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A NEW APPROACH TO CURVE COUPLES WITH BISHOP FRAME

Faik BABADAĞ[1] and Ali ATASOY[2]

[1]Department of Mathematics, Kırıkkale University, Kırıkkale, TÜRKİYE
[2]Keskin Vocational School, Kırıkkale University, Kırıkkale, TÜRKİYE

Abstract. This paper presents a detailed study of a new generation of the Bishop frame with components including three orthogonal unit vectors, which are tangent vector, normal vector and binormal vector. It is a frame field described on a curve in Euclidean space, which is an alternative to the Frenet frame. It is useful for curves for which the second derivative is not available. Moreover, the conditions which the Bishop frame of one curve coincides with the Bishop frame of another curve are defined. It would be valuable to replicate similar approaches in the Bishop frame of one curve coincides with the Bishop frame of another curve.

1. Introduction

In 1975, Bishop [1] defined a frame which is called the Bishop frame as an alternative to Frenet frame. This frame is useful for the curves that the second derivative is not available. The Bishop frame consists of three orthogonal unit vectors. These vectors are tangent vector, normal vector and binormal vector. Unlike the Frenet frame, the Bishop frame does not require the second derivative of the curve to be defined. The curvature functions κ and τ give important information about a curve in the curve theory. For instance, for any curve, if κ and τ are zero, this curve is geodesic. So, with the help of their curvatures, we get information about the shape and properties of a curve. There are different methods for characterization of a curve [2–4]. The position vector of a curve is very important in these methods. The orthogonal coordinate system $\{T, N, B\}$ spanned by tangent (T), basic normal (N) and binormal (B) vectors at each point of a unit-speed curve. The coordinate system $\{T, N, B\}$ is called the Frenet frame. This frame is spanned by osculating plane, rectifying plane and normal plane. The osculating plane is spanned by

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[1] faik.babadag@kku.edu.tr-Corresponding author; 0000-0001-9098-838X
[2] aliatasoy@kku.edu.tr; 0000-0002-1894-7695.

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$\{T, N\}$, the rectifying plane is spanned by $\{T, B\}$ and the normal plane is spanned by $\{N, B\}$, many researchers have done investigation that is related to classical differential geometry topics in the Euclidean space, Minkowski space, dual space, and a new version of Bishop frame and its related applications were given [5–12].

2. Preliminaries

In this section, a brief summary of the concept of Bishop frame is given to provide a background to understand the main idea and the results of this study. The Bishop frame consists of three orthonormal vectors that evolve along a curve, off eringanatural way to describe the curve's orientation in space. Unlike the Frenet-Serret frame, which relies on curvature and torsion defi ned for curves with non-zero curvature and non-vanishing speed, the Bishop frame's adaptability makes it particularly suited for analyzing curves with varying geometric behaviors. Our study develops this adaptability to investigate the cases associated with the tangential, normal, or binormal planes of two distinct curves.

Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ is an arbitrary curve [13–15]. If α is unit speed curve, then $\langle \alpha'(s), \alpha'(s) \rangle = 1$, where s is arc length parameter and

$$\langle \gamma, v \rangle = \gamma_1 v_1 + \gamma_2 v_2 + \gamma_3 v_3$$

is the standard scalar product of $\mathbb{E}^3$. Furthermore, $\|\gamma\| = \sqrt{\langle \gamma, \gamma \rangle}$ is the norm of γ. Bishop frame along the unit speed curve α is denoted by the tangent T, the principal normal N_1 and the binormal N_2 vector fields. Bishop derivative formulas are written as follows:

$$T' = k_1 N_1 + k_2 N_2,\ N_1' = -k_1 T,\ N_2' = -k_2 T$$

where we shall call the set T, N_1, N_2 as Bishop trihedra and k_1 and k_2 (k_2 is not equal to zero) as Bishop curvatures. For all $t \in I$, if a curve γ is considered spherical, then the curve $\gamma(t)$ lies on a sphere of radius $\mathbb{R}$, centered at the origin in $\mathbb{E}^3$. This can be mathematically expressed as $\|\gamma\| = \mathbb{R}$. The normal curves in $\mathbb{E}^3$ are spherical curves [16]. The Bishop frame, an alternative to the traditional Frenet frame, offers a more flexible representation of curves in $\mathbb{E}^3$ by eliminating the torsion component, thus providing a simpler and often more intuitive understanding of curve geometry. This research leverages the Bishop frame to examine the conditions under which two distinct curves in $\mathbb{E}^3$ can share a common plane in their respective Bishop frames. We denote the tangent vector of a curve at a point by T, and the principal normal vector by N_1. The notation $\bar{T} \in Sp\{T, N_1\}$ signifies that the modified tangent vector $\bar{T}$ of one curve lies within the plane spanned by the tangent and principal normal vectors of other curve. This study not only contributes a new perspective to the analysis of curves within Euclidean space but also underscores the utility and versatility of the Bishop frame in uncovering complex geometric relationships. Through a careful and thorough classification of the possible cases where curve pairs share a Bishop frame plane, combined with the derivation of several interesting results, we enhance the existing knowledge base in differential geometry and open the door to future investigations in this rich and captivating field.

3. On Characterizations of the Curve Couples and Bishop Frame

Let α and $\bar{\alpha}$ are any curves in 3-dimensional Euclidean space where they are described on the same open interval $I \subset \mathbb{R}$. The frames $\{T, N_1, N_2\}$ and $\{\bar{T}, \bar{N}_1, \bar{N}_2\}$ are Bishop frames of α and $\bar{\alpha}$, respectively. Let's denote the are length parameters, first Bishop curvatures and second Bishop curvatures for α and $\bar{\alpha}$ with s, k_1, k_2 and $\bar{s}, \bar{k}_1, \bar{k}_2$ respectively. $\{T, N_1\}, \{N_1, N_2\}, \{T, N_2\}$ planes are the osculating plane, the normal plane and the rectifying plane according to Bishop frame, respectively. These planes are donated, respectively, by OB, AB and CA. Let $\bar{\alpha}$ be a unit speed curve with first and second curvatures $\bar{k}_1, \bar{k}_2$ and Bishop vectors $\bar{T}, \bar{N}_1, \bar{N}_2$. Similarly, $\{\bar{T}, \bar{N}_1\}, \{\bar{N}_1, \bar{N}_2\}, \{\bar{T}, \bar{N}_2\}$ planes are the osculating plane, the normal plane and the rectifying plane according to Bishop frame. These planes are denoted with $\overline{OB}, \overline{AB}$ and $\overline{CA}$, respectively. If we take $r = \frac{d\bar{s}}{ds}$, Bishop formulas are

$$T' = k_1 N_1 + k_2 N_2,\ N_1' = -k_1 T,\ N_2' = -k_2 T \tag{1}$$

$$T' = \bar{k}_1 \bar{N}_1 + \bar{k}_2 \bar{N}_2,\ \bar{N}_1' = -\bar{k}_1 \bar{T},\ \bar{N}_2' = -\bar{k}_2 T \tag{2}$$

where $(')$ denotes $\frac{d}{ds}$. Classify whether any Bishop frame plane of a curve is the any Bishop frame plane of another curve or not and we will give the following cases and their related theorems, here a_1 and a_2 are the non-zero functions of the parameter s and the position vectors of $\bar{\alpha}$ and α are $\bar{\gamma}$ and γ.

Table 1. The cases of curve couples

Cases	Bishop plane for α	Bishop plane for $\bar{\alpha}$	Conditions
1	$\mathrm{sp}\{\mathrm{T}, \mathrm{N}_1\} = \mathrm{OB}$	$\mathrm{sp}\{\bar{T}, \bar{N}_1\} = \overline{OB}$	$\mathrm{OB} = \overline{OB}$
2	$\mathrm{sp}\{\mathrm{T}, \mathrm{N}_1\} = \mathrm{OB}$	$\mathrm{sp}\{\bar{N}_1, \bar{N}_2\} = \overline{AB}$	$\mathrm{OB} = \overline{AB}$
3	$\mathrm{sp}\{\mathrm{T}, \mathrm{N}_1\} = \mathrm{OB}$	$\mathrm{sp}\{\bar{T}, \bar{N}_2\} = \overline{CA}$	$\mathrm{OB} = \overline{CA}$
4	$\mathrm{sp}\{\mathrm{N}_1, \mathrm{N}_2\} = \mathrm{AB}$	$\mathrm{sp}\{\bar{T}, \bar{N}_1\} = \overline{OB}$	$\mathrm{AB} = \overline{OB}$
5	$\mathrm{sp}\{\mathrm{N}_1, \mathrm{N}_2\} = \mathrm{AB}$	$\mathrm{sp}\{\bar{N}_1, \bar{N}_2\} = \overline{AB}$	$\mathrm{AB} = \overline{AB}$
6	$\mathrm{sp}\{\mathrm{N}_1, \mathrm{N}_2\} = \mathrm{AB}$	$\mathrm{sp}\{\bar{T}, \bar{N}_2\} = \overline{CA}$	$\mathrm{AB} = \overline{CA}$
7	$\mathrm{sp}\{\mathrm{T}, \mathrm{N}_2\} = \mathrm{CA}$	$\mathrm{sp}\{\bar{T}, \bar{N}_1\} = \overline{OB}$	$\mathrm{CA} = \overline{OB}$
8	$\mathrm{sp}\{\mathrm{T}, \mathrm{N}_2\} = \mathrm{CA}$	$\mathrm{sp}\{\bar{N}_1, \bar{N}_2\} = \overline{AB}$	$\mathrm{CA} = \overline{AB}$
9	$\mathrm{sp}\{\mathrm{T}, \mathrm{N}_2\} = \mathrm{CA}$	$\mathrm{sp}\{\bar{T}, \bar{N}_2\} = \overline{CA}$	$\mathrm{CA} = \overline{CA}$

Now we investigate these possible cases step by step (see TABLE 1):

Case 1. *Assume that $OB = \overline{OB}$. The osculating plane according to Bishop frame plane of a curve α is the osculating plane according to Bishop frame plane of curve $\bar{\alpha}$.*

Theorem 1. *There isn't a curve pair* $(\alpha, \bar{\alpha})$ *in* $\mathbb{E}^3$ *such that the osculating plane according to Bishop frame of* α *is same osculating plane according to Bishop frame of* $\bar{\alpha}$.

Proof. Assume that the osculating plane according to Bishop frame of a curve α is the osculating plane according to Bishop frame of a other curve $\bar{\alpha}$. Then, the relation can be given that

$$\bar{\gamma} = \gamma + a_1 T + a_2 N_1, \ a_1 \neq 0, \ a_2 \neq 0 \tag{3}$$

where $\bar{\gamma}$ and γ are position vectors of the curves $\bar{\alpha}$ and α respectively, a_1 and a_2 are the non-zero functions of the parameter s. If we take the derivative of (3) with respect to s and by using Bishop formulas in (1) and (2), we have

$$\bar{T} = (1 + a_1' - a_2 k_1)\frac{1}{r}T + (a_1 k_1 + a_2')\frac{1}{r}N_1 + a_1 k_2 \frac{1}{r}N_2 \tag{4}$$

If we rewrite in (4) for constant $\lambda = (1 + a_1' - a_2 k_1)\frac{1}{r}$ and $\mu = (a_1 k_1 + a_2')\frac{1}{r}$ then we obtain

$$\bar{T} = \lambda T + \mu N_1 = (1 + a_1' - a_2 k_1)\frac{1}{r}T + (a_1 k_1 + a_2')\frac{1}{r}N_1 + a_1 k_2 \frac{1}{r}N_2 \tag{5}$$

where $\bar{T} \in Sp\{T, N_1\}$. If we multiply (5) by N_2, we find

$$a_1 k_2 \frac{1}{r} = 0.$$

Then, a_1 or k_2 must be zero. This situation contradicts our assumption. □

Case 2. *Suppose that* $OB = \overline{AB}$. *In the osculating plane according to Bishop frame plane of a curve is the normal plane according to Bishop frame plane of other curve.*

Theorem 2. *Let* α *be any unit speed curve in* $\mathbb{E}^3$ *with non-zero first Bishop curvature* k_1*, second Bishop curvature* k_2 *and Bishop vectors* T, N_1, N_2*. If the osculating plane according to Bishop frame of the curve* α *is the normal plane according to Bishop frame of the space curve* $\bar{\alpha}$*, then* $\bar{\alpha}$ *is written as*

$$\bar{\alpha} = \alpha + \frac{d\bar{s}}{ds}\frac{1}{k_2}T + \frac{1}{k_1} + \frac{1}{k_1 k_2^2}\left(\frac{d^2\bar{s}}{ds^2}k_2 - \frac{d\bar{s}}{ds}k_2'\right)N_1.$$

Proof. Assume that the osculating plane according to Bishop frame of a curve α is the normal plane according to Bishop frame of a another curve $\bar{\alpha}$. Then, the relation can be given that

$$\bar{\gamma} = \gamma + a_1 T + a_2 N_1, \ a_1 \neq 0, \ a_2 \neq 0. \tag{6}$$

Taking the derivative of (6) with respect to s and apply (1) and (2). We have

$$\bar{T} = (1 + a_1' - a_2 k_1)\frac{1}{r}T + (a_1 k_1 + a_2')\frac{1}{r}N_1 + a_1 k_2 \frac{1}{r}N_2. \tag{7}$$

Since $N_2^{\perp} = Sp\{T, N_1\} = Sp\{\bar{N}_1, \bar{N}_2\} = \bar{T}^{\perp}$, N_2 and $\bar{T}$ are parallel. If we multiply 7 by N_2 and T, respectively, then we obtain

$$\begin{aligned} a_1 &= \tfrac{r}{k_2} \\ a_2 &= \tfrac{1}{k_1} + \tfrac{1}{k_1 k_2^2}\left(\tfrac{d^2\bar{s}}{ds^2}k_2 - r{k_2}'\right). \end{aligned}$$

□

Case 3. *Assume that $OB = \overline{CA}$. The osculating plane according to Bishop frame plane of a curve is the rectifying plane according to Bishop frame plane of other curve.*

Theorem 3. *There isn't a curve pair $(\alpha, \bar{\alpha})$ in $\mathbb{E}^3$ such that the osculating plane according to Bishop frame of α is the rectifying plane according to Bishop frame of $\bar{\alpha}$.*

Proof. Let's assume that, the osculating plane according to Bishop frame of a curve α be the rectifying plane according to Bishop frame of the curve $\bar{\alpha}$. Then, the relation can be given that

$$\bar{\gamma} = \gamma + a_1 T + a_2 N_1, \ a_1 \neq 0, \ a_2 \neq 0.$$

Taking the derivative of the last equality with respect to s and apply (1) and (2) then, we get

$$\bar{T} = (1 + a_1' - a_2 k_1)\frac{1}{r}T + (a_1 k_1 + a_2')\frac{1}{r}N_1 + a_1 k_2 \frac{1}{r}N_2.$$

For some constant λ and μ, since $\bar{T} \in Sp\{T, N_1\}$, we can write

$$\bar{T} = \lambda T + \mu N_1 = (1 + a_1' - a_2 k_1)\frac{1}{r}T + (a_1 k_1 + a_2')\frac{1}{r}N_1 + a_1 k_2 \frac{1}{r}N_2 \tag{8}$$

if we multiply (8) by N_2, we obtain

$$a_1 k_2 \frac{1}{r} = 0$$

where a_1 or k_2 must be zero. This situation contradicts our assumption. □

Case 4. *Assume that $AB = \overline{OB}$. We examine the case that the normal plane according to Bishop frame plane of a curve is the osculating plane according to Bishop frame plane of other curve.*

Theorem 4. *Let α be any unit speed curve with non-zero first Bishop curvature k_1, second Bishop curvature k_2 and Bishop vectors T, N_1, N_2. If the normal plane according to Bishop frame of the curve α is the osculating plane according to Bishop frame of a other space curve $\bar{\alpha}$, then α is a spherical curve.*

Proof. Let's assume that, the normal plane according to Bishop frame of a curve α is the osculating plane according to Bishop frame of a other curve $\bar{\alpha}$. Then, the relation can be given that

$$\bar{\gamma} = \gamma + a_1 N_1 + a_2 N_2,\ a_1 \neq 0,\ a_2 \neq 0. \tag{9}$$

Let's take the derivative of (9) with respect to s and applying (1) and (2). Then, we get

$$\bar{T} = (1 - a_1 k_1 - a_2 k_2)\frac{1}{r}T + a_1' \frac{1}{r} N_1 + a_2' \frac{1}{r} N_2.$$

For some constant λ and μ, if we take $\bar{T} = \lambda_1 N_1 + \mu_1 N_2$ and $\bar{T} \in \mathrm{Sp}\{N_1, N_2\}$, we can write

$$\lambda_1 N_1 + \mu_1 N_2 = (1 - a_1 k_1 - a_2 k_2)\frac{1}{r}T + a_1' \frac{1}{r} N_1 + a_2' \frac{1}{r} N_2. \tag{10}$$

Now, if we multiply (10) by T, we have

$$a_1 k_1 + a_2 k_2 = 1.$$

Here the curve γ is a spherical curve. □

Case 5. *Assume that $AB = \overline{AB}$. The normal plane according to Bishop frame plane of a curve is the normal plane according to Bishop frame plane of other curve.*

Theorem 5. *Let's take any α and $\bar{\alpha}$ curves with non-zero first Bishop curvature k_1, non-zero second Bishop curvature k_2 in $\mathbb{E}^3$ and Bishop vectors T, N_1, N_2. If the normal plane according to Bishop frame of the curve α is the normal plane according to Bishop frame of a another curve $\bar{\alpha}$, then α is a spherical curve where $r \neq 1$.*

Proof. Let's assume that, the normal plane according to Bishop frame of a curve α be the normal plane according to Bishop frame of the curve $\bar{\alpha}$. Then, the relation can be given that

$$\bar{\gamma} = \gamma + a_1 N_1 + a_2 N_2,\ a \neq 0,\ b \neq 0 \tag{11}$$

Taking the derivative of (11) with respect to s and applying the Bishop formulas given in (1) and (2), we obtain

$$\bar{T} = (1 - a_1 k_1 - a_2 k_2)\frac{1}{r}T + a_1' \frac{1}{r} N_1 + b' \frac{1}{r} N_2. \tag{12}$$

T and $\bar{T}$ are parallel and $T^{\perp} = Sp\{N_1, N_2\} = Sp\{\bar{N}_1, \bar{N}_2\} = \bar{T}^{\perp}$. So, we write

$$< \bar{T}, T >= 1$$

and multiplying (12) by T, we have

$$a_1 k_1 + a_2 k_2 = 1 - r,$$

then, we can write

$$\frac{a_1}{c} k_1 + \frac{a_2}{c} k_2 = 1$$

where $1-r=c$ and $r\neq 1$. Then, the curve α is a spherical curve. □

Case 6. *Suppose that $AB=\overline{CA}$. Now, we give the normal plane according to Bishop frame plane of a curve is the rectifying plane according to Bishop frame plane of other curve.*

Theorem 6. *Let α be unit speed curve in $\mathbb{E}^3$ with non-zero first and second Bishop curvatures k_1, k_2, and Bishop vectors T, N_1, N_2. If the normal plane according to Bishop frame of α is the rectifying plane according to Bishop frame of the curve $\bar{\alpha}$, then α is a spherical curve.*

Proof. Let's assume that, the normal plane according to Bishop frame of a curve α be the rectifying plane according to Bishop frame of another curve $\bar{\alpha}$. Then, the relation can be given that

$$\bar{\gamma}=\gamma+a_1N_1+a_2N_2,\ a_1\neq 0,\ a_2\neq 0.$$

Let's take the derivative of the last equality with respect to s and apply in (1) and (2). Then, we get

$$\bar{T}=(1-a_1k_1-a_2k_2)\frac{1}{r}T+a_1'\frac{1}{r}N_1+a_2'\frac{1}{r}N_2. \tag{13}$$

For some constants λ and μ, we write $\bar{T}=\lambda N_1+\mu N_2$ and $\bar{T}\in \mathrm{Sp}\{N_1,N_2\}$. From (13), we get

$$\lambda N_1+\mu N_2=(1-a_1k_1-a_2k_2)\frac{1}{r}T+a_1'\frac{1}{r}N_1+a_2'\frac{1}{r}N_2. \tag{14}$$

By multiplying (14) by T, we have

$$a_1k_1+a_2k_2=1.$$

□

Case 7. *Let us consider $CA=\overline{OB}$. The rectifying plane according to Bishop frame of a curve α is the osculating plane according to Bishop frame of other curve $\bar{\alpha}$. Then, the relation can be given that*

$$\bar{\gamma}=\gamma+a_1T+a_2N_1,\ a_1\neq 0,\ a_2\neq 0. \tag{15}$$

By taken the derivative of (15) with respect to s and apply in (1) and (2). We will have

$$\bar{T}=\left[(1+a_1'-k_1a_2)T+(a_1k_1+a_2')N_1+a_1k_2N_2\right]\frac{1}{r} \tag{16}$$

where $\bar{T}=\lambda T+\mu N_2$ and for some constant λ and μ, since $\bar{T}\in \mathrm{Sp}\{T,N_2\}$. From (16), we obtain

$$\lambda T+\mu N_2=\left[(1+a_1'-k_1a_2)T+(a_1k_1+a_2')N_1+a_1k_2N_2\right]\frac{1}{r}. \tag{17}$$

Then multiplying (17) by N_1, we get

$$a_1k_1+a_2'=0.$$

Case 8. *Assume that $CA = \overline{AB}$. The rectifying plane according to Bishop frame plane of a curve is the normal plane according to Bishop frame plane of other curve.*

Theorem 7. *Let α be unit speed curve in $\mathbb{E}^3$ with non-zero first and second Bishop curvatures k_1, k_2 and Bishop vectors T, N_1, N_2. If the rectifying plane according to Bishop frame of α is the normal plane according to Bishop frame of other curve $\overline{\alpha}$, then we can write*

$$\overline{\alpha} = \alpha + \frac{1}{k_1}T + [\frac{1}{k_2} + \frac{1}{k_2}(\frac{1}{k_1})']N_2.$$

Proof. Assume that, the rectifying plane according to Bishop frame of a curve α be the normal plane according to Bishop frame of the curve $\overline{\alpha}$. Then, the relation can be given as

$$\overline{\gamma} = \gamma + a_1T + a_2N_2,\ a_1 \neq 0,\ a_2 \neq 0 \tag{18}$$

Taking the derivative of (18) with respect to s and applying the (1) and (2), then, we have

$$\overline{T} = \ [(1 + a_1' - a_2k_2)T + a_1k_1N_1 + (a_1k_2 + a_2')N_2]\ \frac{1}{r}. \tag{19}$$

N_1 and $\overline{T}$ are parallel. Since $N^{\perp} = Sp\{T,\ N_2\} = Sp\{\overline{N}_1, \overline{N}_2\} = \overline{T}^{\perp}$, multiplying (19) by N_1, we get

$$a_1 = \frac{1}{k_1}.$$

If we Multiply (19) by T, we get

$$1 + a_1' - a_2k_2 = 0,$$

$$b = \frac{1}{k_2} + \frac{1}{k_2}(\frac{1}{k_1})'.$$

□

Case 9. *Assume that $CA = \overline{CA}$. The rectifying plane according to Bishop frame plane of a curve is the rectifying plane according to Bishop frame plane of other curve.*

Theorem 8. *There isn't a curve pair $(\alpha, \overline{\alpha})$ in $\mathbb{E}^3$ such that the rectifying plane according to Bishop frame of α is the rectifying plane according to Bishop frame of $\overline{\alpha}$.*

Proof. Assume that the rectifying plane according to Bishop frame of a curve α be the rectifying plane according to Bishop frame of other curve $\overline{\alpha}$. Then, the relation can be given that

$$\overline{\gamma} = \gamma + a_1T + a_2N_2,\ a_1 \neq 0,\ a_2 \neq 0. \tag{20}$$

Taking the derivative of (20) with respect to s and using (1) and (2), then, we write

$$\overline{T} = \ [(1 + a' - bk_2)T + ak_1N_1 + (ak_2 + b')N_2]\ \frac{1}{r}. \tag{21}$$

For some constant λ and μ, we can write $\overline{T} = \lambda T + \mu N_2$. Since $\overline{T} \in Sp\{T,\ N_2\}$, by using (21), we obtain

$$\lambda T + \mu N_2 = [(1 + a_1' - a_2 k_2)T + a_1 k_1 N_1 + (a_1 k_2 + a_2')N_2]\ \frac{1}{r}. \tag{22}$$

If we multiply (22) by N_1, we obtain $a_1 k_1 \frac{1}{r} = 0$ and $a_1 = 0$ or $k_1 = 0$. □

4. Conclusion

Bishop frame is considered as an alternative to Frenet frame in 3-dimensional Euclidean space. This idea is a useful method for curves where the second derivative is not available. Here, the conditions for Bishop frame plane to be the same as Bishop frame plane of another curve for a curve are investigated in 3 dimensional Euclidean space. Thus, nine possible cases are examined. The results found extend previous studies. The Bishop frame provides an alternative framework for studying curves, and the results of this paper demonstrate the usefulness of the Bishop frame in this context.

Overall, this paper provides new insights into the geometry of curves in Euclidean space, and highlights the importance of alternative frames such as the Bishop frame in studying curves.

Author Contribution Statements All authors jointly worked on the results, and they read and approved the final manuscript.

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References

[1] Bishop, L. R., There is more than one way to frame a curve, *The American Mathematical Monthly*, 82(3) (1975), 246-251. https://doi.org/10.2307/2319846
[2] Chen, B. Y., Dillen, F., Rectifying curves as centrodes and extremal curves, *Bull. Inst. Math. Academia Sinica*, 33(2) (2005), 77-90.
[3] Kuhnel, W., Differential Geometry: Curves-Surfaces-Manifolds, Wiesbaden, Germany, 2003.
[4] Millman, R. S., Parker, G. D., Elements of Differential Geometry, Prentice-Hall, New Jersey, 1977.
[5] Chen, B. Y., When does the position vector of a space curve always lie an its rectifying plane?, *The American Mathematical Monthly*, 110 (2003), 147-152. https://doi.org/10.2307/3647775
[6] Azak, A. Z., Masal, M., Bertrand curves and Bishop frame in the 3-Dimensional Euclidean Space, *Sakarya Universty Journal of Science* 21(6) (2017), 1140-1145. https://doi.org/10.16984/saufenbilder.267557
[7] Bükcü, B., Karacan, M. K., The Bishop Darboux rotation axis of the spacelike curve in Minkowski 3-space, *Journal of the Faculty of Science, Ege University*, 30 (2007), 1-5.
[8] Bükcü, B., Karacan, M. K., On the slant helices according to Bishop frame of the timelike curve in Lorentzian space, *Tamkang Journal of Mathematics*, 39(3) (2008), 255-262. https://doi.org/10.5556/j.tkjm.39.2008.18
[9] Bükcü, B., Karacan, M. K., The slant helices according to Bishop frame, *International Journal of Mathematical and Computational Sciences*, 3(11) (2009), 67-70.

[10] Babadağ, F., On similar partner curves in Bishop frames with variable transformations, *International Journal of New Technology and Research (IJNTR)*, 2(4) (2016), 59-64.

[11] Izumiya, S., Takeuchi, N., New special curves and developable surfaces, *Turkish Journal of Mathematics*, 28(2) (2004), 153-164.

[12] Babadağ, F., On dual similar partner curves in dual 3-space, *British Journal of Mathematics and Computer Science*, 17(4) (2016), 1-8. https://doi.org/10.9734/BJMCS/2016/26242

[13] İlarslan, K., Nesovic, E., Some characterizations of rectifying curves in the Euclidean space $\mathbb{E}^4$, *Turkish Journal of Mathematics*, 32(1) (2008), 21-30.

[14] Spivak, M., A Comprehensive Introduction to Differential Geometry, Publish or Perish Inc., 1999.

[15] Carmo, M., Differential Geometry of Curves and Surfaces, New Jersey, NJ, USA Prentice Hall Inc., 1976.

[16] Karakuş, S. Ö., İlarslan, K, Yaylı, Y., A new approach characterization of curve couples in Euclidean 3-space, *Honam Mathematical Journal*, 36(1) (2014), 113-129. https://doi.org/10.5831/HMJ.2014.36.1.113