)$, using double Nörlund summability method, we get*

$$\|\widetilde{t}_{k,l}^{A,B}-\widetilde{g}\|_\infty = \begin{cases} O\left(p_k^{\alpha}+q_l^{\beta}\right), & \alpha,\beta\in(0,1),\\ O\left(p_k\log\frac{\pi}{p_k}+q_l^{\beta}\right), & \beta\in(0,1),\alpha=1,\\ O\left(p_k^{\alpha}+q_l\log\frac{\pi}{q_l}\right), & \alpha\in(0,1),\beta=1,\\ O\left(p_k\log\frac{\pi}{p_k}+q_l\log\frac{\pi}{q_l}\right), & \alpha=\beta=1.\end{cases}$$

Corollary 2.15. *For $\widetilde{g}\in Z(\alpha,\beta)$, using double Cesàro summability method, we get*

$$\|\widetilde{t}_{k,l}^{A,B}-\widetilde{g}\|_\infty = \begin{cases} O\left(\frac{1}{(k+1)^{\alpha}}+\frac{1}{(l+1)^{\beta}}\right), & \alpha,\beta\in(0,1),\\ O\left(\frac{\log(k+1)}{(k+1)}+\frac{1}{(l+1)^{\beta}}\right), & \beta\in(0,1),\alpha=1,\\ O\left(\frac{1}{(k+1)^{\alpha}}+\frac{\log(l+1)}{(l+1)}\right), & \alpha\in(0,1),\beta=1,\\ O\left(\frac{\log(k+1)}{(k+1)}+\frac{\log(l+1)}{(l+1)}\right), & \alpha=\beta=1.\end{cases}$$

Theorem 2.16. *Let $\widetilde{g}(x,y)$ be the conjugate function of 2π-periodic function $g(x,y)$ belonging to $W(L^p(T^2),\xi_1(s),\xi_2(t))$. Then the degree (error) of approximation of function*

$\widetilde{g}(x,y)$ through the double matrix means of its double conjugate Fourier series is given by:

$$\|\widetilde{t}_{k,l}^{A,B}-\widetilde{g}\|_p = O\left((k+1)^{\mu}\,\xi_1\left(\frac{\pi}{k+1}\right)\right)+O\left((l+1)^{\nu}\,\xi_2\left(\frac{\pi}{l+1}\right)\right),$$

provided the positive increasing functions $\xi_1(s)$ and $\xi_2(t)$ satisfy the conditions (2.8) and (2.9).

Proof. Following the proof of Theorem 2.12 and using inequality (2.5), we have

$$\begin{aligned}
\|\widetilde{t}_{k,l}^{A,B}-\widetilde{g}\|_p &= \int_0^{\pi}\int_0^{\pi}\|\varphi(s,t)\|_p\{2\pi\,\widetilde{R}_k(s)\,\widetilde{S}_l(t)-\cot\tfrac{s}{2}\,\widetilde{S}_l(t)-\cot\tfrac{t}{2}\,\widetilde{R}_k(s)\}ds\,dt\\
&= \left(\int_0^{\frac{\pi}{k+1}}\int_0^{\frac{\pi}{l+1}}+\int_{\frac{\pi}{k+1}}^{\pi}\int_0^{\frac{\pi}{l+1}}+\int_0^{\frac{\pi}{k+1}}\int_{\frac{\pi}{l+1}}^{\pi}+\int_{\frac{\pi}{k+1}}^{\pi}\int_{\frac{\pi}{l+1}}^{\pi}\right)\|\varphi(s,t)\|_p\times\\
&\quad\{2\pi\,\widetilde{R}_k(s)\,\widetilde{S}_l(t)-\cot\tfrac{s}{2}\,\widetilde{S}_l(t)-\cot\tfrac{t}{2}\,\widetilde{R}_k(s)\}ds\,dt\\
&\le \left(\int_0^{\frac{\pi}{k+1}}\int_0^{\frac{\pi}{l+1}}+\int_{\frac{\pi}{k+1}}^{\pi}\int_0^{\frac{\pi}{l+1}}+\int_0^{\frac{\pi}{k+1}}\int_{\frac{\pi}{l+1}}^{\pi}+\int_{\frac{\pi}{k+1}}^{\pi}\int_{\frac{\pi}{l+1}}^{\pi}\right)\left(\frac{Z_{2,x}^{\mu}(g,s)}{s^{\mu}}+\right.\\
&\quad\left.\frac{Z_{2,y}^{\nu}(g,t)}{t^{\nu}}\right)\{2\pi\,\widetilde{R}_k(s)\,\widetilde{S}_l(t)-\cot\tfrac{s}{2}\,\widetilde{S}_l(t)-\cot\tfrac{t}{2}\,\widetilde{R}_k(s)\}ds\,dt\\
&\le \left(\int_0^{\frac{\pi}{k+1}}\int_0^{\frac{\pi}{l+1}}+\int_{\frac{\pi}{k+1}}^{\pi}\int_0^{\frac{\pi}{l+1}}+\int_0^{\frac{\pi}{k+1}}\int_{\frac{\pi}{l+1}}^{\pi}+\int_{\frac{\pi}{k+1}}^{\pi}\int_{\frac{\pi}{l+1}}^{\pi}\right)\left(\frac{L_{1,x}^{\mu}(g,s)}{s^{\mu}}+\right.\\
&\quad\left.\frac{L_{1,y}^{\nu}(g,t)}{t^{\nu}}\right)\{2\pi\,\widetilde{R}_k(s)\,\widetilde{S}_l(t)-\cot\tfrac{s}{2}\,\widetilde{S}_l(t)-\cot\tfrac{t}{2}\,\widetilde{R}_k(s)\}ds\,dt\\
&= \left(\int_0^{\frac{\pi}{k+1}}\int_0^{\frac{\pi}{l+1}}+\int_{\frac{\pi}{k+1}}^{\pi}\int_0^{\frac{\pi}{l+1}}+\int_0^{\frac{\pi}{k+1}}\int_{\frac{\pi}{l+1}}^{\pi}+\int_{\frac{\pi}{k+1}}^{\pi}\int_{\frac{\pi}{l+1}}^{\pi}\right)\times\\
&\quad\left\{\left(\frac{\xi_1(s)}{s^{\mu}}\,|\widetilde{R}_k(s)|\,|\widetilde{S}_l(t)|\,ds\,dt+\frac{\xi_2(t)}{t^{\nu}}\,|\widetilde{R}_k(s)|\,|\widetilde{S}_l(t)|\,ds\,dt\right)\right.\\
&\quad+\left(\frac{\xi_1(s)}{s^{\mu}}\,|\cot\tfrac{s}{2}|\,|\widetilde{S}_l(t)|\,ds\,dt+\frac{\xi_2(t)}{t^{\nu}}\,|\cot\tfrac{s}{2}|\,|\widetilde{S}_l(t)|\,ds\,dt\right)\\
&\quad\left.+\left(\frac{\xi_1(s)}{s^{\mu}}\,|\widetilde{R}_k(s)|\,|\cot\tfrac{t}{2}|\,ds\,dt+\frac{\xi_2(t)}{t^{\nu}}\,|\widetilde{R}_k(s)|\,|\cot\tfrac{t}{2}|\,ds\,dt\right)\right\}\\
&= \left(\widetilde{I}_1+\widetilde{I}_2+\widetilde{I}_3+\widetilde{I}_4\right)\times\left\{\widetilde{J}_1+\widetilde{J}_2+\widetilde{J}_3\right\},\ \text{(say)}.
\end{aligned}$$

Similarly, following the proof of Theorem 2.12, we get

$$\|\widetilde{t}_{k,l}^{A,B}-\widetilde{g}\|_p = O\left((k+1)^{\mu}\,\xi_1\left(\frac{\pi}{k+1}\right)\right)+O\left((l+1)^{\nu}\,\xi_2\left(\frac{\pi}{l+1}\right)\right).$$

This completes the proof of Theorem 2.16. □

Corollary 2.17. *For $\widetilde{g}\in Lip(\alpha,\beta;p)$, using double Cesàro summability method, we get*

$$\|\widetilde{t}_{k,l}^{A,B}-\widetilde{g}\|_p = \begin{cases} O\left(\frac{1}{(k+1)^{\alpha}}+\frac{1}{(l+1)^{\beta}}\right), & \alpha,\beta\in(0,1),\\ O\left(\frac{\log(k+1)}{(k+1)}+\frac{1}{(l+1)^{\beta}}\right), & \beta\in(0,1),\alpha=1,\\ O\left(\frac{1}{(k+1)^{\alpha}}+\frac{\log(l+1)}{(l+1)}\right), & \alpha\in(0,1),\beta=1,\\ O\left(\frac{\log(k+1)}{(k+1)}+\frac{\log(l+1)}{(l+1)}\right), & \alpha=\beta=1.\end{cases}$$

Corollary 2.18. *For $\widetilde{g} \in Lip(\alpha,\beta)$, using double Cesàro summability method, we get*

$$\|\widetilde{t}_{k,l}^{A,B} - \widetilde{g}\|_\infty = \begin{cases} O\left(\frac{1}{(k+1)^\alpha} + \frac{1}{(l+1)^\beta}\right), & \alpha,\beta \in (0,1), \\ O\left(\frac{\log(k+1)}{(k+1)} + \frac{1}{(l+1)^\beta}\right), & \beta \in (0,1), \alpha = 1, \\ O\left(\frac{1}{(k+1)^\alpha} + \frac{\log(l+1)}{(l+1)}\right), & \alpha \in (0,1), \beta = 1, \\ O\left(\frac{\log(k+1)}{(k+1)} + \frac{\log(l+1)}{(l+1)}\right), & \alpha = \beta = 1. \end{cases}$$

Remark 2.19. In the results of Móricz and Rhoades [11, Theorem 4 (4.12), Corollary 4 (4.17)], and Móricz and Shi [12, Theorem 3], the error of approximation for functions $\widetilde{g}$ belonging to $Z(\alpha,\beta)$ is worse by a factor of " log " in both k and l if $0 < \alpha,\beta < 1$, and there is an additional " $\log^2$ " factor in both k and l if $\alpha = \beta = 1$. Móricz and Shi [12, Theorem 5] improved the error for functions $\widetilde{g}$ belonging to $Lip(\alpha,\beta)$ and found that the error contains only " log " factors in both k and l for all $0 < \alpha,\beta \leq 1$. In contrast, our results (Corollary 2.14, 2.15, and 2.18) show that the factor " log " in both k and l disappears when $0 < \alpha,\beta < 1$. Moreover, in the case of $\alpha = \beta = 1$, the extra " $\log^2$ " factor in both k and l reduces to " log ". Therefore, the results given in this paper are improved versions of the results given by Móricz and Rhoades [11], and Móricz and Shi [12].

3. Results for functions of several variables

In this section, we extend the above results for functions of several variables by introducing the weighted Lipschitz class $W(L^p(T^N), \xi_1(s_1), \ldots, \xi_N(s_N))$ and weighted Zygmund class $Z(L^p(T^N), \xi_1(s_1), \ldots, \xi_N(s_N))$.

Throughout this section, we use the following notations:

- $T^N = (-\pi,\pi) \times (-\pi,\pi) \times \ldots \times (-\pi,\pi)$($N$-times), where $N \in \mathbb{N}$.
- $M = \{1, 2, 3, \ldots, N-1, N\}$, where $N \in \mathbb{N}$.
- $h_1, h_2 \cdots, h_N \in \{-1, 1\}$.

Let $g(x_1, x_2, \ldots, x_N)$ be a complex valued function, 2π-periodic in each variable, integrable over the N-dimensional torus T^N.

The N-multiple Fourier series of a function $g(x_1, x_2, \ldots, x_N) \in L^p(T^N)$ is given by

$$g(x_1, x_2, \ldots, x_N) \sim \sum_{l_1 \in \mathbb{Z}} \sum_{l_2 \in \mathbb{Z}} \cdots \sum_{l_N \in \mathbb{Z}} c_{(x_1,x_2,\ldots,x_N)} e^{i(l_1x_1 + l_2x_2 + \ldots + l_Nx_N)}, \tag{3.1}$$

where

$$c_{(x_1,x_2,\ldots,x_N)} = \frac{1}{(2\pi)^N} \overbrace{\int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi}}^{N\text{-times}} g(x_1, \ldots, x_N) e^{-i(l_1x_1 + \ldots + l_Nx_N)}\, dx_1 \ldots dx_N.$$

The conjugate series of (3.1) is given by

$$\sum_{l_1 \in \mathbb{Z}} \sum_{l_2 \in \mathbb{Z}} \cdots \sum_{l_N \in \mathbb{Z}} c_{(x_1,x_2,\ldots,x_N)} \prod_{m=1}^{N} (-i \operatorname{sign} l_m)\, e^{i(l_m x_m)}, \tag{3.2}$$

(conjugate with respect to $x_1, x_2, \ldots, x_N$).

The conjugate function of $g(x_1, x_2, \ldots, x_N)$, denoted by $\widetilde{g}(x_1, x_2, \ldots, x_N)$, is defined as

$$\widetilde{g}(x_1, x_2, \ldots, x_N) = \frac{1}{\pi^N} \overbrace{\int_0^{\pi} \cdots \int_0^{\pi}}^{N\text{-times}} \prod_{m=1}^{N} \frac{\Delta g(x_1, \ldots, x_N; s_1, \ldots, s_N)}{\tan \frac{s_m}{2}} ds_m,$$

where

$$\Delta g(x_1,\ldots,x_N;s_1,\quad \ldots,s_N)= \sum_{h_1}\cdots\sum_{h_N}(h_1\times\cdots\times h_N)g(x_1+h_1s_1,\ldots,x_N+h_Ns_N).$$

Here the integrals is taken in the "Principal value" sense at the point $x_1 = x_2 = \ldots = x_N = 0$.
Let $\{\widetilde{s}_{k_1,\ldots,k_N}\}$ be the sequence of $(k_1+1,\ldots,k_N+1)^{\text{th}}$ partial sums corresponding to the conjugate Fourier series (3.2), which is defined as

$$\widetilde{s}_{k_1,\ldots,k_N}=\sum_{l_1=0}^{k_1}\sum_{l_2=0}^{k_2}\cdots\sum_{l_N=0}^{k_N}c_{(x_1,x_2,\ldots,x_N)}\prod_{m=1}^{N}(-i\,\mathrm{sign}\,l_m)\,e^{i(l_mx_m)}.$$

Define

$$\widetilde{t}^{A_1,\ldots,A_N}_{k_1,\ldots,k_N}=\sum_{i_1=0}^{k_1}\cdots\sum_{i_N=0}^{k_N}(a_{k_1,i_1}\times\ldots\times a_{k_N,i_N})\,\widetilde{s}_{i_1,\ldots,i_N},\quad \text{for } k_1,\ldots,k_N\in\mathbb{N}\cup\{0\},$$

where $A_1 \equiv (a_{k_1,i_1}),\ldots,\ A_N \equiv (a_{k_N,i_N})$ are N lower triangular matrices with non-decreasing and non-negative entries with respect to $i_1, i_2, \ldots, i_N$ such that $a_{k_m,-1} = 0, A_{k_m,\sigma_m} = \sum_{i_m=0}^{\sigma_m} a_{k_m,i_m}$ and $A_{k_m,k_m} = 1$, for each $m \in M$. If $\widetilde{t}^{A_1,\ldots,A_N}_{k_1,\ldots,k_N} \to s_1$ as $k_1,\ldots,k_N \to \infty$, then the conjugate series (3.2) is said to be summable to s_1 by the N-multiple matrix means.
The regularity conditions of the N-multiple matrix means are same as given in [2]. We can obtain particular cases of N-multiple matrix means by changing (a_{k_m,i_m}), for each $m \in M$, as given in section 2.
The space of Lebesgue functions on T^N is denoted by $L^p(T^N), p \geq 1$ and the norm on it is defined by

$$\|g\|_p=\left\{\frac{1}{(2\pi)^N}\overbrace{\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}}^{N\text{-times}}|g(x_1,\ldots,x_N)|^p dx_1\ldots dx_N\right\}^{\frac{1}{p}}\quad(1\leq p<\infty),$$

$$\text{and } \|g\|_\infty=\sup_{-\pi\leq x_1,x_2,\ldots,x_N\leq\pi}|g(x_1,x_2,\ldots,x_N)|.$$

Let $L_1^{\mu_1,\ldots,\mu_N}(g,s_1,\ldots,s_N)$ be a weighted integral modulus of continuity of function g, which is defined as

$$L_1^{\mu_1,\ldots,\mu_N}(g,s_1,\ldots,s_N)= \sup_{|\theta_m|\leq s_m}\left\{\|\{g(x_1+\theta_1,\ldots,x_N+\theta_N)-g(x_1,\ldots,x_N)\}w(s_1,\ldots,s_N)\|_p\right\},$$

where $w(s_1,\ldots,s_N)$ is a weight function, defined by

$$w(s_1,\ldots,s_N)=\sqrt{\frac{\sum_{m=1}^N s_m^2\,\sin^{2\mu_m}(\frac{x_m}{2})}{\sum_{m=1}^N s_m^2}},\tag{3.3}$$

for $\mu_1,\mu_2,\ldots,\mu_N \geq 0$ and $s_1^2+\ldots+s_N^2 \neq 0$.
Here, we introduce a new weighted Lipschitz class for N-variables as follows:

Definition 3.1 $(W(L^p(T^N),\xi_1(s_1),\ldots,\xi_N(s_N))$-class)**.** For positive increasing functions $\xi_1(s_1),\xi_2(s_2),\ldots,\xi_N(s_N)$, the weighted Lipschitz class, denoted by $W(L^p(T^N),\xi_1(s_1),\ldots,\xi_N(s_N))$, is defined as

$$W(L^p(T^N), \xi_1(s_1), \quad \dots, \xi_N(s_N)) = \Big\{ g \in L^p(T^N) : L^{\mu_1}_{1,x_1}(g, s_1) = O(\xi_1(s_1)), \\ L^{\mu_2}_{1,x_2}(g, s_2) = O(\xi_2(s_2)), \dots, L^{\mu_N}_{1,x_N}(g, s_N) = O(\xi_N(s_N)) \Big\},$$

where $L^{\mu_m}_{1,x_m}(g, s_m)$, for each $m \in M$, is partial weighted integral moduli of continuity of function g, which is defined as

$$L^{\mu_m}_{1,x_m}(g, s_m) = \sup_{|\theta_m| \le s_m} \{ \| \{ g(x_1, \dots, x_m + \theta_m, \dots, x_N) - g(x_1, \dots, x_N) \} \sin^{\mu_m}(\tfrac{x_m}{2}) \|_p \}.$$

Definition 3.2 ($Lip(\alpha_1, \dots, \alpha_N; p)$-class)**.** For $\alpha_1, \alpha_2, \dots, \alpha_N \in (0, 1]$, the $Lip(\alpha_1, \dots, \alpha_N; p)$-class is defined as

$$Lip(\alpha_1, \dots, \alpha_N; p) = \Big\{ g \in L^p(T^N) : L^p_{1,x_1}(g, s_1) = O(s_1^{\alpha_1}), \\ L^p_{1,x_2}(g, s_2) = O(s_2^{\alpha_2}), \dots, L^p_{1,x_N}(g, s_N) = O(s_N^{\alpha_N}) \Big\},$$

where $L^p_{1,x_m}(g, s_m)$, for each $m \in M$, is partial integral moduli of continuity of function g, which is defined by

$$L^p_{1,x_m}(g, s_m) = \sup_{|\theta_m| \le s_m} \{ \| \{ g(x_1, \dots, x_m + \theta_m, \dots, x_N) - g(x_1, \dots, x_N) \} \|_p \}.$$

Remark 3.3. Definition 3.2 is an extension of $Lip(\alpha, \beta; p)$-class [16] for function of N-variables.

Definition 3.4 ($Lip(\alpha_1, \dots, \alpha_N)$-class)**.** For $\alpha_1, \alpha_2, \dots, \alpha_N \in (0, 1]$, the $Lip(\alpha_1, \dots, \alpha_N)$-class is defined as

$$Lip(\alpha_1, \dots, \alpha_N) = \Big\{ g : T^N \to \mathbb{R} : L_{1,x_1}(g, s_1) = O(s_1^{\alpha_1}), \\ L_{1,x_2}(g, s_2) = O(s_2^{\alpha_2}), \dots, L_{1,x_N}(g, s_N) = O(s_N^{\alpha_N}) \},$$

where $L_{1,x_m}(g, s_m)$, for each $m \in M$, is partial moduli of continuity of function g, which is defined by

$$L_{1,x_m}(g, s_m) = \sup_{x_1, \dots, x_N} \sup_{|\theta_m| \le s_m} \{ | g(x_1, \dots, x_m + \theta_m, \dots, x_N) - g(x_1, \dots, x_N) | \}.$$

Remark 3.5. Definition 3.4 is an extension of $Lip(\alpha, \beta)$-class [12] for function of N-variables.

Remark 3.6. If we take $\mu_m = 0$ and $\xi_m(s_m) = s_m^{\alpha_m}$, $\alpha_m \in (0, 1]$, for each $m \in M$ in Definition 3.1, then $W(L^p(T^N), \xi_1(s_1), \dots, \xi_N(s_N))$ reduces to $Lip(\alpha_1, \dots, \alpha_N; p)$. If we take $p \to \infty$ in Definition 3.2, then $Lip(\alpha_1, \dots, \alpha_N; p)$ reduces to $Lip(\alpha_1, \dots, \alpha_N)$. Then, we can write

$$Lip(\alpha_1, \dots, \alpha_N) \subseteq Lip(\alpha_1, \dots, \alpha_N; p) \subseteq W(L^p(T^N), \xi_1(s_1), \dots, \xi_N(s_N)).$$

Here, we define the total weighted integral modulus of symmetric smoothness of a function g by

$$Z_2^{\mu_1, \dots, \mu_N}(g, s_1, \dots, s_N) = \sup_{|\theta_m| \le s_m} \Big\{ \| \psi(\theta_1, \theta_2, \dots, \theta_N) w(s_1, \dots, s_N) \|_p \Big\},$$

where $w(s_1, \dots, s_N)$ is weight function, defined as (3.3), and
$\psi(\theta_1, \theta_2, \dots, \theta_N) = \sum_{h_1} \cdots \sum_{h_N} g(x_1 + \theta_1 h_1, \dots, x_N + \theta_N h_N) - 2^N g(x_1, \dots, x_N)$.
Now, we introduce a new weighted Zygmund class for N-variables as follows:

Definition 3.7 ($Z(L^p(T^N), \xi_1(s_1), \dots, \xi_N(s_N))$-class)**.** For positive increasing functions $\xi_1(s_1), \xi_2(s_2), \dots, \xi_N(s_N)$, the weighted Zygmund class, denoted by $Z(L^p(T^N), \xi_1(s_1), \dots, \xi_N(s_N))$, is defined as

$$Z(L^p(T^N), \xi_1(s_1), \dots, \xi_N(s_N)) = \Big\{ g \in L^p(T^N) : Z^{\mu_1}_{2,x_1}(g, s_1) = O(\xi_1(s_1)),$$
$$Z^{\mu_2}_{2,x_2}(g, s_2) = O(\xi_2(s_2)), \dots, Z^{\mu_N}_{2,x_N}(g, s_N) = O(\xi_N(s_N)) \Big\},$$

where $Z^{\mu_m}_{2,x_m}(g, s_m)$, for each $m \in M$, is partial weighted integral moduli of smoothness of function g, which is defined by

$$Z^{\mu_m}_{2,x_m}(g, s_m) = \sup_{|\theta_m| \le s_m} \{\| \{\phi(\theta_m)\} \sin^{\mu_m}(\tfrac{x_m}{2}) \|_p\},$$

where

$$\phi(\theta_m) = g(x_1, \dots, x_m + \theta_m, \dots, x_N) + g(x_1, \dots, x_m - \theta_m, \dots, x_N) - 2g(x_1, \dots, x_N). \quad (3.4)$$

Definition 3.8 ($Z(\alpha_1, \dots, \alpha_N; p)$-class)**.** For $\alpha_1, \alpha_2, \dots, \alpha_N \in (0, 2]$, the $Z(\alpha_1, \dots, \alpha_N; p)$-class is defined as

$$Z(\alpha_1, \dots, \alpha_N; p) = \Big\{ g \in L^p(T^N) : Z^p_{2,x_1}(g, s_1) = O(s_1^{\alpha_1}),$$
$$Z^p_{2,x_2}(g, s_2) = O(s_2^{\alpha_2}), \dots, Z^p_{2,x_N}(g, s_N) = O(s_N^{\alpha_N}) \Big\},$$

where $Z^p_{2,x_m}(g, s_m)$, for each $m \in M$, is partial integral moduli of smoothness of function g, which is defined by

$$Z^p_{2,x_m}(g, s_m) = \sup_{|\theta_m| \le s_m} \{\|\phi(\theta_m)\|_p\},$$

where $\phi(\theta_m)$ is the same as (3.4).

Remark 3.9. Definition 3.8 is an extension of $Z(\alpha, \beta; p)$-class [16] for function of N-variables.

Definition 3.10 ($Z(\alpha_1, \dots, \alpha_N)$-class)**.** For $\alpha_1, \alpha_2, \dots, \alpha_N \in (0, 2]$, the $Z(\alpha_1, \dots, \alpha_N)$-class is defined as

$$Z(\alpha_1, \dots, \alpha_N) = \Big\{ g : T^N \to \mathbb{R} : Z_{2,x_1}(g, s_1) = O(s_1^{\alpha_1}),$$
$$Z_{2,x_2}(g, s_2) = O(s_2^{\alpha_2}), \dots, Z_{2,x_N}(g, s_N) = O(s_N^{\alpha_N}) \},$$

where $Z_{2,x_m}(g, s_m)$, for each $m \in M$, is partial moduli of smoothness of function g, which is defined by

$$Z_{2,x_m}(g, s_m) = \sup_{x_1, \dots, x_N} \sup_{|\theta_m| \le s_m} \{|\phi(\theta_m)|\},$$

where $\phi(\theta_m)$ is the same as (3.4).

Remark 3.11. Definition 3.10 is an extension of $Z(\alpha, \beta)$-class [12] for function of N-variables.

Remark 3.12. If we take $\mu_m = 0$ and $\xi_m(s_m) = s_m^{\alpha_m}$, $\alpha_m \in (0, 2]$, for each $m \in M$ in Definition 3.7, then $Z(L^p(T^N), \xi_1(s_1), \dots, \xi_N(s_N))$ reduces to $Z(\alpha_1, \dots, \alpha_N; p)$. If we take $p \to \infty$ in in Definition 3.8, then $Z(\alpha_1, \dots, \alpha_N; p)$ reduces to $Z(\alpha_1, \dots, \alpha_N)$. Then, we can write

$$Z(\alpha_1, \dots, \alpha_N) \subseteq Z(\alpha_1, \dots, \alpha_N; p) \subseteq Z(L^p(T^N), \xi_1(s_1), \dots, \xi_N(s_N)).$$

Following section 2, we can write

$$Z^{\mu_m}_{2,x_m}(g, s_m) \le N L^{\mu_m}_{1,x_m}(g, s_m), \text{ for each } m \in M. \quad (3.5)$$

From (3.5), we obtain the relationship between the weighted Lipschitz class and Zygmund class as

$$W(L^p(T^N), \xi_1(s_1), \dots, \xi_N(s_N)) \subseteq Z(L^p(T^N), \xi_1(s_1), \dots, \xi_N(s_N)).$$

We also write

$$\varphi(\theta_1,\theta_2,\ldots,\theta_N)=\sum_{h_1}\cdots\sum_{h_N}\left[(h_1\times\cdots\times h_N)\ g(x_1+\theta_1h_1,\ldots,x_N+\theta_Nh_N)\right],$$

$\widetilde{R}_{k_m}(s_m) = \displaystyle\sum_{i_m=0}^{k_m} \frac{a_{k_m,k_m-i_m}\cos(2k_m-2i_m+1)\frac{s_m}{2}}{2\pi\,\sin\frac{s_m}{2}}$, and $\sigma_m := \left[\frac{1}{s_m}\right]$, is the integral part of $\frac{1}{s_m}$, for each $m\in M$.

Note 3.13. We can prove the following inequalities:

$$|\varphi(\theta_1,\theta_2,\ldots,\theta_N)|\le N\left(\sum_{m=1}^{N} Z_{2,x_m}(g,s_m)\right), \tag{3.6}$$

and

$$\|\varphi(\theta_1,\theta_2,\ldots,\theta_N)\|_p\le N\left(\sum_{m=1}^{N}\frac{Z_{2,x_m}^{\mu_m}(g,s_m)}{s_m^{\mu_m}}\right),\ s_m\neq 0,\text{ for each } m\in M. \tag{3.7}$$

We extend Lemma 2.10 and 2.11 for N-variable as follows:

Lemma 3.14. $|\widetilde{R}_{k_m}(s_m)| = O\left(\frac{1}{s_m}\right)$*, for* $0<s_m\le\frac{\pi}{k_m+1}$*, for each* $m\in M$*.*

Lemma 3.15. $|\widetilde{R}_{k_m}(s_m)| = O\left(\frac{A_{k_m,k_m-\sigma_m}}{s_m}\right)$*, for* $\frac{\pi}{k_m+1}<s_m\le\pi$*, for each* $m\in M$*.*

Theorem 3.16. *Let* $\widetilde{g}(x_1,x_2,\ldots,x_N)$ *be the conjugate function of* 2π*-periodic function* $g(x_1,x_2,\ldots,x_N)$ *belonging to* $Z(L^p(T^N),\xi_1(s_1),\ldots,\xi_N(s_N))$*. Then the degree (error) of approximation of function* $\widetilde{g}(x_1,x_2,\ldots,x_N)$ *through the* N*-multiple matrix means of its* N*-multiple conjugate Fourier series is given by:*

$$\|\widetilde{t}_{k_1,\ldots,k_N}^{A_1,\ldots,A_N}-\widetilde{g}\|_p = O\left(\sum_{m=1}^{N}(k_m+1)^{\mu_m}\,\xi_m\left(\frac{\pi}{k_m+1}\right)\right),$$

provided the positive increasing function $\xi_m(s_m)$ *satisfies the following condition for each* $m\in M$*:*

$$\left(\int_{\frac{\pi}{k_m+1}}^{\pi}\left(\frac{\xi_m(s_m)}{s_m^{\mu_m+1}}\right)^p ds_m\right)^{\frac{1}{p}} = O\left((k_m+1)^{\mu_m+1-\frac{1}{p}}\,\xi_m\left(\frac{\pi}{k_m+1}\right)\right). \tag{3.8}$$

Proof. We have

$$\widetilde{s}_{k_1,\ldots,k_N}-\widetilde{g} \quad = \frac{1}{(2\pi)^N}\overbrace{\int_0^{\pi}\cdots\int_0^{\pi}}^{N\text{-times}}\varphi(\theta_1,\theta_2,\ldots,\theta_N)\times$$
$$\left\{\prod_{m=1}^{N}\frac{(\cos\frac{s_m}{2}-\cos(k_m+\frac{1}{2})s_m)}{\sin\frac{s_m}{2}}-\prod_{m=1}^{N}\frac{\cos\frac{s_m}{2}}{\sin\frac{s_m}{2}}\right\}ds_1\ldots ds_N.$$

Then

$$\widetilde{t}^{A_1,\dots,A_N}_{k_1,\dots,k_N} - \widetilde{g} = \sum_{i_1=0}^{k_1} \dots \sum_{i_N=0}^{k_N} (a_{k_1,i_1} \times \dots \times a_{k_N,i_N}) \{ \widetilde{s}_{i_1,\dots,i_N} - \widetilde{g} \}$$

$$= \overbrace{\int_0^\pi \dots \int_0^\pi}^{N\text{-times}} \frac{\varphi(\theta_1,\theta_2,\dots,\theta_N)}{(2\pi)^N} \sum_{i_1=0}^{k_1} \dots \sum_{i_N=0}^{k_N} (a_{k_1,i_1} \times \dots \times a_{k_N,i_N}) \times$$

$$\left\{ (-1)^N \prod_{j_1=1}^{N} \frac{\cos(i_{j_1}+\frac{1}{2})s_{j_1}}{\sin\frac{s_{j_1}}{2}} + (-1)^{N-1} \sum_{j_1=1}^{N} \left(\cot\frac{s_{j_1}}{2} \prod_{\substack{j_2=1\\ j_1\neq j_2}}^{N} \frac{\cos(i_{j_2}+\frac{1}{2})s_{j_2}}{\sin\frac{s_{j_2}}{2}} \right) + \right.$$

$$\left. \dots - \sum_{j_{n-1}=1}^{N} \left(\frac{\cos(i_{j_{n-1}}+\frac{1}{2})s_{j_{n-1}}}{\sin\frac{s_{j_{n-1}}}{2}} \prod_{\substack{j_n=1\\ j_{n-1}\neq j_n}}^{N} \cot\frac{s_{j_n}}{2} \right) \right\} ds_1 \dots ds_N$$

$$= \overbrace{\int_0^\pi \dots \int_0^\pi}^{N\text{-times}} \frac{\varphi(\theta_1,\theta_2,\dots,\theta_N)}{(2\pi)^N} \left\{ (-2\pi)^N \prod_{j_1=1}^{N} \widetilde{R}_{k_{j_1}}(s_{j_1}) + (-2\pi)^{N-1} \sum_{j_1=1}^{N} \Big(\cot\frac{s_{j_1}}{2} \right.$$

$$\left. \prod_{\substack{j_2=1\\ j_1\neq j_2}}^{N} \widetilde{R}_{k_{j_2}}(s_{j_2}) \Big) + \dots - \sum_{j_{n-1}=1}^{N} \left(\widetilde{R}_{k_{j_{n-1}}}(s_{j_{n-1}}) \prod_{\substack{j_n=1\\ j_{n-1}\neq j_n}}^{N} \cot\frac{s_{j_n}}{2} \right) \right\} ds_1 \dots ds_N$$

Using (3.7), (3.8), Lemma 3.14, 3.15, and following the proof of Theorem 2.12, we obtain

$$\|\widetilde{t}^{A_1,\dots,A_N}_{k_1,\dots,k_N} - \widetilde{g}\|_p = O\left(\sum_{m=1}^{N} (k_m+1)^{\mu_m} \xi_m \left(\frac{\pi}{k_m+1} \right) \right).$$

This completes the proof of Theorem 3.16. □

Theorem 3.17. *Let $\widetilde{g}(x_1,x_2,\dots,x_N)$ be the conjugate function of 2π-periodic function $g(x_1,x_2,\dots,x_N)$ belonging to $W(L^p(T^N),\xi_1(s_1),\dots,\xi_N(s_N))$. Then the degree (error) of approximation of function $\widetilde{g}$ through the N-multiple matrix means of its N-multiple conjugate Fourier series is given by:*

$$\|\widetilde{t}^{A_1,\dots,A_N}_{k_1,\dots,k_N} - \widetilde{g}\|_p = O\left(\sum_{m=1}^{N} (k_m+1)^{\mu_m} \xi_m \left(\frac{\pi}{k_m+1} \right) \right),$$

provided the positive increasing function $\xi_m(s_m)$ satisfies the condition (3.8) for each $m \in M$.

Proof. Using (3.5), and following the proof of Theorem 2.12 and 3.16, we obtain

$$\|\widetilde{t}^{A_1,\dots,A_N}_{k_1,\dots,k_N} - \widetilde{g}\|_p = O\left(\sum_{m=1}^{N} (k_m+1)^{\mu_m} \xi_m \left(\frac{\pi}{k_m+1} \right) \right).$$

This completes the proof of Theorem 3.17. □

4. Conclusion

In this paper, we introduced a new weighted Lipschitz class and Zygmund class for N-variables. For $N = 2$, the weighted Lipschitz class $W(L^p(T^2),\xi_1(s),\xi_2(t))$ and the weighted Zygmund class $Z(L^p(T^2),\xi_1(s),\xi_2(t))$ are generalizations of the Lipschitz classes $Lip(\alpha,\beta)$ and $Lip(\alpha,\beta;p)$ and the Zygmund classes $Z(\alpha,\beta)$ and $Z(\alpha,\beta;p)$, respectively. Also, we derive results on the degree (error) of approximation of functions, conjugate to the functions of several variables belonging to these weighted Lipschitz class and Zygmund class, using multiple matrix means (for $N \geq 2$). The followings are the particular cases of the results of this paper :

- Second part of Theorem 4 and corollary 4 of Móricz and Rhoades [11] are particular cases of our corollary 2.14 and 2.15, respectively.
- Theorem 3 and Theorem 5 of Móricz and Shi [12] are particular case of our corollary 2.15 and 2.18, respectively.

Acknowledgment. This work was supported by Sardar Vallabhbhai National Institute of Technology, Surat-395007, Gujarat [Grant No.: 2020-21/Seed Money/26], and the Council of Scientific and Industrial Research (CSIR), New Delhi, India [Award No.: 09/1007(0008)/2020-EMR-I].

References

[1] Deepmala, L.N. Mishra and V.N. Mishra, *Trigonometric approximation of signals (functions) belonging to the $W(L_r, \xi(t)), (r \geq 1)-$class by $(E, q)(q > 0)$-means of the conjugate series of its Fourier series*, Glob. J. Math. Sci. **2** (2), 6169, 2014.

[2] U. Değer, *On Approximation by matrix means of the multiple Fourier Series in the Hölder Metric*, Kyungpook Math. J. **56**, 57-68, 2016.

[3] S. Lal, *Double matrix summability of double Fourier series*, Int. J. Math. Anal. **3** (34), 16691681, 2009.

[4] V.N. Mishra, K. Khatria and L.N. Mishra, *Using linear operators to approximate signals of $Lip(\alpha, p), (p \geq 1)$-class*, Filomat, **27** (2), 353-363, 2013.

[5] V.N. Mishra, K. Khatri and L.N. Mishra, Deepmala, *Trigonometric approximation of periodic Signals belonging to generalized weighted Lipschitz $W'(L_r, \xi(t)), (r \geq 1)-$class by Nörlund-Euler $(N, p_n)(E, q)$ operator of conjugate series of its Fourier series*, J. Class. Anal. **5** (2), 91-105, 2014.

[6] V.N. Mishra and L.N. Mishra, *Trigonometric approximation of signals (functions) in L^p-norm*, Int. J. Contemp. Math. Sci. **7** (19), 909-918, 2012.

[7] L.N. Mishra, V.N. Mishra and K. Khatri, Deepmala, *On the trigonometric approximation of signals belonging to generalized weighted Lipschitz $W(L^r, \xi(t))(r \geq 1)-$class by matrix $(C^1.N_p)$ operator of conjugate series of its Fourier series*, Appl. Math. Comput. **237**, 252-263, 2014.

[8] A. Mishra, V.N. Mishra and M. Mursaleen, *Trigonometric approximation of functions $f(x, y)$ of generalized Lipschitz class by double Hausdorff matrix summability method*, Adv. Difference Equ. **2020** (1), 681, 2020.

[9] M.L. Mittal and B.E. Rhoades, *Approximation by matrix means of double Fourier series to continuous functions in two variables*, Rad. Mat. **9** (1), 7799, 1999.

[10] F. Móricz and B.E. Rhoades, *Approximation by Nörlund means of double Fourier series for Lipschitz functions*, J. Approx. Theory **50** (4), 341358, 1987.

[11] F. Móricz and B.E. Rhoades, *Approximation by Nörlund means of double Fourier series to continuous functions in two variables*, Constr. Approx. **3** (1), 281296, 1987.

[12] F. Móricz and X. Shi, *Approximation to continuous functions by Cesàro means of double Fourier series and conjugate series*, J. Approx. Theory **49** (4), 346377, 1987.

[13] H.K. Nigam and M.K. Saha, *Characterizations of double Hausdorff matrices and best approximation of conjugate of a function in generalized Hölder space*, Filomat **36** (15), 5003-5028, 2022.

[14] H.K. Nigam and K. Sharma, *On double summability of double conjugate Fourier series*, Int. J. Math. Math. Sci. **2012**, 2012.

[15] Y.K. Patel and R.G. Vyas, *On approximation of double Fourier series and its conjugate series for functions in mixed Lebesgue space* $L_{\vec{p}}, \vec{p} \in [1, \infty]^2$, Proyecciones. **42** (2), 433-444, 2023.

[16] A. Rathore and U. Singh, *Approximation of certain bivariate functions by almost Euler means of double Fourier series*, J. Inequal. Appl. **2018** (1), 89, 2018.

[17] S.K. Srivastava and S. Devaiya, *Error of approximation of functions, conjugate to the functions belonging to weighted Lipschitz class using matrix means*, IAENG Int. J. Appl. Math. **51** (4), 837-841, 2021.