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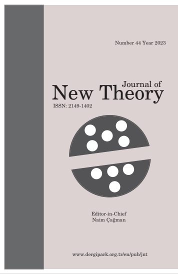

On Dual Quaternions with $k-$Generalized Leonardo Components

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Abstract − In this paper, we define a one-parameter generalization of Leonardo dual quaternions, namely $k-$generalized Leonardo-like dual quaternions. We introduce the properties of k-generalized Leonardo-like dual quaternions, including relations with Leonardo, Fibonacci, and Lucas dual quaternions. We investigate their characteristic relations, involving the Binet-like formula, the generating function, the summation formula, Catalan-like, Cassini-like, d'Ocagne-like, Tagiuri-like, and Hornsberger-like identities. The crucial part of the present paper is that one can reduce the calculations of Leonardo-like dual quaternions by considering k. For $k = 1$, these results are generalizations of the ones for ordered Leonardo quadruple numbers. Finally, we discuss the need for further research.

Keywords *Leonardo sequence, recurrence relations, dual quaternions*

Mathematics Subject Classification (2020) 11B37, 11B39

1. Introduction

The well-known Fibonacci sequence $\{F_n\}_{n\geq 2}$ and the Lucas sequence $\{L_n\}_{n\geq 2}$ are defined recursively by $F_n = F_{n-1} + F_{n-2}$, and $L_n = L_{n-1} + L_{n-2}$ with initial conditions $F_0 = 0$, $F_1 = 1$, and $L_0 = 2$, $L_1 = 1$, respectively [1]. The Binet's formulas of the Fibonacci and Lucas sequences are as follows, respectively:

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \tag{1}$$

and

$$L_n = \alpha^n + \beta^n$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ are roots of characteristic equation $x^2 - x - 1 = 0$ [1]. Many generalizations of the Fibonacci and Lucas sequences have been studied by several researchers. In this study, we consider the Leonardo sequence. The Leonardo sequence $\{Le_n\}_{n\geq 2}$ is defined non-homogeneous recursively by

$$Le_n = Le_{n-1} + Le_{n-2} + 1$$

or

$$Le_{n+1} = 2Le_n - Le_{n-2}$$

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with initial conditions $Le_0 = Le_1 = 1$ and $Le_2 = 3$ [2]. The Binet-like formula of the Leonardo sequence is

$$Le_n = 2\left(\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}\right)-1$$

There exist many identities between Fibonacci, Lucas, and Leonardo numbers. For $n \geq 0$, the fundamental relationships between the Fibonacci, Lucas, and Leonardo sequences are [2]:

$$Le_n = 2F_{n+1} - 1$$

$$Le_n = 2\left(\frac{L_n + L_{n+2}}{5}\right)-1$$

$$Le_{n+3} = \frac{L_{n+1}+L_{n+7}}{5} - 1$$

and

$$Le_n = L_{n+2} - F_{n+2} - 1$$

Although the Fibonacci, Lucas, and Leonardo sequences are closely related, they exhibit distinct characteristic properties. Several different properties and generalizations of the Leonardo sequence were previously studied by various researchers [3–15]. Recently, a one-parameter generalized Leonardo sequence has been defined as non-homogeneous recursively by

$$\mathsf{Le}_n^{(k)} = \mathsf{Le}_{n-1}^{(k)} + \mathsf{Le}_{n-2}^{(k)} + k, \quad n \geq 2 \tag{2}$$

with the initial conditions $\mathsf{Le}_0^{(k)} = \mathsf{Le}_1^{(k)} = 1$. Here, k is a fixed positive integer [4]. The Binet-like formula of the $k-$generalized Leonardo sequence is

$$\mathsf{Le}_n^{(k)} = (k+1)\left(\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}\right)-k$$

The $k-$generalized Leonardo sequence is related to the Fibonacci and Lucas sequences. For $n \geq 0$, the fundamental relationships between Fibonacci, Lucas, and the $k-$generalized Leonardo sequences are [4]:

$$\mathsf{Le}_n^{(k)} = (k+1)F_{n+1} - k \tag{3}$$

and

$$\mathsf{Le}_n^{(k)} = (k+1)\left(L_n - F_{n-1}\right) - k$$

The summation formulas of the $k-$generalized Leonardo numbers are [4]:

$$\sum_{s=0}^{n} \mathsf{Le}_s^{(k)} = \mathsf{Le}_{n+2}^{(k)} - k\left(n+1\right) - 1 \tag{4}$$

$$\sum_{s=0}^{n} \mathsf{Le}_{2s}^{(k)} = \mathsf{Le}_{2n+1}^{(k)} - kn \tag{5}$$

and

$$\sum_{s=0}^{n} \mathsf{Le}_{2s+1}^{(k)} = \mathsf{Le}_{2n+2}^{(k)} - k(n+2) \tag{6}$$

The $k-$generalized Leonardo sequence is the key concept of the present paper. For $k = 1$, this sequence is the classical Leonardo sequence, i.e., $\mathsf{Le}_n^{(1)} = Le_n$. In this case, we may omit the superscript (1) in the notation.

There are several ways to define new special sequences but the most popular method is to define a sequence with different hypercomplex number components. Horadam [16] defined the real quaternions with the classic Fibonacci sequence $\{QF_n\}_{n\geq 2}$ and the classic Lucas sequence $\{QL_n\}_{n\geq 2}$ recursively by

$$QF_n = F_n + F_{n+1}e_1 + F_{n+2}e_2 + F_{n+3}e_3$$

and

$$QL_n = L_n + L_{n+1}e_1 + L_{n+2}e_2 + L_{n+3}e_3$$

where F_n and L_n are the n-th classic Fibonacci and Lucas numbers, respectively. Here, the quaternionic units $\{e_1, e_2, e_3\}$ satisfy the following multiplication rules:

$${e_1}^2 = {e_2}^2 = {e_3}^2 = -1, \quad e_1e_2 = -e_2e_1 = e_3, \quad e_2e_3 = -e_3e_2 = e_1, \quad \text{and} \quad e_3e_1 = -e_1e_3 = e_2$$

In general, a real quaternion q is of the form $q = a + e_1b + e_2c + e_3d$ where $a, b, c, d \in \mathbb{R}$. The quaternions form a four-dimensional associative and non-commutative algebra over the real numbers. For a deeper discussion of the quaternions, see [17–19]. Changing conditions in the multiplication rules produces different type of quaternions. In this study, we consider the dual quaternions. The dual quaternionic units obey the following multiplication rules:

$${e_1}^2 = {e_2}^2 = {e_3}^2 = 0 \quad \text{and} \quad e_1e_2 = -e_2e_1 = e_2e_3 = -e_3e_2 = e_3e_1 = -e_1e_3 = 0 \tag{7}$$

Yüce et al. [20] defined the dual quaternions with the Fibonacci sequence $\{\hat{\mathcal{F}}_n\}_{n\geq 2}$ and the Lucas sequence $\{\hat{\mathcal{L}}_n\}_{n\geq 2}$ recursively by

$$\hat{\mathcal{F}}_n = F_n + F_{n+1}e_1 + F_{n+2}e_2 + F_{n+3}e_3$$

and

$$\hat{\mathcal{L}}_n = L_n + L_{n+1}e_1 + L_{n+2}e_2 + L_{n+3}e_3$$

where F_n and L_n are the n-th classic Fibonacci and Lucas numbers, respectively. Here, the non-real dual quaternionic units $\{e_1, e_2, e_3\}$ satisfy Equation 7. For a deeper discussion of the dual quaternions, see [21–27]. Nurkan et al. [13] defined the dual quaternions with the classic Leonardo sequence $\{\hat{\mathcal{L}}e_n\}_{n\geq 2}$ recursively by

$$\hat{\mathcal{L}}e_n = Le_n + Le_{n+1}e_1 + Le_{n+2}e_2 + Le_{n+3}e_3$$

where Le_n is the n-th classic Leonardo number.

Considering all these details, a natural question is whether the paper [13] can be generalized. In this study, we aim to determine the dual quaternions with the $k-$generalized Leonardo sequence. We consider the coefficients of the dual quaternions as the $k-$generalized Leonardo sequence.

2. The $k-$Generalized Leonardo-like Dual Quaternion Sequence

This section introduces the dual quaternions with one parameter generalized Leonardo sequence and investigates their characteristic properties.

Definition 2.1. The n-th $k-$generalized Leonardo-like dual quaternion is defined by

$$\hat{\mathsf{Le}}_n^{(k)} = \mathsf{Le}_n^{(k)} + \mathsf{Le}_{n+1}^{(k)}\, e_1 + \mathsf{Le}_{n+2}^{(k)}\, e_2 + \mathsf{Le}_{n+3}^{(k)}\, e_3 \tag{8}$$

where $\mathsf{Le}_n^{(k)}$ is the n-th $k-$generalized Leonardo number, k is a fixed positive integer, and $\{e_1, e_2, e_3\}$ are the set of all the dual quaternionic units. The $k-$generalized Leonardo-like dual quaternion sequence is denoted by $\left\{\hat{\mathsf{Le}}_n^{(k)}\right\}_{n\geq 2}$.

Note that, if $k = 1$, the generalized Leonardo-like dual quaternion sequence $\left\{\hat{\mathsf{Le}}_n^{(k)}\right\}_{n\geq 2}$ is the Leonardo quadruple sequence. In this case, we may omit the superscript (1) in the notation.

Let $\hat{\mathsf{Le}}_n^{(k)}$ and $\hat{\mathsf{Le}}_m^{(k)}$ be two $k-$generalized Leonardo-like dual quaternions. The $k-$generalized Leonardo number $\mathsf{Le}_n^{(k)}$ is called scalar (real) part of $\hat{\mathsf{Le}}_n^{(k)}$ and denoted by $\mathcal{S}_{\hat{\mathsf{Le}}_n^{(k)}}$ and the vector

$$\mathsf{Le}_{n+1}^{(k)}\, e_1 + \mathsf{Le}_{n+2}^{(k)}\, e_2 + \mathsf{Le}_{n+3}^{(k)}\, e_3$$

is called the pure part of $\hat{\mathsf{Le}}_n^{(k)}$ and denoted by $\mathcal{V}_{\hat{\mathsf{Le}}_n^{(k)}}$. The addition is defined component-wise as

$$\hat{\mathsf{Le}}_n^{(k)} \pm \hat{\mathsf{Le}}_m^{(k)} = \left(\mathsf{Le}_n^{(k)} \pm \mathsf{Le}_m^{(k)}\right) + \left(\mathsf{Le}_{n+1}^{(k)} \pm \mathsf{Le}_{m+1}^{(k)}\right) e_1 + \left(\mathsf{Le}_{n+2}^{(k)} \pm \mathsf{Le}_{m+2}^{(k)}\right) e_2 + \left(\mathsf{Le}_{n+3}^{(k)} \pm \mathsf{Le}_{m+3}^{(k)}\right) e_3$$

whereas multiplication is defined by

$$\begin{aligned}\hat{\mathsf{Le}}_n^{(k)}\hat{\mathsf{Le}}_m^{(k)} = & \left(\mathsf{Le}_n^{(k)}\mathsf{Le}_m^{(k)}\right) + \left(\hat{\mathsf{Le}}_n^{(k)}\hat{\mathsf{Le}}_{m+1}^{(k)} + \hat{\mathsf{Le}}_m^{(k)}\hat{\mathsf{Le}}_{n+1}^{(k)}\right) e_1 + \left(\hat{\mathsf{Le}}_n^{(k)}\hat{\mathsf{Le}}_{m+2}^{(k)} + \hat{\mathsf{Le}}_m^{(k)}\hat{\mathsf{Le}}_{n+2}^{(k)}\right) e_2 \\ & + \left(\hat{\mathsf{Le}}_n^{(k)}\hat{\mathsf{Le}}_{m+3}^{(k)} + \hat{\mathsf{Le}}_m^{(k)}\hat{\mathsf{Le}}_{n+3}^{(k)}\right) e_3\end{aligned}$$

or

$$\hat{\mathsf{Le}}_n^{(k)}\hat{\mathsf{Le}}_m^{(k)} = \mathcal{S}_{\hat{\mathsf{Le}}_n^{(k)}}\mathcal{S}_{\hat{\mathsf{Le}}_m^{(k)}} + \mathcal{S}_{\hat{\mathsf{Le}}_n^{(k)}}\mathcal{V}_{\hat{\mathsf{Le}}_m^{(k)}} + \mathcal{S}_{\hat{\mathsf{Le}}_m^{(k)}}\mathcal{V}_{\hat{\mathsf{Le}}_n^{(k)}}$$

The conjugate and norm of any $k-$generalized Leonardo-like dual quaternion $\hat{\mathsf{Le}}_n^{(k)}$ is given by

$$\overline{\hat{\mathsf{Le}}_n^{(k)}} = \mathcal{S}_{\hat{\mathsf{Le}}_n^{(k)}} - \mathcal{V}_{\hat{\mathsf{Le}}_n^{(k)}} = \hat{\mathsf{Le}}_n^{(k)} - \hat{\mathsf{Le}}_{n+1}^{(k)} e_1 - \hat{\mathsf{Le}}_{n+2}^{(k)}\, e_2 - \hat{\mathsf{Le}}_{n+3}^{(k)} e_3$$

and

$$||\hat{\mathsf{Le}}_n^{(k)}|| = \hat{\mathsf{Le}}_n^{(k)}\, \overline{\hat{\mathsf{Le}}_n^{(k)}} = \left(\hat{\mathsf{Le}}_n^{(k)}\right)^2 \in \mathbb{R}$$

respectively.

Theorem 2.2. The recurrence relation of the $k-$generalized Leonardo-like dual quaternion sequence is

$$\hat{\mathsf{Le}}_n^{(k)} = \hat{\mathsf{Le}}_{n-1}^{(k)} + \hat{\mathsf{Le}}_{n-2}^{(k)} + \hat{\mathcal{K}}, \quad n \geq 2$$

where $\hat{\mathcal{K}} = k\,(1 + e_1 + e_2 + e_3)$ with initial conditions $\hat{\mathsf{Le}}_0^{(k)} = 1 + e_1 + (2+k)\, e_2 + (3+2k)\, e_3$ and $\hat{\mathsf{Le}}_1^{(k)} = 1 + (2+k)\, e_1 + (3+2k)\, e_2 + (5+4k)\, e_3$.

Proof.
From Definition 2.1, it follows that

$$\begin{aligned}\hat{\mathsf{Le}}_{n-1}^{(k)} + \hat{\mathsf{Le}}_{n-2}^{(k)} + \hat{\mathcal{K}} = & \left(\mathsf{Le}_{n-1}^{(k)} + \mathsf{Le}_{n-2}^{(k)} + k\right) + \left(\mathsf{Le}_n^{(k)} + \mathsf{Le}_{n-1}^{(k)} + k\right) e_1 + \left(\mathsf{Le}_{n+1}^{(k)} + \mathsf{Le}_n^{(k)} + k\right) e_2 \\ & + \left(\mathsf{Le}_{n+2}^{(k)} + \mathsf{Le}_{n+1}^{(k)} + k\right) e_3\end{aligned}$$

By applying Equation 2, we complete the proof. □

Throughout this paper, let $\hat{\mathcal{K}} = k\,(1 + e_1 + e_2 + e_3)$.

Theorem 2.3. The other recurrence relation of $\left\{\hat{\mathsf{Le}}_n^{(k)}\right\}_{n\geq 2}$ is

$$\hat{\mathsf{Le}}_{n+1}^{(k)} = 2\hat{\mathsf{Le}}_n^{(k)} - \hat{\mathsf{Le}}_{n-2}^{(k)}$$

Proof.
By Theorem 2.2, the proof is straightforward. □

Afterward, we state the Binet-like formula for the $k-$generalized Leonardo-like dual quaternion $\hat{\mathsf{Le}}_n^{(k)}$. Thus, we derive some well-known mathematical properties.

Theorem 2.4. The Binet-like formula of the $k-$generalized Leonardo-like dual quaternion $\hat{\mathsf{Le}}_n^{(k)}$ is

$$\hat{\mathsf{Le}}_n^{(k)} = (k+1)\left(\frac{\alpha^*\alpha^{n+1} - \beta^*\beta^{n+1}}{\alpha-\beta}\right) - \hat{\mathcal{K}} \tag{9}$$

where $\alpha^* = 1+\alpha e_1+\alpha^2 e_2+\alpha^3 e_3$ and $\beta^* = 1+\beta e_1+\beta^2 e_2+\beta^3 e_3$.

Proof.
By using Definition 2.1 and Equation 3,

$$\begin{aligned}\hat{\mathsf{Le}}_n^{(k)} &= \mathsf{Le}_n^{(k)} + \mathsf{Le}_{n+1}^{(k)}\, e_1 + \mathsf{Le}_{n+2}^{(k)}\, e_2 + \mathsf{Le}_{n+3}^{(k)}\, e_3 \\ &= (k+1)\left(F_{n+1} + F_{n+2}\, e_1 + F_{n+3}\, e_2 + F_{n+4}\, e_3\right) - k\left(1+e_1+e_2+e_3\right)\end{aligned}$$

Applying the Binet's formula of the Fibonacci sequence in Equation 1 and then taking $\alpha^* = 1+\alpha e_1 + \alpha^2 e_2+\alpha^3 e_3$, $\beta^* = 1+\beta e_1+\beta^2 e_2+\beta^3 e_3$, and $\hat{\mathcal{K}} = k\left(1+e_1+e_2+e_3\right)$,

$$\begin{aligned}\hat{\mathsf{Le}}_n^{(k)} &= (k+1)\left(\frac{\alpha^{n+1}\left(1+\alpha e_1+\alpha^2 e_2+\alpha^3 e_3\right) - \beta^{n+1}\left(1+\beta e_1+\beta^2 e_2+\beta^3 e_3\right)}{\alpha-\beta}\right) - \hat{\mathcal{K}} \\ &= (k+1)\left(\frac{\alpha^*\alpha^{n+1} - \beta^*\beta^{n+1}}{\alpha-\beta}\right) - \hat{\mathcal{K}}\end{aligned}$$

is obtained. □

Here, we state some relations between $k-$generalized Leonardo-like dual quaternions, Fibonacci dual quaternions, Lucas dual quaternions, and Fibonacci and Lucas numbers.

Theorem 2.5. Let $\hat{\mathsf{Le}}_n^{(k)}$ be the n-th $k-$generalized Leonardo-like dual quaternion, $\hat{\mathcal{F}}_n$ be the n-th Fibonacci dual quaternion, and $\hat{\mathcal{L}}_n$ be the n-th Lucas dual quaternion. For positive integers n, m, r, and t with $n \geq r$ and $n \geq m$, the following relations hold:

i. $\hat{\mathsf{Le}}_n^{(k)} = (k+1)\hat{\mathcal{F}}_{n+1} - \hat{\mathcal{K}}$

ii. $\hat{\mathsf{Le}}_n^{(k)} = (k+1)(\hat{\mathcal{L}}_n - \hat{\mathcal{F}}_{n-1}) - \hat{\mathcal{K}}$

iii. $\hat{\mathsf{Le}}_{n+r}^{(k)} + \hat{\mathsf{Le}}_{n-r}^{(k)} = (k+1)\begin{cases} L_r\hat{\mathcal{F}}_{n+1} - 2\hat{\mathcal{K}}, & r=2t \\ F_r\hat{\mathcal{L}}_{n+1} - 2\hat{\mathcal{K}}, & r=2t+1\end{cases}$

iv. $\hat{\mathsf{Le}}_{n+r}^{(k)} - \hat{\mathsf{Le}}_{n-r}^{(k)} = (k+1)\begin{cases} F_r\hat{\mathcal{L}}_{n+1}, & r=2t \\ L_r\hat{\mathcal{F}}_{n+1}, & r=2t+1\end{cases}$

v. $\hat{\mathsf{Le}}_{n+m}^{(k)} + (-1)^m\hat{\mathsf{Le}}_{n-m}^{(k)} = L_m\hat{\mathsf{Le}}_n^{(k)} + \hat{\mathcal{K}}(L_m - (-1)^m - 1)$

vi. $\hat{\mathsf{Le}}_{n+m}^{(k)} - (-1)^m\hat{\mathsf{Le}}_{n-m}^{(k)} = (k+1)\, F_m\hat{\mathcal{L}}_{n+1} - \hat{\mathcal{K}}\left(1-(-1)^m\right)$

Proof.
Let $\hat{\mathsf{Le}}_n^{(k)}$ be the n-th $k-$generalized Leonardo-like dual quaternion, $\hat{\mathcal{F}}_n$ be the n-th Fibonacci dual quaternion, and $\hat{\mathcal{L}}_n$ be the n-th Lucas dual quaternion.

i. According to Equation 3,

$$\begin{aligned}\hat{\mathsf{Le}}_n^{(k)} &= ((k+1)\,F_{n+1} - k) + ((k+1)\,F_{n+2} - k)\,e_1 + ((k+1)\,F_{n+3} - k)\,e_2 + ((k+1)\,F_{n+4} - k)\,e_3 \\ &= (k+1)\,\hat{\mathcal{F}}_{n+1} - \hat{\mathcal{K}}\end{aligned}$$

iii.

$$\hat{\mathsf{Le}}_{n+r}^{(k)} + \hat{\mathsf{Le}}_{n-r}^{(k)} = \left(\mathsf{Le}_{n+r}^{(k)} + \mathsf{Le}_{n-r}^{(k)}\right) + \left(\mathsf{Le}_{n+1+r}^{(k)} + \mathsf{Le}_{n+1-r}^{(k)}\right) e_1 + \left(\mathsf{Le}_{n+2+r}^{(k)} + \mathsf{Le}_{n+2-r}^{(k)}\right) e_2 + \left(\mathsf{Le}_{n+3+r}^{(k)} + \mathsf{Le}_{n+3-r}^{(k)}\right) e_3$$

Considering Equation 3,

$$\hat{\mathsf{Le}}_{n+r}^{(k)} + \hat{\mathsf{Le}}_{n-r}^{(k)} = (k+1)\left(\left(F_{n+1+r} + F_{n+1-r}\right) + \left(F_{n+2+r} + F_{n+2-r}\right) e_1 + \left(F_{n+3+r} + F_{n+3-r}\right) e_2 + \left(F_{n+4+r} + F_{n+4-r}\right) e_3\right) - 2\hat{\mathcal{K}}$$

By using the definition of Fibonacci dual quaternion and the following relation of Fibonacci numbers (see [1])

$$F_{n+r} + F_{n-r} = \begin{cases} F_n L_r, & r = 2t \\ F_r L_n, & r = 2t+1 \end{cases}$$

we complete the proof.

v. From Theorem 2.5(i) and the Binet's formulas of Fibonacci and Lucas numbers,

$$\begin{aligned} \hat{\mathsf{Le}}_{n+m}^{(k)} + (-1)^m \hat{\mathsf{Le}}_{n-m}^{(k)} &= \left((k+1)\hat{\mathcal{F}}_{n+m+1} - \hat{\mathcal{K}}\right) + (-1)^m \left((k+1)\hat{\mathcal{F}}_{n-m+1} - \hat{\mathcal{K}}\right) \\ &= (k+1)\hat{\mathcal{F}}_{n+1} L_m + \hat{\mathcal{K}}\left(-1 - (-1)^m\right) \\ &= \left((k+1)\hat{\mathcal{F}}_{n+1} - \hat{\mathcal{K}}\right) L_m + \hat{\mathcal{K}}(L_m - (-1)^m - 1) \\ &= L_m \hat{\mathsf{Le}}_n^{(k)} + \hat{\mathcal{K}}(L_m - (-1)^m - 1) \end{aligned}$$

□

Corollary 2.6. Using the identities *iii* and *iv* presented in Theorem 2.5, the following basic identities are obtained:

i. $\hat{\mathsf{Le}}_{n+1}^{(k)} + \hat{\mathsf{Le}}_{n-1}^{(k)} = (k+1)\hat{\mathcal{L}}_{n+1} - 2\hat{\mathcal{K}}$

ii. $\hat{\mathsf{Le}}_{n+1}^{(k)} - \hat{\mathsf{Le}}_{n-1}^{(k)} = (k+1)\hat{\mathcal{F}}_{n+1}$

iii. $\hat{\mathsf{Le}}_{n+2}^{(k)} + \hat{\mathsf{Le}}_{n-2}^{(k)} = 3(k+1)\hat{\mathcal{F}}_{n+1} - 2\hat{\mathcal{K}}$

iv. $\hat{\mathsf{Le}}_{n+2}^{(k)} - \hat{\mathsf{Le}}_{n-2}^{(k)} = (k+1)\hat{\mathcal{L}}_{n+1}$

Theorem 2.7. Let $\hat{\mathsf{Le}}_n^{(k)}$ be the n-th $k-$generalized Leonardo-like dual quaternion. Then, the following relations hold:

i. $\hat{\mathsf{Le}}_n^{(k)} - \hat{\mathsf{Le}}_{n+1}^{(k)} e_1 - \hat{\mathsf{Le}}_{n+2}^{(k)} e_2 - \hat{\mathsf{Le}}_{n+3}^{(k)} e_3 = \mathsf{Le}_n^{(k)}$

ii. $\hat{\mathsf{Le}}_n^{(k)} + \overline{\hat{\mathsf{Le}}_n^{(k)}} = 2\mathsf{Le}_n^{(k)}$

iii. $\left(\hat{\mathsf{Le}}_n^{(k)}\right)^2 = 2\mathsf{Le}_n^{(k)} \hat{\mathsf{Le}}_n^{(k)} - \left(\mathsf{Le}_n^{(k)}\right)^2$

iv. $\hat{\mathsf{Le}}_n^{(k)}\overline{\hat{\mathsf{Le}}_n^{(k)}} + \hat{\mathsf{Le}}_{n+1}^{(k)}\overline{\hat{\mathsf{Le}}_{n+1}^{(k)}} = (k+1)\,\mathsf{Le}_{2n+2}^{(k)} - 2k\,\mathsf{Le}_{n+2}^{(k)} + k\,(k+1)$

v. $\hat{\mathsf{Le}}_{n+1}^{(k)}\overline{\hat{\mathsf{Le}}_{n+1}^{(k)}} - \hat{\mathsf{Le}}_{n-1}^{(k)}\overline{\hat{\mathsf{Le}}_{n-1}^{(k)}} = (k+1)\,\mathsf{Le}_{2n+1}^{(k)} - 2k\,\mathsf{Le}_n^{(k)} - k\,(k-1)$

Proof.

Let $\hat{\mathsf{Le}}_n^{(k)}$ be the n-th $k-$generalized Leonardo-like dual quaternion.

iv. Using Equation 3 and the Binet's formula of the Fibonacci sequence in Equation 1,

$$\begin{aligned}
\hat{\mathsf{Le}}_n^{(k)}\overline{\hat{\mathsf{Le}}_n^{(k)}}+\hat{\mathsf{Le}}_{n+1}^{(k)}\overline{\hat{\mathsf{Le}}_{n+1}^{(k)}} &= \left(\mathsf{Le}_n^{(k)}\right)^2+\left(\mathsf{Le}_{n+1}^{(k)}\right)^2\\
&= \left((k+1)F_{n+1}-k\right)^2+\left((k+1)F_{n+2}-k\right)^2\\
&= (k+1)^2\left(F_{n+1}^2+F_{n+2}^2\right)-2k(k+1)\left(F_{n+1}+F_{n+2}\right)+2k^2\\
&= (k+1)^2F_{2n+3}-2k(k+1)F_{n+3}+2k^2\\
&= (k+1)\left((k+1)F_{2n+3}-k\right)+k(k+1)-2k\left((k+1)F_{n+3}-k\right)\\
&= (k+1)\,\mathsf{Le}_{2n+2}^{(k)}-2k\,\mathsf{Le}_{n+2}^{(k)}+k(k+1)
\end{aligned}$$

is obtained.

v. Applying Equation 3 and the Binet's formula of the Fibonacci sequence in Equation 1,

$$\begin{aligned}
\hat{\mathsf{Le}}_{n+1}^{(k)}\overline{\hat{\mathsf{Le}}_{n+1}^{(k)}}-\hat{\mathsf{Le}}_{n-1}^{(k)}\overline{\hat{\mathsf{Le}}_{n-1}^{(k)}} &= \left(\mathsf{Le}_{n+1}^{(k)}\right)^2-\left(\mathsf{Le}_{n-1}^{(k)}\right)^2\\
&= \left((k+1)F_{n+2}-k\right)^2-\left((k+1)F_n-k\right)^2\\
&= (k+1)^2\left(F_{n+2}^2-F_n^2\right)+2k(k+1)\left(F_n-F_{n+2}\right)\\
&= (k+1)\left((k+1)F_{2n+2}-k\right)+k(k+1)-2k\left((k+1)F_{n+1}-k\right)-2k^2\\
&= (k+1)\,\mathsf{Le}_{2n+1}^{(k)}-2k\,\mathsf{Le}_n^{(k)}-k(k-1)
\end{aligned}$$

is obtained.

□

Theorem 2.8. Let $\hat{\mathsf{Le}}_n^{(k)}$ be the n-th $k-$generalized Leonardo-like dual quaternion. For $n \geq 2$, the generating function $\mathsf{G}(x)=\sum_{n=0}^{\infty}\hat{\mathsf{Le}}_n^{(k)}$ is as follows:

$$\begin{aligned}
\mathsf{G}(x) &= \frac{\hat{\mathsf{Le}}_0^{(k)}+\left(\hat{\mathsf{Le}}_1^{(k)}-2\hat{\mathsf{Le}}_0^{(k)}\right)x+\left(\hat{\mathsf{Le}}_2^{(k)}-2\hat{\mathsf{Le}}_1^{(k)}\right)x^2}{1-2x+x^3}\\
&= \frac{\left(1-x+kx^2\right)+\left(1+kx-x^2\right)e_1+\left(2+k-x+x^2\right)e_2+\left(3+2k-x+(-2-k)x^2\right)\,e_3}{1-2x+x^3}
\end{aligned}$$

where $1-2x+x^3 \neq 0$.

Proof.
The proof is similar to the proof of the generating function of the Leonardo sequence in [2]. □

Theorem 2.9. For $n \geq 0$, the following summation formulas are satisfied:

i. $\sum_{s=0}^{n}\hat{\mathsf{Le}}_s^{(k)}=\hat{\mathsf{Le}}_{n+2}^{(k)}-\hat{\mathcal{K}}(n+2)+(k-1)-2e_1+(-k-3)e_2+(-5-3k)e_3$

ii. $\sum_{s=0}^{n}\hat{\mathsf{Le}}_{2s}^{(k)}=\hat{\mathsf{Le}}_{2n+1}^{(k)}-\hat{\mathcal{K}}n-(2k)e_1+(-k-1)e_2+(-3k-1)e_3$

iii. $\sum_{s=0}^{n}\hat{\mathsf{Le}}_{2s+1}^{(k)}=\hat{\mathsf{Le}}_{2n+2}^{(k)}-\hat{\mathcal{K}}n-(2k)+(-k-1)e_1+(-3k-1)e_2+(-3k-3)e_3$

Proof.
Let $n \geq 0$.

i. Using Equations 4 and 8,

$$\begin{aligned}\sum_{s=0}^{n}\hat{\mathsf{Le}}_s^{(k)} &= \sum_{s=0}^{n}\mathsf{Le}_s^{(k)} + \left(\sum_{s=0}^{n}\mathsf{Le}_{s+1}^{(k)}\right)e_1 + \left(\sum_{s=0}^{n}\mathsf{Le}_{s+2}^{(k)}\right)e_2 + \left(\sum_{s=0}^{n}\mathsf{Le}_{s+3}^{(k)}\right)e_3 \\ &= \left(\mathsf{Le}_{n+2}^{(k)} - k(n+1) - 1\right) + \left(\sum_{s=0}^{n+1}\mathsf{Le}_s^{(k)} - \mathsf{Le}_0^{(k)}\right)e_1 + \left(\sum_{s=0}^{n+2}\mathsf{Le}_s^{(k)} - \mathsf{Le}_0^{(k)} - \mathsf{Le}_1^{(k)}\right)e_2 \\ &\quad + \left(\sum_{s=0}^{n+3}\mathsf{Le}_s^{(k)} - \mathsf{Le}_0^{(k)} - \mathsf{Le}_1^{(k)} - \mathsf{Le}_2^{(k)}\right)e_3 \\ &= \left(\mathsf{Le}_{n+2}^{(k)} - k(n+1) - 1\right) + \left(\mathsf{Le}_{n+3}^{(k)} - k(n+2) - 1 - 1\right)e_1 \\ &\quad + \left(\mathsf{Le}_{n+4}^{(k)} - k(n+3) - 1 - 1 - 1\right)e_2 + \left(\mathsf{Le}_{n+5}^{(k)} - k(n+4) - 1 - 1 - 1 - (2+k)\right)e_3 \\ &= \hat{\mathsf{Le}}_{n+2}^{(k)} - \hat{\mathcal{K}}(n+2) + (k-1) - 2e_1 + (-k-3)e_2 + (-5-3k)e_3\end{aligned}$$

ii. Using Equations 5, 6, and 8,

$$\begin{aligned}\sum_{s=0}^{n}\hat{\mathsf{Le}}_{2s}^{(k)} &= \sum_{s=0}^{n}\mathsf{Le}_{2s}^{(k)} + \left(\sum_{s=0}^{n}\mathsf{Le}_{2s+1}^{(k)}\right)e_1 + \left(\sum_{s=0}^{n}\mathsf{Le}_{2s+2}^{(k)}\right)e_2 + \left(\sum_{s=0}^{n}\mathsf{Le}_{2s+3}^{(k)}\right)e_3 \\ &= \left(\mathsf{Le}_{2n+1}^{(k)} - kn\right) + \left(\mathsf{Le}_{2n+2}^{(k)} - k(n+2)\right)e_1 + \left(\sum_{s=0}^{n+1}\mathsf{Le}_{2s}^{(k)} - \mathsf{Le}_0^{(k)}\right)e_2 + \left(\sum_{s=0}^{n+1}\mathsf{Le}_{2s+1}^{(k)} - \mathsf{Le}_1^{(k)}\right)e_3 \\ &= \left(\mathsf{Le}_{2n+1}^{(k)} - kn\right) + \left(\mathsf{Le}_{2n+2}^{(k)} - k(n+2)\right)e_1 + \left(\mathsf{Le}_{2n+3}^{(k)} - k(n+1) - 1\right)e_2 \\ &\quad + \left(\mathsf{Le}_{2n+4}^{(k)} - k(n+3) - 1\right)e_3 \\ &= \hat{\mathsf{Le}}_{2n+1}^{(k)} - \hat{\mathcal{K}}n + (-2k)e_1 + (-k-1)e_2 + (-3k-1)e_3\end{aligned}$$

□

Theorem 2.10 (The Honsberger-like identity)**.** Let $\hat{\mathsf{Le}}_n^{(k)}$ be the n-th $k-$generalized Leonardo-like dual quaternion. For positive integers n and m,

$$\begin{aligned}\hat{\mathsf{Le}}_n^{(k)}\hat{\mathsf{Le}}_m^{(k)} + \hat{\mathsf{Le}}_{n+1}^{(k)}\hat{\mathsf{Le}}_{m+1}^{(k)} &= (k+1)\left(2\hat{\mathsf{Le}}_{n+m+2}^{(k)} + k\right) - k\,\hat{\mathsf{Le}}_{n+2}^{(k)} - k\,\hat{\mathsf{Le}}_{m+2}^{(k)} - (k+1)\,\mathsf{Le}_{n+m+2}^{(k)} \\ &\quad - (e_1+e_2+e_3)\left(k\,\mathsf{Le}_{n+2}^{(k)} + k\,\mathsf{Le}_{m+2}^{(k)} - 2k(k+1)\right)\end{aligned}$$

where $\mathsf{Le}_n^{(k)}$ is the n-th $k-$generalized Leonardo number.

Proof.
Let $\hat{\mathsf{Le}}_n^{(k)}$ be the n-th $k-$generalized Leonardo-like dual quaternion and n and m be positive integers. Then,

$$\begin{aligned}\hat{\mathsf{Le}}_n^{(k)}\hat{\mathsf{Le}}_m^{(k)} + \hat{\mathsf{Le}}_{n+1}^{(k)}\hat{\mathsf{Le}}_{m+1}^{(k)} &= \left(\mathsf{Le}_n^{(k)}\,\mathsf{Le}_m^{(k)} + \mathsf{Le}_{n+1}^{(k)}\,\mathsf{Le}_{m+1}^{(k)}\right) \\ &\quad + \left(\left(\mathsf{Le}_n^{(k)}\,\mathsf{Le}_{m+1}^{(k)} + \mathsf{Le}_{n+1}^{(k)}\,\mathsf{Le}_{m+2}^{(k)}\right) + \left(\mathsf{Le}_{n+1}^{(k)}\,\mathsf{Le}_m^{(k)} + \mathsf{Le}_{n+2}^{(k)}\,\mathsf{Le}_{m+1}^{(k)}\right)\right)e_1 \\ &\quad + \left(\left(\mathsf{Le}_n^{(k)}\,\mathsf{Le}_{m+2}^{(k)} + \mathsf{Le}_{n+1}^{(k)}\,\mathsf{Le}_{m+3}^{(k)}\right) + \left(\mathsf{Le}_{n+2}^{(k)}\,\mathsf{Le}_m^{(k)} + \mathsf{Le}_{n+3}^{(k)}\,\mathsf{Le}_{m+1}^{(k)}\right)\right)e_2 \\ &\quad + \left(\left(\mathsf{Le}_n^{(k)}\,\mathsf{Le}_{m+3}^{(k)} + \mathsf{Le}_{n+1}^{(k)}\,\mathsf{Le}_{m+4}^{(k)}\right) + \left(\mathsf{Le}_{n+3}^{(k)}\,\mathsf{Le}_m^{(k)} + \mathsf{Le}_{n+4}^{(k)}\,\mathsf{Le}_{m+1}^{(k)}\right)\right)e_3\end{aligned}$$

We conclude from Equation 3 and $F_nF_m + F_{n+1}F_{m+1} = F_{n+m+1}$ [1] that

$$\hat{\mathsf{Le}}_n^{(k)}\hat{\mathsf{Le}}_m^{(k)} + \hat{\mathsf{Le}}_{n+1}^{(k)}\hat{\mathsf{Le}}_{m+1}^{(k)} = (k+1)^2\left(2\hat{\mathcal{F}}_{n+m+3} - F_{n+m+3}\right) - k(k+1)(F_{n+3}+F_{m+3})(e_1+e_2+e_3)$$
$$-k(k+1)\left(\hat{\mathcal{F}}_{m+3}+\hat{\mathcal{F}}_{n+3}\right) + 4k^2(1+e_1+e_2+e_3) - 2k^2$$

Then, applying Theorem 2.5(i),

$$\hat{\mathsf{Le}}_n^{(k)}\hat{\mathsf{Le}}_m^{(k)} + \hat{\mathsf{Le}}_{n+1}^{(k)}\hat{\mathsf{Le}}_{m+1}^{(k)} = (k+1)\left(2\hat{\mathsf{Le}}_{n+m+2}^{(k)} + k\right) - k\,\hat{\mathsf{Le}}_{n+2}^{(k)} - k\,\hat{\mathsf{Le}}_{m+2}^{(k)} - (k+1)\,\mathsf{Le}_{n+m+2}^{(k)}$$
$$-(e_1+e_2+e_3)\left(k\,\mathsf{Le}_{n+2}^{(k)} + k\,\mathsf{Le}_{m+2}^{(k)} - 2k(k+1)\right)$$

is obtained. □

Across this study, let $\alpha^*\beta^* = 1 + e_1 + 3e_2 + 4e_3$.

Theorem 2.11 (The Catalan-like identity)**.** Let $\hat{\mathsf{Le}}_n^{(k)}$ be the n-th $k-$generalized Leonardo-like dual quaternion. For positive integers n and r with $n \geq r$,

$$\left(\hat{\mathsf{Le}}_n^{(k)}\right)^2 - \hat{\mathsf{Le}}_{n+r}^{(k)}\hat{\mathsf{Le}}_{n-r}^{(k)} = (k+1)^2\alpha^*\beta^*(-1)^{n-r+1}(F_r)^2 - \hat{\mathcal{K}}\left(2\hat{\mathsf{Le}}_n^{(k)} - \hat{\mathsf{Le}}_{n+r}^{(k)} - \hat{\mathsf{Le}}_{n-r}^{(k)}\right)$$

where F_n is the n-th Fibonacci number.

Proof.
By using the Binet-like formula in Equation 9,

$$\left(\hat{\mathsf{Le}}_n^{(k)}\right)^2 - \hat{\mathsf{Le}}_{n+r}^{(k)}\hat{\mathsf{Le}}_{n-r}^{(k)} = \left((k+1)\left(\frac{\alpha^*\alpha^{n+1} - \beta^*\beta^{n+1}}{\alpha-\beta}\right) - \hat{\mathcal{K}}\right)^2$$
$$-\left((k+1)\left(\frac{\alpha^*\alpha^{n+r+1} - \beta^*\beta^{n+r+1}}{\alpha-\beta}\right) - \hat{\mathcal{K}}\right)\left((k+1)\left(\frac{\alpha^*\alpha^{n-r+1} - \beta^*\beta^{n-r+1}}{\alpha-\beta}\right) - \hat{\mathcal{K}}\right)$$
$$= \frac{(k+1)^2}{5}\alpha^*\beta^*\left(\alpha^{n-r+1}\beta^{n-r+1}\right)\left(\alpha^{2r}+\beta^{2r}-2\alpha^r\beta^r\right)$$
$$-\hat{\mathcal{K}}\left(2\left(\hat{\mathsf{Le}}_n^{(k)} + \hat{\mathcal{K}}\right) - \left(\hat{\mathsf{Le}}_{n+r}^{(k)} + \hat{\mathcal{K}}\right) - \left(\hat{\mathsf{Le}}_{n-r}^{(k)} + \hat{\mathcal{K}}\right)\right)$$
$$= (k+1)^2\alpha^*\beta^*(-1)^{n-r+1}(F_r)^2 - \hat{\mathcal{K}}\left(2\hat{\mathsf{Le}}_n^{(k)} - \hat{\mathsf{Le}}_{n+r}^{(k)} - \hat{\mathsf{Le}}_{n-r}^{(k)}\right)$$

□

Theorem 2.12 (The Cassini-like identity)**.** Let $\hat{\mathsf{Le}}_n^{(k)}$ be the n-th $k-$generalized Leonardo-like dual quaternion. For positive integer n with $n \geq 3$,

$$\hat{\mathsf{Le}}_{n-1}^{(k)}\hat{\mathsf{Le}}_{n+1}^{(k)} - \left(\hat{\mathsf{Le}}_n^{(k)}\right)^2 = (k+1)^2(-1)^{n+1}\alpha^*\beta^* - \hat{\mathcal{K}}\,\hat{\mathsf{Le}}_{n-3}^{(k)} - \hat{\mathcal{K}}^2$$

Proof.
From the Cassini identity and the recurrence relation of the Fibonacci dual quaternion sequence (see [20]),

$$\hat{\mathsf{Le}}_{n-1}^{(k)}\hat{\mathsf{Le}}_{n+1}^{(k)} - \left(\hat{\mathsf{Le}}_n^{(k)}\right)^2 = \left((k+1)\hat{\mathcal{F}}_n - \hat{\mathcal{K}}\right)\left((k+1)\hat{\mathcal{F}}_{n+2} - \hat{\mathcal{K}}\right) - \left((k+1)\hat{\mathcal{F}}_{n+1} - \hat{\mathcal{K}}\right)^2$$
$$= ((k+1))^2\left(\hat{\mathcal{F}}_n\,\hat{\mathcal{F}}_{n+2} - \left(\hat{\mathcal{F}}_{n+1}\right)^2\right) - \hat{\mathcal{K}}(k+1)\left(\hat{\mathcal{F}}_n + \hat{\mathcal{F}}_{n+2} - 2\hat{\mathcal{F}}_{n+1}\right)$$
$$= (k+1)^2(-1)^{n+1}(1+e_1+3e_2+4e_3) - \hat{\mathcal{K}}(k+1)\left(\left(\hat{\mathcal{F}}_{n+2} - \hat{\mathcal{F}}_{n+1}\right) - \left(\hat{\mathcal{F}}_{n+1} - \hat{\mathcal{F}}_n\right)\right)$$
$$= (k+1)^2(-1)^{n+1}(1+e_1+3e_2+4e_3) - \hat{\mathcal{K}}\left((k+1)\hat{\mathcal{F}}_{n-2} - \hat{\mathcal{K}}\right) - \hat{\mathcal{K}}^2$$
$$= (k+1)^2(-1)^{n+1}(1+e_1+3e_2+4e_3) - \hat{\mathcal{K}}\,\hat{\mathsf{Le}}_{n-3}^{(k)} - \hat{\mathcal{K}}^2$$

□

Theorem 2.13 (The d'Ocagne-like identity)**.** Let $\hat{\mathsf{Le}}_n^{(k)}$ be the n-th $k-$generalized Leonardo-like dual quaternion. For positive integers m and n,

$$\hat{\mathsf{Le}}_m^{(k)}\hat{\mathsf{Le}}_{n+1}^{(k)} - \hat{\mathsf{Le}}_{m+1}^{(k)}\hat{\mathsf{Le}}_n^{(k)} = (k+1)^2\,\alpha^*\beta^*\,(-1)^{n+1}\,F_{m-n} + \hat{\mathcal{K}}\left(\hat{\mathsf{Le}}_{m-1}^{(k)} - \hat{\mathsf{Le}}_{n-1}^{(k)}\right)$$

where F_n is the n-th Fibonacci number.

Proof.
Applying the Binet-like formula in Equation 9,

$$\begin{aligned}
\hat{\mathsf{Le}}_m^{(k)}\hat{\mathsf{Le}}_{n+1}^{(k)} - \hat{\mathsf{Le}}_{m+1}^{(k)}\hat{\mathsf{Le}}_n^{(k)} &= \left((k+1)\left(\frac{\alpha^*\alpha^{m+1}-\beta^*\beta^{m+1}}{\alpha-\beta}\right)-\hat{\mathcal{K}}\right)\left((k+1)\left(\frac{\alpha^*\alpha^{n+2}-\beta^*\beta^{n+2}}{\alpha-\beta}\right)-\hat{\mathcal{K}}\right)\\
&\quad -\left((k+1)\left(\frac{\alpha^*\alpha^{m+2}-\beta^*\beta^{m+2}}{\alpha-\beta}\right)-\hat{\mathcal{K}}\right)\left((k+1)\left(\frac{\alpha^*\alpha^{n+1}-\beta^*\beta^{n+1}}{\alpha-\beta}\right)-\hat{\mathcal{K}}\right)\\
&= \frac{(k+1)^2}{\alpha-\beta}\,\alpha^*\beta^*\left(\alpha^{m+1}\beta^{n+1}-\alpha^{n+1}\beta^{m+1}\right)\\
&\quad -\hat{\mathcal{K}}\left(\left(\hat{\mathsf{Le}}_m^{(k)}+\hat{\mathcal{K}}\right)+\left(\hat{\mathsf{Le}}_{n+1}^{(k)}+\hat{\mathcal{K}}\right)-\left(\hat{\mathsf{Le}}_{m+1}^{(k)}+\hat{\mathcal{K}}\right)-\left(\hat{\mathsf{Le}}_n^{(k)}+\hat{\mathcal{K}}\right)\right)\\
&= (k+1)^2\,\alpha^*\beta^*\alpha^{n+1}\beta^{n+1}\frac{(\alpha^{m-n}-\beta^{m-n})}{\alpha-\beta}+\hat{\mathcal{K}}\left(\left(\hat{\mathsf{Le}}_{m+1}^{(k)}-\hat{\mathsf{Le}}_m^{(k)}\right)-\left(\hat{\mathsf{Le}}_{n+1}^{(k)}-\hat{\mathsf{Le}}_n^{(k)}\right)\right)\\
&= (k+1)^2\,\alpha^*\beta^*\,(-1)^{n+1}\,F_{m-n}+\hat{\mathcal{K}}\left(\hat{\mathsf{Le}}_{m-1}^{(k)}-\hat{\mathsf{Le}}_{n-1}^{(k)}\right)
\end{aligned}$$

□

Theorem 2.14 (The Tagiuri-like identity)**.** Let $\hat{\mathsf{Le}}_n^{(k)}$ be the n-th $k-$generalized Leonardo-like dual quaternion. For positive integers $n, n+r$ and $n+s$,

$$\hat{\mathsf{Le}}_{n+r}^{(k)}\hat{\mathsf{Le}}_{n+s}^{(k)} - \hat{\mathsf{Le}}_n^{(k)}\hat{\mathsf{Le}}_{n+r+s}^{(k)} = \frac{(k+1)^2}{5}\alpha^*\beta^*(-1)^{n+1}(L_{r+s}-(-1)^sL_{r-s}) + \hat{\mathcal{K}}(\hat{\mathsf{Le}}_n^{(k)} + \hat{\mathsf{Le}}_{n+r+s}^{(k)} - \hat{\mathsf{Le}}_{n+r}^{(k)} - \hat{\mathsf{Le}}_{n+s}^{(k)})$$

where L_n is the n-th Lucas number.

Proof.
The proof is straightforward from applying the Binet-like formula in Equation 9. □

Note that the d'Ocagne-like, Catalan-like, and Cassini-like identities are the special cases of the Tagiuri-like identity.

3. Conclusion

Taking $k = 1$ gives the analogous relations for the Leonardo sequence with the dual-quaternions coefficients. Hence, we can say that our main results presented here generalize the paper [13]. These results can trigger further research on the subjects of the Leonardo sequence and the dual quaternions. Additionally, this study opens the door for future research on sequences; for instance, one may define non-commutative quaternions (real, split, semi-split, etc.) with the $k-$generalized Leonardo sequence.

Author Contributions

All the authors equally contributed to this work. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

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