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Araştırma Makalesi / Research Article

A Novel Numerical Approach for Solving the Newell-Whitehead Equation

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Keywords
Fitzhugh-Nagumo equation; Newell-Whitehead equation; collocation; finite element; septic B-spline.

Abstract

Numerical solutions of Newell-Whitehead equation are investigated by collocation method in this study. Since higher order functions produce better approximations, septic B-spline basis functions is used for analysis and approximation. Error norms are calculated for the adequacy and effectiveness of the current method. Unconditional stability is proved using Von-Neumann theory. The numerical results are obtained and the comparisons are presented in the tables. Additionally, simulations of all numerical results are plotted to show the numerical behavior of the solution. Numerical results make the method more convenient and systematically handle the nonlinear solution process. The numerical solutions found make the method attractive and reliable for the solution of Fitzhugh-Nagumo type equations.

Newell-Whitehead Denkleminin Çözümü için Yeni Bir Sayısal Yaklaşım

Anahtar kelimeler
Fitzhugh-Nagumo denklemi; Newell-Whitehead denklemi; kollokasyon; sonlu elemanlar yöntemi; septik B-spline.

Öz

Bu çalışmada Newell-Whitehead denkleminin sayısal çözümleri kollokasyon yöntemi ile elde edilmiştir. Daha yüksek dereceli fonksiyonlar daha iyi yaklaşımlar ürettiğinden, analiz ve yaklaşım için septik B-spline baz fonksiyonları kullanılmıştır. Mevcut yöntemin yeterliliği ve etkinliği için hata normları hesaplanmıştır. Koşulsuz kararlılık, Von-Neumann teorisi kullanılarak kanıtlanmıştır. Sayısal sonuçlar elde edilmiş ve yapılan karşılaştırmalar tablolar halinde sunulmuştur. Ek olarak, çözümün sayısal davranışını göstermek için tüm sayısal sonuçların grafikleri çizilmiştir. Sayısal sonuçlar, yöntemi daha uygun hale getirir ve doğrusal olmayan çözüm sürecini sistematik olarak ele alır. Bulunan sayısal çözümler, kollokasyon yöntemini Fitzhugh-Nagumo tipi denklemlerin çözümü için oldukça ilgi çekici ve güvenilir kılmaktadır.

1. Introduction

The study of nonlinear evolution equations (NLEEs) is of great importance in many areas of mathematical physics and fluid dynamics. Recently, many scientists have paid great attention to the nonlinear parabolic partial differential equation (NPPDE), namely Fitzhugh-Nagumo (F-N) because of its importance in mathematical physics. In 1952, this nonlinear mathematical model was first proposed by Huxley and Hodgkin (Hodgkin and Huxley 1952). This model was an important model of biological systems to explain phenomena such as transmission of nerve impulses, flame propagation, population genetics, branching Brownian motion process, ionic current flows for axonal membranes, nuclear reactor theory (Abbasbandy 2008, Nagumo *et al.* 1962, Li and Guo 2006, Shih *et al.* 2005). But it was very difficult to analyze this model analytically. Later, in 1961, Fitzhugh and Nagumo developed a simplified version of this model, which they called F-N equation. The F-N equation given as below

$$u_t - u_{xx} = u(u-\alpha)(1-u), \tag{1}$$

is a nonlinear reaction-diffusion equation that describes nerve-impulse propagation (FitzHugh 1961). In the literature, the F-N equation is solved using various numerical and exact methods (Hariharan and Kannan 2010, Nucci and Clarkson 1992, Chen *et al.* 2003, Teodoro 2012, Ali *et al.* 2020, Devi and Yadav 2022). By taking $\alpha=-1$, Equation (1) transforms to the following Newell-Whitehead (N-W) equation:

$$u_t - u_{xx} = u - u^3, \tag{2}$$

The N-W equation is one of NLEEs and solved several researchers (Ezzati and Shakibi 2011). Kheiri obtained analytical solutions of the N-W equation by Homotopy Analysis Method (HAM) and Homotopy Pade Method (HPadeM) (Kheiri *et al*. 2011). Hariharan introduced a Legendre wavelet-based approximation method to solve the N-W equation (Hariharan 2014). Inan et al. found some analytical solutions of the F-N and N-W equations using the extended tanh extension method (İnan *et al.* 2021).

In order to find the numerical solution of actual life problems and in various fields of science, it is very important to choose an appropriate numerical approach. The Finite Element Method (FEM), which is the most competent method in solving boundary-value problems in approximation theory, is remarkable (Karakoç *et al* 2023 and Kutluay et al 2022). In this method, B-spline-based sequencing method has been chosen as interpolation functions in terms of its programmable computational approach and easy applicability (Karakoç *et al.* 2022). In this study, collocation finite element method is applied to obtain the numerical solution of the N-W equation.

One of the important goals of this article is to develop different numerical solutions of the Newell-Whitehead equation. For these aims, we begin in Section 2 introducing B-spline functions and applying the method to the equation. Section 3 includes stability analysis of the numerical technique. In Section 4, test problems taken from the literature have been solved and the obtained results are given in the tabular form as well as plotted graphically. The article ends with a brief conclusion.

2. Septic B-Splines

Septic B-spline functions $\phi_m(x), m=-3(3)N$, at the nodes x_m are defined over the solution interval $[a,b]$:

$$\phi_m(x)=\begin{cases} a^7, & [x_{m-4},x_{m-3}] \\ a^7-8b^7, & [x_{m-3},x_{m-2}] \\ a^7-8b^7+28c^7, & [x_{m-2},x_{m-1}] \\ a^7-8b^7+28c^7-56d^7, & [x_{m-1},x_m] \\ e^7-8f^7+28g^7-56h^7, & [x_m,x_{m+1}] \\ e^7-8f^7+28g^7, & [x_{m+1},x_{m+2}] \\ e^7-8f^7, & [x_{m+2},x_{m+3}] \\ e^7, & [x_{m+3},x_{m+4}] \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

where $a=(x-x_{m-4})^7$, $b=(x-x_{m-3})^7$, $c=(x-x_{m-2})^7$, $d=(x-x_{m-1})^7$, $e=(x_{m+4}-x)^7$, $f=(x_{m+3}-x)^7$, $g=(x_{m+2}-x)^7$, h$=(x_{m+1}-x)^7$ (Prenter 1975). Among the others, collocation is well-known technique to improve numerical methods because of its various intended feature (Baker 1976). In collocation method, $u_{numeric}(x,t)$ corresponding to the $u_{exact}(x,t)$ is written as a linear combination of septic B-splines as shown below:

$$u_{numeric}(x,t)=\sum_{m=-3}^{N+3}\phi_m(x)\vartheta_m(t). \tag{4}$$

When $h\text{l}=x-x_m, 0\le \text{l} \le 1$ transformation is made in the finite region $[x_m, x_{m+1}]$ the region turns to an interval of [0,1]. Thus the septic B-spline functions in the new region [0,1] are obtained as follows:

$$
\begin{aligned}
\phi_{m-3} &= 1-7l+21l^2-35l^3+35l^4-21l^5+7l^6-l^7,\\
\phi_{m-2} &= 120-392l+504l^2-280l^3+84l^5-42l^6+7l^7,\\
\phi_{m-1} &= 1191-1715l+315l^2+665l^3-315l^4-105l^5+105l^6-21l^7,\\
\phi_{m} &= 2416-1680l+560l^4-140l^5+35l^7,\\
\phi_{m+1} &= 1191+1715l+315l^2-665l^3-315l^4+105l^5+105l^6-35l^7,\\
\phi_{m+2} &= 120+392l+504l^2+280l^3-84l^5-42l^6+21l^7,\\
\phi_{m+3} &= 1+7l+21l^2+35l^3+35l^4+21l^5+7l^6-l^7,\\
\phi_{m+4} &= l^7.
\end{aligned} \tag{5}
$$

Using the equalities given by (4) and (5), the following expressions are obtained:

$$
\begin{aligned}
u_{numeric}(x_m,t) &= \vartheta_{m-3}+120\vartheta_{m-2}+1191\vartheta_{m-1}+2416\vartheta_m\\
&\quad +1191\vartheta_{m+1}+120\vartheta_{m+2}+\vartheta_{m+3}\\
u_m' &= \frac{7}{h}(-\vartheta_{m-3}-56\vartheta_{m-2}-245\vartheta_{m-1}\\
&\quad +245\vartheta_{m+1}+56\vartheta_{m+2}+\vartheta_{m+3}),\\
u_m'' &= \frac{42}{h^2}(\vartheta_{m-3}+24\vartheta_{m-2}+15\vartheta_{m-1}-80\vartheta_m\\
&\quad +15\vartheta_{m+1}+24\vartheta_{m+2}+\vartheta_{m+3}),\\
u_m''' &= \frac{210}{h^3}(-\vartheta_{m-3}-8\vartheta_{m-2}+19\vartheta_{m-1}-19\vartheta_{m+1}\\
&\quad +8\vartheta_{m+2}+\vartheta_{m+3}).
\end{aligned} \tag{6
$$

2.1 Analysis of the Method on Newell-Whitehead Equation

In this section, putting (4) and (6) in Equation (2) and making some simplifications, the next system of ordinary differential equations (ODEs) are reached:

$$
\begin{aligned}
&\dot{\vartheta}_{m-3}+120\dot{\vartheta}_{m-2}+1191\dot{\vartheta}_{m-1}+2416\dot{\vartheta}_m\\
&+1191\dot{\vartheta}_{m+1}+120\dot{\vartheta}_{m+2}+\dot{\vartheta}_{m+3}\\
&-\frac{42}{h^2}(\vartheta_{m-3}+24\vartheta_{m-2}+15\vartheta_{m-1}-80\vartheta_m\\
&+15\vartheta_{m+1}+24\vartheta_{m+2}+\vartheta_{m+3})\\
&+(Z_m+1)(\vartheta_{m-3}+120\vartheta_{m-2}+1191\vartheta_{m-1}\\
&+2416\vartheta_m+1191\vartheta_{m+1}+120\vartheta_{m+2}+\vartheta_{m+3})=0,
\end{aligned} \tag{7}
$$

in which $\dot{\vartheta}=\frac{d\vartheta}{dt}$ and

$$
\begin{aligned}
Z_m = u^2 &= (\vartheta_{m-3}+120\vartheta_{m-2}+1191\vartheta_{m-1}+2416\vartheta_m\\
&\quad +1191\vartheta_{m+1}+120\vartheta_{m+2}+\vartheta_{m+3})^2,
\end{aligned}
$$

Replacing $\dot{\vartheta}_i$ by forward difference approximation

$$\dot{\vartheta}_i=\frac{\vartheta_i^{n+1}-\vartheta_i^n}{\Delta t}$$

and ρ_i by Crank-Nicolson formulation

$$\vartheta_i=\frac{\vartheta_i^{n+1}+\vartheta_i^n}{2},$$

then the system of ODEs (7) reduces to a system of nonlinear equations

$$
\begin{aligned}
&\lambda_1\vartheta_{m-3}^{n+1}+\lambda_2\vartheta_{m-2}^{n+1}+\lambda_3\vartheta_{m-1}^{n+1}+\lambda_4\vartheta_m^{n+1}\\
&+\lambda_5\vartheta_{m+1}^{n+1}+\lambda_6\vartheta_{m+2}^{n+1}+\lambda_7\vartheta_{m+3}^{n+1}\\
&=+\lambda_7\vartheta_{m-3}^{n}+\lambda_6\vartheta_{m-2}^{n}+\lambda_5\vartheta_{m-1}^{n}+\lambda_4\vartheta_m^{n}\\
&+\lambda_3\vartheta_{m+1}^{n}+\lambda_2\vartheta_{m+2}^{n}+\lambda_1\vartheta_{m+3}^{n}
\end{aligned} \tag{8}
$$

where

$$
\begin{aligned}
\lambda_1 &= [1-A+BZ_m-B],\\
\lambda_2 &= [120-24A+120BZ_m-120B],\\
\lambda_3 &= [1191-15A+1191BZ_m-1191B],\\
\lambda_4 &= [2416+80A+2416BZ_m-2416B],\\
\lambda_5 &= [1191-15A+1191BZ_m-1191B],\\
\lambda_6 &= [120-24A+120BZ_m-120B],\\
\lambda_7 &= [1-A+BZ_m-B]
\end{aligned} \tag{9}
$$

and

$$A=\frac{21}{h^2}\Delta t,\ B=\frac{1}{2}\Delta t,\quad m=0,1,\ldots,N-1. \tag{10}$$

To assure a unique solution, it is necessary to eliminate unknown parameters $\vartheta_{-3},\vartheta_{-2},\vartheta_{-1}$, $\vartheta_{N+1},\vartheta_{N+2}$ and ϑ_{N+3} from the resulting system (8). This procedure can be easily applied using the values of u and boundary conditions, and then

$$Pd^{n+1}=Qd^n \tag{11}$$

is obtained where $d^n=(\sigma_0,\sigma_1,\ldots,\sigma_N)^T$.

3. Stability of the Method

For the stability analysis, Von Neumann technique has been used. Amplification factor ξ of a typical Fourier mode of amplitude is determined as:

$$\sigma_m^n=\xi^n e^{imkh}. \tag{12}$$

Using Equation (12) in the (8),

$$\xi = \frac{\cos a_1}{\cos a_2} \tag{13}$$

is obtained and in which

$$\begin{aligned} a_1 &= (2-\alpha+\mu-\beta)\cos(3kh) \\ &+(240-24\alpha+120\mu-120\beta)\cos(2kh) \\ &+(2382-15\alpha+1191\mu-1191\beta)\cos(kh)+\kappa, \\ a_2 &= (2+\alpha-\mu+\beta)\cos(3kh) \\ &+(240+24+\alpha-120\mu+120\beta)\cos(2kh) \\ &+(2382+15\alpha-1191\mu+1191\beta)\cos(kh)+\kappa, \end{aligned}$$

where

$$\begin{aligned} &\alpha = 2A, \mu = 2BZ_m, \beta = 2B, \\ &\kappa = 2416+80A+2416BZ_m-2416B, \end{aligned}$$

so that $|\xi| \le 1$, which proves unconditional stability of the linearized numerical scheme for the N-W equation.

4. Numerical Experiments and Discussions

In this section, the proposed scheme is applied for solution of N-W equation for various values of the time and space division and we approximate them using the described scheme. Firstly, the F-N equation has an exact solution of the form

$$u(x,t) = \frac{1}{2} + \frac{1}{2}\tanh[\frac{1}{2\sqrt{2}}(x-\frac{2\alpha-1}{\sqrt{2}})t]. \tag{14}$$

F-N equation is considered the following boundary-initial conditions

$$\begin{aligned} u(x,0) &= \frac{1}{2}+\frac{1}{2}\tanh[\frac{x}{2\sqrt{2}}], \\ u(a,t) &= \frac{1}{2}+\frac{1}{2}\tanh[\frac{1}{2\sqrt{2}}(a-\frac{2\alpha-1}{\sqrt{2}})t], \\ u(b,t) &= \frac{1}{2}+\frac{1}{2}\tanh[\frac{1}{2\sqrt{2}}(b-\frac{2\alpha-1}{\sqrt{2}})t] \end{aligned} \tag{15}$$

where $u \to 0$ as $x \to \mp\infty$.

The F-N equation is reduced to the N-W equation by taking $\alpha = -1$. To demonstrate the validity of our numerical scheme, the interval of the problem is chosen as $x \in [-1,1]$ and various time steps considering the studies in the literature.

We will use the error norms, widely used in the literature, namely L_2 and L_∞ in order to check the efficiency and accuracy of our method:

$$\begin{aligned} L_2 &= \|u_{exact}-u_{numeric}\|_2 \\ &= \sqrt{h\sum_{j=1}^{N}\left|(u_{exact})_j-(u_{numeric})_j\right|^2}, \\ L_\infty &= \|u_{exact}-u_{numeric}\|_\infty \\ &= \max_j\left|(u_{exact})_j-(u_{numeric})_j\right|, \quad j=1,2,\ldots,N. \end{aligned} \tag{16}$$

In simulation calculations in order to comply with the literature, as typical values $\Delta t = 0.1$, $\Delta t = 0.01$ with $h = 0.1$ and $h = 0.01$ were chosen. In Table 1 and Table 2, the values of the error norms L_2 and L_∞ calculated over these values for time levels and step sizes are presented. When the tables are examined, the calculated error norms L_2 and L_∞ are found to be satisfactorily small. We can say that the increase in the number of time divisions has positive effects on the numerical results. In those tables, it is clearly seen that the collocation method combined with the division methods is better, faster and more reliable than the other methods.

Table 1. The error norms for $h = 0.1, h = 0.01$ and $\Delta t = 0.01$.

	$\Delta t = 0.01, h = 0.1$		$\Delta t = 0.01, h = 0.01$	
t	L_2	L_∞	L_2	L_∞
10	0.8828×10^{-6}	1.4451×10^{-6}	9.778×10^{-7}	1.4940×10^{-6}
20	1.3890×10^{-6}	2.0655×10^{-6}	7.220×10^{-7}	0.9835×10^{-6}
30	1.3894×10^{-6}	2.0656×10^{-6}	8.452×10^{-7}	1.4006×10^{-6}
40	1.3915×10^{-6}	2.0658×10^{-6}	9.976×10^{-7}	1.4608×10^{-6}
50	1.3890×10^{-6}	2.0660×10^{-6}	8.790×10^{-7}	1.5913×10^{-6}

Table 2. The error norms for $h = 0.1, h = 0.01$ and $\Delta t = 0.1$.

	$\Delta t = 0.1, h = 0.1$		$\Delta t = 0.1, h = 0.01$	
t	L_2	L_∞	L_2	L_∞
10	7.082×10^{-7}	3.240×10^{-7}	2.8680×10^{-5}	1.5624×10^{-5}
20	6.172×10^{-7}	2.538×10^{-7}	2.6989×10^{-5}	1.4440×10^{-5}
30	6.031×10^{-7}	1.765×10^{-7}	2.5485×10^{-5}	1.3300×10^{-5}
40	6.176×10^{-7}	1.893×10^{-7}	2.4081×10^{-5}	1.2296×10^{-5}
50	6.027×10^{-7}	1.962×10^{-7}	2.2819×10^{-5}	1.1391×10^{-5}

If we analyze Figure 1, we can clearly see the two-dimensional case of the upper half bell-shaped wave solutions generated at times $t=1$, $t=5$ and $t=10$. We see the three-dimensional shape and contour graph of the wave solutions in the selected time intervals in Figure 2.

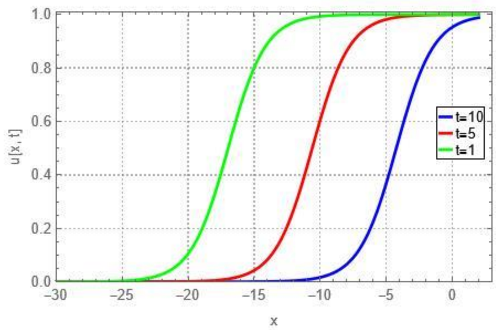

Figure 1. Numerical behaviours (2D) of N-W equation for $t=1, t=5$ and $t=10$.

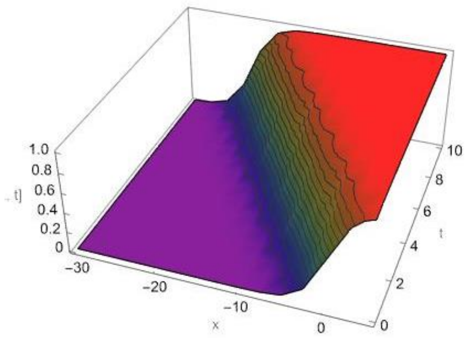

Figure 2. Numerical behaviours (3D) of N-W equation for $\Delta t=0.01$ and $h=0.1$.

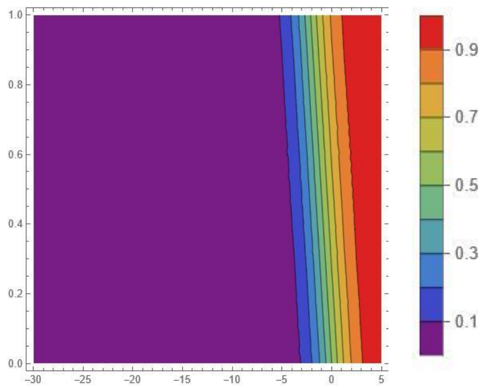

Figure 3. Contour graph of N-W equation for $\Delta t=0.01$ and $h=0.1$.

4. Conclusion

We derived the formulation of the collocation finite element method for the N-W equation in the article. We consider the equation with the approximate solution which converges faster to the exact solution. The method has been successfully used to discover the approximate solution of the N-W equation. The obtained numerical results are quite satisfactory for large time steps as well as for space steps, and the absolute error map provides very few errors that can be neglected. It is obvious that numerical algorithm is unconditionally stable. From both the numerical and graphical presentation it can be concluded that the method is quite efficient for solving the wide variety of NLEEs that arise in various disciplines.

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