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Research Article

On k-quasi-$(m, n, \mathbf{C})$-isosymmetric operators

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Abstract

We study the set of k-quasi-$(m, n, \mathbf{C})$-isosymmetric operators. This family extends the set of $(m, , n, \mathbf{C})$-isosymmetric operators. In the present article, we give operator matrix representation of k-quasi-$(m, , n, \mathbf{C})$-isosymmetric operator in order to obtain some structural properties for such operators. We show that if $\mathbf{R}$ is a k-quasi-$(m, n, \mathbf{C})$-isosymmetric, then $\mathbf{R}^q$ is a k-quasi-$(m, n, \mathbf{C})$-isosymmetric operator. We show that the product of a k_1-quasi-$(m_1, n_1, \mathbf{C})$-isosymmetric and a k_2-quasi-$(m_2, n_2, \mathbf{C})$-isosymmetric which are $\mathbf{C}$-double commuting is a $\max\{k_1, k_2\}$-quasi-$(m_1 + m_2 - 1, n_1 + n_2 - 1, \mathbf{C})$-isosymmetry under suitable conditions. In particular, we prove the stability of perturbation of k-quasi-$(m, n, \mathbf{C})$-isosymmetric operator by a nilpotent operator of order p under suitable conditions. Moreover, we give some results about the joint approximate spectrum of a k-quasi-$(m, n, \mathbf{C})$-isosymmetric operator.

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1. Introduction

Let $\mathcal{B}[\mathcal{K}]$ be the set of all bounded linear operators on a separable complex Hilbert space $\mathcal{K}$ with inner product $\langle \cdot \mid \cdot \rangle$ and denote by $\mathbf{I}$ be the identity of $\mathcal{B}[\mathcal{K}]$. For an operator $\mathbf{R} \in \mathcal{B}[\mathcal{K}]$, we denote by $ran(\mathbf{R})$ its range, $\ker(\mathbf{R})$ its kernel, and $\mathbf{R}^*$ its adjoint. Recall from [16] that a conjugation on $\mathcal{K}$ is a map $\mathbf{C} : \mathcal{K} \longrightarrow \mathcal{K}$ which is antilinear, involutive ($\mathbf{C}^2 = \mathbf{I}$. Moreover, $\mathbf{C}$ satisfies the following properties:

$$\begin{cases} \langle \mathbf{C}x \mid \mathbf{C}y \rangle = \langle y \mid x \rangle \quad \text{for } all\ x, y \in \mathcal{K}, \\ \mathbf{CRC} \in \mathcal{B}[\mathcal{K}] \quad \text{for every } \mathbf{R} \in \mathcal{B}[\mathcal{K}], \\ (\mathbf{CRC})^r = \mathbf{CR}^r\mathbf{C} \quad \text{for all } r \in \mathbb{N}, \\ (\mathbf{CRC})^* = \mathbf{CR}^*\mathbf{C}. \end{cases}$$

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See [5,12] for properties of conjugation operators.

In this work, $\mathbb{N} = \{1, 2, \cdots\}$, $\mathbb{N}_0 = \{0, 1, 2, \cdots\}$ and n, $m \in \mathbb{N}$.

During the past years, the m-isometric operators term has known a great interest on the part of researchers in the field of operator theory, by the works that has been published in this aspect. It should be noted that most of these works are dependent on the following definition, which is due to Agler [2]. An operator $\mathbf{R} \in \mathcal{B}[\mathcal{K}]$ is said to be m-isometric if

$$\sum_{0\le k\le m} (-1)^k \binom{m}{k} \mathbf{R}^{*m-k}\mathbf{R}^{m-k} = 0, \tag{1.1}$$

or

$$\sum_{0\le k\le m} (-1)^k \binom{m}{k} \|\mathbf{R}^{m-k}x\|^2 = 0 \qquad \forall\ x \in \mathcal{K}. \tag{1.2}$$

(For more detail, see [2–4,6,7,17–19,27] about the theory of m-isometries).
As extensions of the concepts of m-isometric operators on Hilbert spaces, some authors have introduced and studied in different papers the following classes of operators.

(1) $(m, \mathbf{C})$-isometric operator that is an operator $\mathbf{R} \in \mathcal{B}[\mathcal{K}]$ satisfies

$$\sum_{0\le k\le m} (-1)^k \binom{m}{k} \mathbf{R}^{*m-k}\mathbf{C}\mathbf{R}^{m-k}\mathbf{C} = 0, \tag{1.3}$$

for some $m \in \mathbb{N}$ and some conjugation $\mathbf{C}$ ([9,11,21]).

(2) n-quasi-m-isometric operator that is an operator $\mathbf{R} \in \mathcal{B}[\mathcal{K}]$ satisfies

$$\mathbf{R}^{*n}\Big(\sum_{0\le k\le m} (-1)^{m-k} \binom{m}{k} \mathbf{R}^{*m-k}\mathbf{R}^{m-k}\Big)\mathbf{R}^n = 0, \tag{1.4}$$

for some $n \in \mathbb{N}$ and $m \in \mathbb{N}$ ([8,15,24,26]).

(3) n-quasi-$(m, \mathbf{C})$-isometric operator that is an operator $\mathbf{R} \in \mathcal{B}[\mathcal{K}]$ satisfies

$$\mathcal{Q}_m(\mathbf{R};\mathbf{C}) := \mathbf{R}^{*n}\Big(\sum_{0\le k\le m} (-1)^{m-k} \binom{m}{k} \mathbf{R}^{*m-k}\mathbf{C}\mathbf{R}^{m-k}\mathbf{C}\Big)\mathbf{R}^n = 0, \tag{1.5}$$

for some conjugation $\mathbf{C}$ and some $n \in \mathbb{N}$ and $m \in \mathbb{N}$ ([1,22,25]).

It is well known that the properties of powers, products and perturbations of the members of each of the classes cited above has been discussed ([9,22,24–26]).

The readers are invited to see the reference list and citations guide for more detailed information.

From [10,13], an operator $\mathbf{R} \in \mathcal{B}[\mathcal{K}]$ is said to be n-complex symmetric if there exists a conjugation operator $\mathbf{C}$ such that

$$\sum_{0\le k\le n} (-1)^{n-k} \binom{n}{k} \mathbf{R}^{*k}\mathbf{C}\mathbf{R}^{n-k}\mathbf{C} = 0. \tag{1.6}$$

for some $n \in \mathbb{N}$. However, $\mathbf{R}$ is said k-quasi-n-symmetric ([1]), if

$$\mathbf{R}^{*k}\Big(\sum_{0\le j\le n} (-1)^{n-j} \binom{n}{j} \mathbf{R}^{*j}\mathbf{C}\mathbf{R}^{n-j}\mathbf{C}\Big)\mathbf{R}^k = 0, \tag{1.7}$$

for some conjugation operator $\mathbf{C}$ and for some integers $k, n \in \mathbb{N}$.

Definition 1.1 ([29, 30]). An operator $\mathbf{R} \in \mathcal{B}[\mathcal{K}]$ is called (m, n)-isosymmetric if $\mathbf{R}$ satisfies

$$\begin{aligned}\sum_{0\leq j\leq m}(-1)^j\binom{m}{j}\mathbf{R}^{*(m-j)}\Big(\sum_{0\leq k\leq n}(-1)^k\binom{n}{k}\mathbf{R}^{*(n-k)}\mathbf{R}^k\Big)\mathbf{R}^{m-j}\\ = \sum_{0\leq k\leq n}(-1)^k\binom{n}{k}\mathbf{R}^{*(n-k)}\Big(\sum_{0\leq j\leq m}(-1)^j\binom{m}{j}\mathbf{R}^{*(m-j)}R^{m-j}\Big)R^k\\ = 0.\end{aligned}$$

Remark 1.2. Every m-isometric operator is an (m, n)-isosymmetric and every n-symmetric operator is an (m, n)-isosymmetric operator.

For $\mathbf{R} \in \mathcal{B}[\mathcal{K}]$ and $\mathbf{C} \in \mathcal{C}[\mathcal{K}]$. Following [14], we put

$$\begin{cases}\Lambda_m(\mathbf{C};\mathbf{R}) := \displaystyle\sum_{0\leq k\leq m}(-1)^k\binom{m}{k}\mathbf{R}^{*m-k}\mathbf{C}\mathbf{R}^k\mathbf{C},\\ \Upsilon_m(\mathbf{C};\mathbf{R}) := \displaystyle\sum_{0\leq k\leq m}(-1)^k\binom{m}{k}\mathbf{R}^{*m-k}\mathbf{C}\mathbf{R}^{m-k}\mathbf{C}\end{cases}$$

Note that $\Lambda_1(\mathbf{C};\mathbf{R}) = \mathbf{R}^* - \mathbf{CRC}$ and $\Upsilon_1(\mathbf{C};\mathbf{R}) = \mathbf{R}^*\mathbf{CRC} - \mathbf{I}$.

Definition 1.3 ([14]). An operator $\mathbf{R} \in \mathcal{B}[\mathcal{K}]$ is said to be an $(m, n, \mathbf{C})$-isosymmetric operator for some conjugation $\mathbf{C} \in \mathcal{C}[\mathcal{K}]$ if $\mathcal{Q}_{m,n}(\mathbf{C};\mathbf{R}) = 0$ where

$$\begin{aligned}\mathcal{Q}_{m,n}(\mathbf{C};\mathbf{R}) : &= \sum_{0\leq k\leq m}(-1)^k\binom{m}{k}\mathbf{R}^{*m-k}\Lambda_n(\mathbf{C};\mathbf{R})\mathbf{C}\mathbf{R}^{m-k}\mathbf{C}\\ &= \sum_{0\leq k\leq n}(-1)^k\binom{n}{k}\mathbf{R}^{*(n-k)}\Upsilon_m(\mathbf{C};\mathbf{R})\mathbf{C}\mathbf{R}^k\mathbf{C}\end{aligned}$$

Remark 1.4.

$$\mathcal{Q}_{m+1,n}(\mathbf{C};\mathbf{R}) = \mathbf{R}^*\mathcal{Q}_{m,n}(\mathbf{C};\mathbf{R})(\mathbf{CRC}) - \mathcal{Q}_{m,n}(\mathbf{C};\mathbf{R}) \tag{1.8}$$

and

$$\mathcal{Q}_{m,n+1}(\mathbf{C};\mathbf{R}) = \mathbf{R}^*\mathbf{Q}_{m,n}(\mathbf{C};\mathbf{R}) - \mathcal{Q}_{m,n}(\mathbf{C};\mathbf{R})(\mathbf{CRC}). \tag{1.9}$$

The outline of the paper is as follows. In section two, we introduce the concept of k-$(m, n, \mathbf{C})$-isosymmetric operators. Some properties of these families are studied. Section three is devoted to the study of some spectral properties of k- $(m, n, \mathbf{C})$-isosymmetric operators.

2. k-quasi-$(m, n, \mathbf{C})$-isosymmetric operators

In the present section, we give the definition and basic properties of k-quasi-$(m, n, \mathbf{C})$-isosymmetric operators.

Definition 2.1 ([23]). An operator $\mathbf{R} \in \mathcal{B}[\mathcal{K}]$ is said to be a k-quasi-(m, n)-isosymmetric operator for some positive integers m, n and k, if $\mathbf{R}$ satisfies

$$\begin{aligned}\mathbf{R}^{*k}\Big(\sum_{0\leq j\leq m}(-1)^j\binom{m}{j}\mathbf{R}^{*(m-j)}\Big(\sum_{0\leq r\leq n}(-1)^r\binom{n}{r}\mathbf{R}^{*(n-r)}R^r\Big)\mathbf{R}^{m-j}\Big)\mathbf{R}^k\\ = \mathbf{R}^{*k}\Big(\sum_{0\leq r\leq n}(-1)^r\binom{n}{r}\mathbf{R}^{*(n-r)}\Big(\sum_{0\leq j\leq m}(-1)^j\binom{m}{j}\mathbf{R}^{*(m-j)}\mathbf{R}^{m-j}\Big)\mathbf{R}^r\Big)R^k = 0.\end{aligned}$$

Definition 2.2. An operator $\mathbf{R} \in \mathcal{B}[\mathcal{K}]$ is said to be a k-quasi-$(m,n,\mathbf{C})$-isosymmetric operator for some conjugation $\mathbf{C} \in \mathcal{C}[\mathcal{K}]$ if

$$\begin{aligned}
\mathcal{Q}_{m,n,k}(\mathbf{C};\mathbf{R}) &:= \mathbf{R}^{*k}\mathcal{Q}_{m,n}(\mathbf{C};\mathbf{R})\mathbf{R}^k \\
&= \mathbf{R}^{*k}\Big(\sum_{0\le j\le m}(-1)^j\binom{m}{j}\mathbf{R}^{*(m-j)}\Big(\sum_{0\le r\le n}(-1)^r\binom{n}{r}\mathbf{R}^{*(n-r)}\mathbf{C}\mathbf{R}^r\mathbf{C}\Big)\mathbf{C}\mathbf{R}^{m-j}\mathbf{C}\Big)\mathbf{R}^k \\
&= \mathbf{R}^{*k}\Big(\sum_{0\le r\le n}(-1)^r\binom{n}{r}\mathbf{R}^{*(n-r)}\Big(\sum_{0\le j\le m}(-1)^j\binom{m}{j}\mathbf{R}^{*(m-j)}\mathbf{C}R^{m-j}\mathbf{C}\Big)\mathbf{C}\mathbf{R}^r\mathbf{C}\Big)\mathbf{R}^k \\
&= 0,
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{Q}_{m,n}(\mathbf{C};\mathbf{R}) : &= \sum_{0\le k\le m}(-1)^k\binom{m}{k}\mathbf{R}^{*m-k}\Lambda_n(\mathbf{C};\mathbf{R})\mathbf{C}\mathbf{R}^{m-k}\mathbf{C} \\
&= \sum_{0\le k\le n}(-1)^k\binom{n}{k}\mathbf{R}^{*(n-k)}\Upsilon_m(\mathbf{C};\mathbf{R})\mathbf{C}\mathbf{R}^k\mathbf{C}
\end{aligned}$$

Example 2.3. It should be noted that

(1) Every $(m,\mathbf{C})$-isometric operator is a k-quasi-$(m,n,\mathbf{C})$-isosymmetric and every n-complex symmetric operator is a k-quasi-$(m,n,\mathbf{C})$-isosymmetric operator.

(2) Every $(m,n,\mathbf{C})$-isosymmetric operator is a k-quasi-$(m,n,\mathbf{C})$-isosymmetric operator.

Remark 2.4. Let $\mathbf{R} \in \mathcal{B}[\mathcal{K}]$ and $\mathbf{C} \in \mathcal{C}[\mathcal{K}]$ such that $[\mathbf{R},\mathbf{CRC}] = 0$, the following hold.

(1) If $\mathbf{R}$ is k-quasi-$(m,\mathbf{C})$-isometric operator, then it is k-quasi-$(m,n,\mathbf{C})$-isosymmetric operator.

(2) If $\mathbf{R}$ is a k-quasi-n-complex symmetric operator, then it is a k-quasi-$(m,n,\mathbf{C})$- isosymmetric operator.

Remark 2.5. When $n = m = k = 1$, 1-quasi-$(1,1,\mathbf{C})$-isosymmetric operator is a quasi-complex isosymmetric i.e; an operator $\mathbf{R}$ is quasi complex isosymmetric if and only if

$$\mathbf{R}^*\Big(\mathbf{R}^{*2}\mathbf{CRC} - \mathbf{R}^*\mathbf{C}\mathbf{R}^2\mathbf{C} - \mathbf{R}^* + \mathbf{CRC}\Big)\mathbf{R} = 0.$$

Proposition 2.6. *If* $\mathbf{R}$ *is a k-quasi-$(m,n,\mathbf{C})$-isosymmetric operator for some conjugation* $\mathbf{C}$ *such that* $[\mathbf{R},\mathbf{CRC}] = 0$*, then* $\mathbf{R}$ *is a k'-quasi-$(p,q,\mathbf{C})$-isosymmetric for all $q \ge n$, $p \ge m$ and $k' \ge k$.*

Proof. Since $[\mathbf{R},\mathbf{CRC}] = 0$, it follows that

$$\begin{aligned}
&\mathbf{R}^{*k}\mathcal{Q}_{m+1,n}(\mathbf{C};\mathbf{R})\mathbf{R}^k \\
&\quad= \mathbf{R}^{*k}\Big(\mathbf{R}^*\mathcal{Q}_{m,n}(\mathbf{C};\mathbf{R})(\mathbf{CRC}) - \mathcal{Q}_{m,n}(\mathbf{C};\mathbf{R})\Big)\mathbf{R}^k \\
&\quad= \mathbf{R}^*\mathbf{R}^{*k}\mathcal{Q}_{m,n}(\mathbf{C};\mathbf{R})\mathbf{R}^k(\mathbf{CRC}) - \mathbf{R}^{*k}\mathcal{Q}_{m,n}(\mathbf{C};\mathbf{R})\mathbf{R}^k
\end{aligned}$$

and

$$\begin{aligned}
&\mathbf{R}^{*k}\mathcal{Q}_{m,n+1}(\mathbf{C};\mathbf{R})\mathbf{R}^k \\
&\quad= \mathbf{R}^{*k}\Big(\mathbf{R}^*\mathbf{Q}_{m,n}(\mathbf{C};\mathbf{R}) - \mathcal{Q}_{m,n}(\mathbf{C};\mathbf{R})(\mathbf{CRC})\Big)\mathbf{R}^k
\end{aligned}$$

$$= \mathbf{R}^*\mathbf{R}^{*k}\mathbf{Q}_{m,n}(\mathbf{C};\mathbf{R})\mathbf{R}^k - \mathbf{R}^{*k}\mathcal{Q}_{m,n}(\mathbf{C};\mathbf{R})\mathbf{R}^k(\mathbf{CRC}).$$

Based on the above identities, the required result is achieved. □

Proposition 2.7. *Let $\mathbf{R} \in \mathcal{B}[\mathcal{K}]$ and $\mathbf{C} \in \mathcal{C}[\mathcal{K}]$, then the following are equivalent.*

(1) $\mathbf{R}$ *is a k-quasi-$(m,n,\mathbf{C})$-isosymmetric operator.*

(2) $\langle \mathcal{Q}_{m,n}(\mathbf{C};\mathbf{R})w \mid w\rangle = 0, \ \forall\, w \in \overline{ran(\mathbf{R}^k)}$.

Proof. (1) $\Longrightarrow$ (2). If $\mathbf{R}$ is a k-quasi-$(m,n,\mathbf{C})$-isosymmetric operator, then $\mathbf{R}^{*k}\mathcal{Q}_{m,n}(\mathbf{C};\mathbf{R})\mathbf{R}^k = 0$. Depending on the known properties, we can obtain

$$\begin{aligned}\mathbf{R}^{*k}\mathcal{Q}_{m,n}(\mathbf{C};\mathbf{R})\mathbf{R}^k = 0 &\Longrightarrow \left\langle \mathbf{R}^{*k}\mathcal{Q}_{m,n}(\mathbf{C};\mathbf{R})\mathbf{R}^k w \mid w\right\rangle = 0, \ \forall\, w \in \mathcal{K}\\ &\Longrightarrow \left\langle \mathcal{Q}_{m,n}(\mathbf{C};\mathbf{R})\mathbf{R}^k w \mid \mathbf{R}^k w\right\rangle = 0, \ \forall\, w \in \mathcal{K}\\ &\Longrightarrow \langle \mathcal{Q}_{m,n}(\mathbf{C};\mathbf{R})w \mid w\rangle = 0, \ \forall\, w \in \overline{ran(\mathbf{R}^k)}.\end{aligned}$$

(2) $\Longrightarrow$ (1). With similar steps, we have

$$\begin{aligned}&\langle \mathcal{Q}_{m,n}(\mathbf{C};\mathbf{R})w \mid w\rangle = 0, \ \forall\, w \in \overline{ran(\mathbf{R}^k)}\\ &\Longrightarrow \left\langle \mathcal{Q}_{m,n}(\mathbf{C};\mathbf{R})\mathbf{R}^k u \mid \mathbf{R}^k u\right\rangle = 0, \ \forall\, u \in \mathcal{K}\\ &\Longrightarrow \left\langle \mathbf{R}^{*k}\mathcal{Q}_{m,n}(\mathbf{C};\mathbf{R})\mathbf{R}^k u \mid u\right\rangle = 0, \ \forall\, u \in \mathcal{K}\\ &\Longrightarrow \mathbf{R}^{*k}\mathcal{Q}_{m,n}(\mathbf{C};\mathbf{R})\mathbf{R}^k = 0,\end{aligned}$$

which implies that $\mathbf{R}$ is a k-quasi-$(m,n,\mathbf{C})$-isosymmetirc operator. □

Corollary 2.8. *Let $\mathbf{R} \in \mathcal{B}[\mathcal{K}]$ and $k_1, k_2 \in \mathbb{N}_0$. If $\overline{ran(\mathbf{R}^{k_1})} = \overline{ran(\mathbf{R}^{k_2})}$, then $\mathbf{R}$ is a k_1-quasi-$(m,n,\mathbf{C})$-isosymmetric operator if and only if $\mathbf{R}$ is a k_2-quasi-$(m,n,\mathbf{C})$-isosymmetric operator for some conjugation $\mathbf{C} \in \mathcal{C}[\mathcal{K}]$.*

Proof. Referring to Proposition 2.7, we find that

$$k_1 - \text{quasi} - (m,n,\mathbf{C}) - \text{isosymmetric} \ \Leftrightarrow \langle \mathcal{Q}_{m,n}(\mathbf{C};\mathbf{R})w \mid w\rangle = 0, \ \forall\, w \in \overline{ran(\mathbf{R}^{k_1})}.$$

Using the condition $\overline{ran(\mathbf{R}^{k_1})} = \overline{ran(\mathbf{R}^{k_2})}$, we may write

$$\begin{aligned}\langle \mathcal{Q}_{m,n}(\mathbf{C};\mathbf{R})w \mid w\rangle = 0, \ \forall\, w \in \overline{ran(\mathbf{R}^{k_1})} &\Leftrightarrow \langle \mathcal{Q}_{m,n}(\mathbf{C};\mathbf{R})w \mid u\rangle = 0, \ \forall\, u \in \overline{ran(\mathbf{R}^{k_2})}\\ &\Leftrightarrow k_2 - \text{quasi} - (m,n,\mathbf{C}) - \text{isosymmetric}.\end{aligned}$$

□

Theorem 2.9. *Let $\mathbf{R} \in \mathcal{B}[\mathcal{K}]$ be a k-quasi-$(m,n,\mathbf{C})$-isosymmetric operator for some $\mathbf{C} \in \mathcal{C}[\mathcal{K}]$. If $\ker(\mathbf{R}^{*r}) = \ker(\mathbf{R}^{*(r+1)})$ for some $1 \le r \le k-1$, then $\mathbf{R}$ is a r-quasi-$(m,n,\mathbf{C})$-isosymmetric operator.*

Proof. From the assumptions $\ker(\mathbf{R}^{*r}) = \ker(\mathbf{R}^{*(r+1)})$ and $r \le k-1$, it follows that $\ker(\mathbf{R}^{*r}) = \ker(\mathbf{R}^{*k})$. We get $\overline{ran(\mathbf{R}^k)} = \overline{ran(\mathbf{R}^r)}$. Applying Corollary 2.8, the desired conclusion will be obtained. □

Theorem 2.10. *Let $\mathbf{R} \in \mathcal{B}[\mathcal{K}]$ and $\mathbf{C} = \mathbf{C}_1 \oplus \mathbf{C}_2$ be a conjugation on $\mathcal{K}$ where $\mathbf{C}_1$ and $\mathbf{C}_2$ are conjugations on $\overline{ran(\mathbf{R}^k)}$ and $\ker(\mathbf{R}^{*k})$, respectively. If $\mathbf{R}^k$ does not have a dense range, then the following statements are equivalent:*

(1) $\mathbf{R}$ *is a k-quasi-$(m,n,\mathbf{C})$-isosymmetric operator.*

(2) $\mathbf{R} = \begin{pmatrix} R_1 & R_2 \\ 0 & R_3 \end{pmatrix}$ *on $\mathcal{K} = \overline{ran(\mathbf{R}^k)} \oplus \ker(\mathbf{R}^{*k})$, where R_1 is a $(m,n,\mathbf{C}_1)$-isosymmetric operator and $R_3^k = 0$.*

Proof. (1) $\Rightarrow$ (2) Let $\mathbf{P}$ be the projection onto $\overline{ran(\mathbf{R}^k)}$. Since $\mathbf{R}$ is an k-quasi-$(m,n,\mathbf{C})$-isosymmetric operator, it follows that

$$\mathbf{P}\left(\mathcal{Q}_{m,n}(\mathbf{C},\mathbf{R}\right)\mathbf{P}=0,$$

then $\mathcal{Q}_{m,n}(\mathbf{C_1},\mathbf{R_1})=0$. Hence R_1 is an $(m,n,\mathbf{C}_1)$-isosymmetric on $\overline{ran(\mathbf{R}^k)}$. On the other hand, for any $x=x_1\oplus x_2\in\mathcal{K}=\overline{ran(\mathbf{R}^k)}\oplus\ker(\mathbf{R}^{*k})$, we have

$$\left\langle R_3^k x_2, x_2\right\rangle=\left\langle \mathbf{R}^k(I-P)x,(I-P)x\right\rangle=\left\langle (I-P)x,\mathbf{R}^{*k}(I-P)x\right\rangle=0.$$

Hence $R_3^k=0$.

(2) $\Rightarrow$ (1) Assume that $\mathbf{R}=\begin{pmatrix} R_1 & R_2\\ 0 & R_3\end{pmatrix}$ on $\mathcal{K}=\overline{ran(\mathbf{R}^k)}\oplus\ker(\mathbf{R}^{*k})$, such that R_1 is a $(m,n,\mathbf{C}_1)$-isosymmetric operator and $R_3^k=0$. We know that for all $r\in\mathbb{N}$, we have

$$\mathbf{R}^r=\begin{pmatrix} R_1^r & \sum\limits_{i=0}^{r-1}R_1^iR_2R_3^{r-1-i}\\ 0 & R_3^r\end{pmatrix}.$$

Therefore

$$\mathbf{R}^k\mathbf{R}^{*k}=\begin{pmatrix} R_1^kR_1^{*k} & 0\\ 0 & 0\end{pmatrix},$$

since $R_3^k=0$. On the other hand, by a simple calculation we get

$$\mathcal{Q}_{m,n}(\mathbf{C};\mathbf{R})=\begin{pmatrix} \mathcal{Q}_{m,n}(\mathbf{C_1};R_1) & A\\ B & D\end{pmatrix}$$

such that $A,B,D\in\mathcal{B}[\mathcal{K}]$.

$$\mathbf{R}^k\mathbf{R}^{*k}\mathcal{Q}_{m,n}(\mathbf{C};\mathbf{R})\mathbf{R}^k\mathbf{R}^{*k}=\begin{pmatrix} R_1^kR_1^{*k}\mathcal{Q}_{m,n}(\mathbf{C_1};R_1)R_1^kR_1^{*k} & 0\\ 0 & 0\end{pmatrix}.$$

Since R_1 is $(m,n,\mathbf{C}_1)$-isosymmetric operator, then $\mathcal{Q}_{m,n}(\mathbf{C_1};\mathbf{R_1})=0$. Therefore,

$$\mathbf{R}^k\mathbf{R}^{*k}\mathcal{Q}_{m,n}(\mathbf{C};\mathbf{R})\mathbf{R}^k\mathbf{R}^{*k}=0.$$

Consequently, for all $x\in\mathcal{K}$, we have

$$\begin{aligned} 0 &= \left\langle \mathbf{R}^k\mathbf{R}^{*k}\mathcal{Q}_{m,n}(\mathbf{C};\mathbf{R})\mathbf{R}^k\mathbf{R}^{*k}x \mid x\right\rangle\\ &= \left\langle \mathbf{R}^{*k}\mathcal{Q}_{m,n}(\mathbf{C};\mathbf{R})\mathbf{R}^k\left(\mathbf{R}^{*k}x\right)\mid\left(\mathbf{R}^{*k}x\right)\right\rangle.\end{aligned}$$

As a result that $\mathbf{R}^{*k}\mathcal{Q}_{m,n}(\mathbf{C};\mathbf{R})\mathbf{R}^k=0$, therefore $\mathbf{R}$ is a k-quasi-$(m,n,\mathbf{C})$-isosymmetric operator. □

Corollary 2.11. *Let* $\mathbf{R}\in\mathcal{B}[\mathcal{K}]$ *be a* k*-quasi-*$(m,n,\mathbf{C})$*-isosymmetric operator, where* $\mathbf{C}=\mathbf{C}_1\oplus\mathbf{C}_2$ *be a conjugation on* $\mathcal{K}$ *with* $\mathbf{C}_1$ *and* $\mathbf{C}_2$ *are conjugations on* $\overline{ran(\mathbf{R}^k)}$ *and* $\ker(\mathbf{R}^{*k})$*, respectively. If the restriction* $\mathbf{R}|_{\overline{ran(\mathbf{R}^k)}}$ *is invertible, then* $\mathbf{R}$ *is similar to a direct sum of an* $(m,n,\mathbf{C}_1)$*-isosymmetric operator and a nilpotent operator with index of nilpotence less than or equal* k*.*

Proof. According to that $\mathbf{R}$ is a k-quasi-$(m,n,\mathbf{C})$-isosymmetric operator and to the decomposition

$$\mathbf{R}=\begin{pmatrix} R_1 & R_2\\ 0 & R_3\end{pmatrix}\quad\text{on } \mathcal{K}=\overline{ran(\mathbf{R}^k)}\oplus\ker(\mathbf{R}^{*k}).$$

We have from Theorem 2.10 that R_1 is $(m,n,\mathbf{C}_1)$-isosymmetric operator and R_3 is nilpotent. By the fact that R_1 is invertible, we have $0\notin\sigma(R_1)$. Hence, $\sigma(R_1)\cap\sigma(R_3)=\emptyset$. By

Rosenblum's Corollary [28], there exists $S \in \mathcal{B}[\mathcal{K}]$ for which $R_1S - SR_3 = R_2$. Therefore $\mathbf{R}$ can be written as

$$\mathbf{R} = \begin{pmatrix} I & S \\ 0 & I \end{pmatrix}^{-1} \begin{pmatrix} R_1 & 0 \\ 0 & R_3 \end{pmatrix} \begin{pmatrix} I & S \\ 0 & I \end{pmatrix}.$$

□

Theorem 2.12. *Let* $\mathbf{R} = \begin{pmatrix} \mathbf{R}_1 & \mathbf{R}_2 \\ 0 & \mathbf{R}_3 \end{pmatrix} \in \mathcal{B}[\mathcal{K} \oplus \mathcal{K}]$ *and* $\mathbf{C} \in \mathcal{C}[\mathcal{K}]$*. If* $\mathbf{R}_1$ *is a surjective* $(m,n,\mathbf{C})$*-isosymmetric operator and* $R_3^k = 0$*, then* $\mathbf{R}$ *is similar to a* k*-quasi-*$(m,n,\mathbf{C}\oplus\mathbf{C})$*-isosymmetric operator.*

Proof. Under the assumptions that $\mathbf{R}_1$ is surjective and $\mathbf{R}_3^k = 0$, we have $\sigma_s(\mathbf{R}_1) \cap \sigma_a(\mathbf{R}_3) = \emptyset$ where $\sigma_s(.)$ is the surjective spectrum and $\sigma_a(.)$ is the approximate spectrum. From the statement (c) in [20, Theorem 3.5.1], there exist an operator $\mathbf{S} \in \mathcal{B}[\mathcal{K}]$ for which $\mathbf{R}_1\mathbf{S} - \mathbf{S}\mathbf{R}_3 = \mathbf{R}_2$. Therefore, we can write

$$\begin{pmatrix} \mathbf{R}_1 & \mathbf{R}_2 \\ 0 & \mathbf{R}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{I} & \mathbf{S} \\ 0 & \mathbf{I} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{R}_1 & 0 \\ 0 & \mathbf{R}_3 \end{pmatrix} \begin{pmatrix} \mathbf{I} & \mathbf{S} \\ 0 & \mathbf{I} \end{pmatrix},$$

it follows that $\mathbf{R}$ is similar to $\mathbf{A} = \begin{pmatrix} \mathbf{R}_1 & 0 \\ 0 & \mathbf{R}_3 \end{pmatrix}$.

In fact, since $\mathbf{R}_1$ is $(m,n,\mathbf{C})$-isosymmetric and $\mathbf{R}_3^k = 0$, we obtain

$$\mathcal{Q}_{m,n,k}(\mathbf{C}\oplus\mathbf{C},\mathbf{A}) = \begin{pmatrix} \mathcal{Q}_{m,n,k}(\mathbf{C},\mathbf{R}_1) & 0 \\ 0 & 0 \end{pmatrix} = 0.$$

Therefore, $\mathbf{R}$ is similar to a k-quasi-$(m,n,\mathbf{C}\oplus\mathbf{C})$-isosymmetric operator. □

Proposition 2.13. *Let* $\mathbf{R}$ *be a* k*-quasi-*$(m,n,\mathbf{C})$*-isosymmetric operator, then* $\mathbf{R}^q$ *is also for all* $q \in \mathbb{N}$ *where* $\mathbf{C} = \mathbf{C}_1 \oplus \mathbf{C}_2$ *be a conjugation on* $\mathcal{K}$ *with* $\mathbf{C}_1$ *and* $\mathbf{C}_2$ *are conjugations on* $\overline{ran(\mathbf{R}^k)}$ *and* $\ker(\mathbf{R}^{*k})$*, respectively.*

Proof. If $\overline{ran(\mathbf{R}^k)} = \mathcal{K}$ it is obvious. Else, by Theorem 2.10 we write the matrix representation of $\mathbf{R}$ on $\mathcal{K} = \overline{ran(\mathbf{R}^k)} \oplus \ker(\mathbf{R}^{*k})$ as follows $\mathbf{R} = \begin{pmatrix} R_1 & R_2 \\ 0 & R_3 \end{pmatrix}$ where $R_1 = \mathbf{R}_{|\overline{ran(\mathbf{R}^k)}}$ is an $(m,n,\mathbf{C_1})$-isosymmetric operator and $R_3^k = 0$. We notice that

$$\mathbf{R}^q = \begin{pmatrix} R_1^q & \sum_{i=0}^{q-1} R_1^i R_2 R_3^{q-1-i} \\ 0 & R_3^q \end{pmatrix},$$

where $(R_3^q)^k = 0$ and since R_1 is an $(m,n,\mathbf{C_1})$-isosymmetric operator, then according to [14, Theorem 3.3], we get, R_1^q is an $(m,n,\mathbf{C_1})$-isosymmetric operator. Hence, by Theorem 2.10 $\mathbf{R}^q$ is k-quasi-$(m,n,\mathbf{C})$-isosymmetric operator. □

Theorem 2.14. *Let* $\mathbf{R},\mathbf{T}$ *be in* $\mathcal{B}[\mathcal{K}]$ *and* $\mathbf{C} \in \mathcal{C}[\mathcal{K}]$ *such that*

$$[\mathbf{R},\mathbf{T}] = [\mathbf{T}^*,\mathbf{CRC}] = [\mathbf{R},\mathbf{CRC}] = [\mathbf{T},\mathbf{CTC}] = 0.$$

Let k_1,k_2,m_1,m_2,n_1,n_2 *be positive integers. If* $\mathbf{R}$ *is a* k_1*-quasi-*$(m_1,n_1,\mathbf{C})$*-isosymmetric operator and* $\mathbf{T}$ *is a* k_2*-quasi-*$(m_1,\mathbf{C})$*-isometric operator and, a* k_2*-quasi-*n_2*-complex symmetric with conjugation* $\mathbf{C}$*. Then* $\mathbf{RT}$ *is a* k*-quasi-*$(m,n,\mathbf{C})$*-isosymmetric operator, with* $m = m_1 + m_2 - 1$*,* $n = n_1 + n_2 - 1$ *and* $k = \max\{k_1,k_2\}$

Proof. Under the assumptions $[\mathbf{R},\mathbf{T}]=[\mathbf{T}^*,\mathbf{CRC}]=0$ and taking into account [14, Theorem 3.6] it follows that

$$\begin{aligned}
&(\mathbf{RT})^{*k}\,\mathcal{Q}_{m,n}(\mathbf{C};\mathbf{RT})\,(\mathbf{RT})^{k}\\
&\quad=\mathbf{R}^{*k}\mathbf{T}^{*k}\left(\sum_{i=0}^{m_1+m_2-1}\sum_{j=0}^{n_1+n_2-1}\binom{m_1+m_2-1}{i}\binom{n_1+n_2-1}{j}\right.\\
&\qquad\left.\times\,\mathbf{R}^{*j+i}\Big(\mathcal{Q}_{m_1+m_2-1-i,n_1+n_2-1-j}(\mathbf{C};\mathbf{R})\mathcal{Q}_{i,j}(\mathbf{C};\mathbf{T})\mathbf{R}^{i}\mathbf{T}^{n_1+n_2-1-j}\Big)\right)\mathbf{R}^{k}\mathbf{T}^{k}\\
&\quad=\sum_{i=0}^{m_1+m_2-1}\sum_{j=0}^{n_1+n_2-1}\binom{m_1+m_2-1}{i}\binom{n_1+n_2-1}{j}\\
&\qquad\times\,\mathbf{R}^{*j+i}\Big(\mathbf{R}^{*k}\mathcal{Q}_{m_1+m_2-1-i,n_1+n_2-1-j}(\mathbf{C};\mathbf{R})\mathbf{R}^{k}\Big)\Big(\mathbf{T}^{*k}\mathcal{Q}_{i,j}(\mathbf{C};\mathbf{T})\mathbf{T}^{k}\Big)\mathbf{R}^{i}\mathbf{T}^{n_1+n_2-1-j}\\
&\quad=\sum_{i=0}^{m_1+m_2-1}\sum_{j=0}^{n_1+n_2-1}\binom{m_1+m_2-1}{i}\binom{n_1+n_2-1}{j}\\
&\qquad\times\,\mathbf{R}^{*j+i}\Big(\mathcal{Q}_{m_1+m_2-1-i,n_1+n_2-1-j,k}(\mathbf{C};\mathbf{R})\Big)\Big(\mathcal{Q}_{i,j,k}(\mathbf{C};\mathbf{T}\Big)\mathbf{R}^{i}\mathbf{T}^{n_1+n_2-1-j}.
\end{aligned}$$

If $i\geq m_2$ or $j\geq n_2$, since $\mathbf{T}$ is a k_2-quasi-$(m_2,\mathbf{C})$-isometric operator and k_2-quasi-n_2-complex symmetric operator and $[\mathbf{T},\mathbf{CTC}]=0$ then, $\mathcal{Q}_{i,j,k}(\mathbf{C};\mathbf{T})=0$ by Remark 2.4 and Proposition 2.6. Else, if $i\leq m_2-1$ and $j\leq n_2-1$, then $m_1+m_2-1-i\geq m_1$ and $n_1+n_2-1-j\geq n_1$. Under the hypotheses $\mathbf{R}$ is a k_1-quasi-$(m_1,n_1,\mathbf{C})$-isosymmetric operator and $[\mathbf{R},\mathbf{CRC}]=0$ by Proposition 2.6, we get that $\mathcal{Q}_{m_1+m_2-1-i,n_1+n_2-1-j,k}(\mathbf{C};\mathbf{R})=0$. □

Proposition 2.15. *Let $\mathbf{R}\in\mathcal{B}[\mathcal{K}]$ be a k_1-quasi-$(m_1,n_1,\mathbf{C})$-isosymmetric operator with a conjugation $\mathbf{C}=\mathbf{C}_1\oplus\mathbf{C}_2$ where $\mathbf{C}_1$ and $\mathbf{C}_2$ are conjugations on $\overline{ran(\mathbf{R}_1^{k})}$ and $\ker(\mathbf{R}^{*k_1})$, respectively. Let $\mathbf{T}\in\mathcal{B}[\mathcal{K}]$ be a k_2-quasi-$(m_2,\mathbf{C})$-isometric and a k_2-quasi-n_2-symmetric operator with a conjugation $\mathbf{C}$. Assume that*

$$[\mathbf{R},\mathbf{T}]=[\mathbf{T}^*,\mathbf{CRC}]=[\mathbf{R},\mathbf{CRC}]=[\mathbf{T},\mathbf{CTC}]=0,$$

and $\overline{ran(\mathbf{R}^{k_1})}=\overline{ran(\mathbf{T}^{k_2})}$. Then $\mathbf{R}^p\mathbf{T}^q$ is a k-quasi-$(m,n,\mathbf{C})$-isosymmetric operator for all positive integers p and q, with $m=m_1+m_2-1$, $n=n_1+n_2-1$ and $k=\max\{k_1,k_2\}$.

Proof. If $\overline{ran(\mathbf{R}^{k_1})}=\mathcal{K}(=\overline{ran(\mathbf{T}^{k_2})})$, it follows that $\mathbf{R}$ is $(m_1,n_1,\mathbf{C})$-isosymmetric operator and so is $\mathbf{R}^p$ by [14, Theorem 3.]. On the other hand $\mathbf{T}$ is $(m_2,\mathbf{C})$-isometry and n_2-complex symmetric with a conjugation $\mathbf{C}$. In view of [9, Theorem 3.7] and [10, Theorem 4.5] it follows that $\mathbf{T}^q$ is $(m_2,\mathbf{C})$-isometry and n_2-complex symmetry with a conjugation $\mathbf{C}$. Applying [14, Theorem 3.6] we obtain that $\mathbf{R}^p\mathbf{T}^q$ is $(m_1+m_2-1,n_1+n_2-1,\mathbf{C})$-isosymmetric operator.

If $\overline{ran(\mathbf{R}^{k_1})}\neq\mathcal{K}(\neq\overline{ran(\mathbf{T}^{k_2})})$. From Proposition 2.13 we have $\mathbf{R}^p$ is k_1-quasi-$(m_1,n_1,\mathbf{C})$-isosymmetric operator. On the other hand, $\mathbf{T}^q$ is k_2-quasi-$(m_2,\mathbf{C})$ isometric (by [25, Theorem 2.2] and it is k_2-quasi-n_2-complex symmetric (by [1, Theorem 2.3]. Elementary calculation shows that $\mathbf{R}^p$ and $\mathbf{T}^q$ satisfy the conditions of Theorem 2.14 and consequently, $\mathbf{R}^p\mathbf{S}^q$ is k-quasi-$(m_1+m_2-1,n_1+n_2-1,\mathbf{C})$-isosymmetric operator. □

Proposition 2.16. *Let $\mathbf{R}\in\mathcal{B}[\mathcal{K}]$ be a k-quasi-$(m,n,\mathbf{C})$-isosymmetric operator for some $\mathbf{C}\in\mathcal{C}[\mathcal{K}]$ and $\mathbf{N}\in\mathcal{B}[\mathcal{K}]$ be a p-nilpotent operator such that $\mathbf{R}$ and $\mathbf{N}$ satisfying $[\mathbf{R},\mathbf{T}]=[\mathbf{T}^*,\mathbf{CRC}]=0$, then $\mathbf{R}+\mathbf{N}$ is $(k+p)$-quasi-$(m+2p-2,n+2p-1,\mathbf{C})$-isosymmetric operator.*

Proof. We know that from [14, Theorem 3.4]

$$\mathcal{Q}_{m+2p-2,n+2p-1}(\mathbf{C};\mathbf{R}+\mathbf{N}) = \sum_{j=0}^{n+2p-1} \sum_{i+l+h=m+2p-2} \binom{n+2p-1}{j}\binom{m+2p-2}{i,l,h} \times (\mathbf{R}^*+\mathbf{N}^*)^i\,\mathbf{N}^{*l}\mathcal{Q}_{h,n+2p-1-j}(\mathbf{C};\mathbf{R})\alpha_j(\mathbf{C};\mathbf{N})\mathbf{R}^l\mathbf{N}^i$$

where $\alpha_j(\mathbf{C};\mathbf{N}) = \sum_{\mu=0}^{j}(-1)^j\binom{j}{\mu}\mathbf{N}^{*j-\mu}\mathbf{C}\mathbf{N}^{\mu}\mathbf{C}$ and $\alpha_j(\mathbf{C};\mathbf{N}) = 0$ if $j \geq 2p$. Therefore

$$\begin{aligned}
&\mathcal{Q}_{m+2p-2,n+2p-1,k+p}(\mathbf{C};\mathbf{R}+\mathbf{N})\\
&= (\mathbf{R}+\mathbf{N})^{*k+p}\,\mathcal{Q}_{m+2p-2,n+2p-1}(\mathbf{C};\mathbf{R}+\mathbf{N})\,(\mathbf{R}+\mathbf{N})^{k+p}\\
&= \sum_{r=0}^{k+p}\binom{k+p}{r}\mathbf{R}^{*k+p-r}\mathbf{N}^{*r}\Big(\sum_{j=0}^{n+2p-1}\sum_{i+l+h=m+2p-2}\binom{n+2p-1}{j}\binom{m+2p-2}{i,l,h}\\
&\quad\times(\mathbf{R}^*+\mathbf{N}^*)^i\,\mathbf{N}^{*l}\mathcal{Q}_{h,n+2p-1-j}(\mathbf{C};\mathbf{R})\alpha_j(\mathbf{C};\mathbf{N})\mathbf{R}^l\mathbf{N}^i\Big)\sum_{r=0}^{k+p}\binom{k+p}{r}\mathbf{R}^{k+p-r}\mathbf{N}^r.
\end{aligned}$$

- If

$$\left.\begin{array}{c} j\geq 2p \Rightarrow \alpha_j(\mathbf{C};\mathbf{N})=0\\ \text{or}\\ i\geq p\Rightarrow \mathbf{N}^i=0\\ \text{or}\\ l\geq p\Rightarrow \mathbf{N}^{*l}=0\\ \text{or}\\ r\geq p+1\Rightarrow \mathbf{N}^r=0 \text{ and } \mathbf{N}^{*r}=0\end{array}\right\}\Rightarrow \mathcal{Q}_{m+2p-2,n+2p-1,k+p}(\mathbf{C};\mathbf{R})=0$$

- Else, if

$$\left.\begin{array}{c} j\leq 2p-1\Rightarrow n+2p-1-j\geq n\\ \left.\begin{array}{l} i\leq p-1\\ l\leq p-1\end{array}\right\}\Rightarrow h=m+2k-2-i-l\geq m\\ r\leq p\Rightarrow k+p-r\geq k\end{array}\right\}\Rightarrow \mathbf{R}^{*k+p-r}\mathcal{Q}_{h,n+2p-1-j}(\mathbf{C};\mathbf{R})\mathbf{R}^{k+p-r}=0.$$

Putting together the above cases, we obtain that $\mathbf{R}+\mathbf{N}$ is is $(k+p)$-quasi-$(m+2p-2,n+2p-1,\mathbf{C})$-isosymmetric operator. □

Corollary 2.17. *Let $\mathbf{R}_j \in \mathcal{B}[\mathcal{K}]$ be a k_j-quasi-$(m_j,n_j,\mathbf{C})$-isosymmetric operator such that $[\mathbf{R}_j,\mathbf{C}\mathbf{R}_j\mathbf{C}]=0$ for $j=1,\cdots,d$ where $\mathbf{C}\in\mathcal{C}[\mathcal{K}]$. Set*

$$\mathbf{S} = \begin{pmatrix} \mathbf{R}_1 & \lambda I & 0 & \cdots\\ 0 & \ddots & \ddots & \ddots\\ \ddots & \ddots & \ddots & \lambda I\\ 0 & \ddots & 0 & \mathbf{R}_d\end{pmatrix} \quad \text{on } \mathcal{K}^{(d)} := \mathcal{K}\oplus\cdots\oplus\mathcal{K},$$

where $\lambda\in\mathbb{C}$, then $\mathbf{S}$ is $(\max\{k_j\}+d)$-quasi- $\big(\max\{m_j\}+2d-2,\max\{n_j\}+2d-1,\mathbf{C}^{(d)}\big)$-isosymmetric operator where $\mathbf{C}^{(d)} := \mathbf{C}\oplus\mathbf{C}\cdots\oplus\mathbf{C}$ is a conjugation on $\mathcal{K}^{(d)}$.

Proof. Consider the matrices

$$\mathbf{R} = \begin{pmatrix} \mathbf{R}_1 & 0 & 0 & \cdots\\ 0 & \ddots & \ddots & \ddots\\ \ddots & \ddots & \ddots & 0\\ 0 & \ddots & 0 & \mathbf{R}_d\end{pmatrix} \quad\text{and}\quad \mathbf{N} = \begin{pmatrix} 0 & \lambda I & 0 & \cdots\\ 0 & \ddots & \ddots & \ddots\\ \ddots & \ddots & \ddots & \lambda I\\ 0 & \ddots & 0 & 0\end{pmatrix}.$$

Obviously we have that $\mathbf{S} = \mathbf{R} + \mathbf{N}$. Since $\mathbf{R}_j$ is a k_j-quasi- $(m_j, n_j, \mathbf{C})$-isosymmetric operator and satisfies $[\mathbf{R}_j, \mathbf{C}\mathbf{R}_j\mathbf{C}] = 0$ for $j = 1, \cdots, d$, it follows from Proposition 2.6 that $\mathbf{R}_j$ is a $\max\{k_j\}$-quasi-$(\max\{m_j\}, \max\{n_j\}, \mathbf{C})$-isosymmetric for $j = 1, \cdots, d$. From which we deduce that $\mathbf{R}$ is a $\max\{k_j\}$-quasi-$(\max\{m_j\}, \max\{n_j\}, \mathbf{C}^{(d)})$-isosymmetric operator. It easy to check that $\mathbf{N}$ is d-nilpotent and $[\mathbf{N}, \mathbf{R}] = [\mathbf{N}^*, \mathbf{C}^{(d)}\mathbf{R}\mathbf{C}^{(d)}] = 0$. This means that $\mathbf{R}$ and $\mathbf{N}$ satisfying the conditions of Proposition 2.16 so we get that $\mathbf{S}$ is $(\max\{k_j\}+d)$-quasi-$(\max\{m_j\}+2d-2, \max\{n_j\}+2d-1, \mathbf{C}^{(d)})$-isosymmetric operator. □

3. Spectral properties

In this section, we will study some spectral properties of a k-quasi-$(m, n, \mathbf{C})$-isosymmetric operator. We will note by $\sigma(\mathbf{R}), \sigma_{ap}(\mathbf{R}), \sigma_p(\mathbf{R})$ the spectrum, the approximate point spectrum and the point spectrum of an operator $\mathbf{R}$, respectively.

Proposition 3.1. *Let $\mathbf{R}$ be a k-quasi-$(m, n, \mathbf{C})$-isosymmetric operator, then $\sigma_{ap}(\mathbf{R}) \subset \partial\mathbb{D} \cup \mathbb{R}$, where $\partial\mathbb{D} = \{ \lambda \in \mathbb{C} \; |\lambda| = 1 \}$.*

Proof. Let $\lambda \in \sigma_{ap}(\mathbf{R})$, then there exists a sequence $(x_i)_{i\geq 0}$, with $\|x_i\| = 1$ such that $(\mathbf{R} - \lambda\mathbf{I})x_i \to 0$ as $i \to +\infty$. We have $(\mathbf{R}^{\mathbf{j}} - \lambda^j\mathbf{I})x_i \to 0$ for all positive integers j. Under the hypothesis $\mathbf{R}$ is a k-quasi-$(m, n, \mathbf{C})$-isosymmetric operator, then

$$\begin{aligned}
0 =& \langle \mathcal{Q}_{m,n,k}(\mathbf{C};\mathbf{R})x_i \mid x_i\rangle \\
=& \left\langle \mathbf{R}^{*k}\mathcal{Q}_{m,n}(\mathbf{C};\mathbf{R})\mathbf{R}^k x_i \mid x_i\right\rangle \\
=& \left\langle \mathcal{Q}_{m,n}(\mathbf{C};\mathbf{R})\mathbf{R}^k x_i \mid \mathbf{R}^k x_i\right\rangle \\
=& \left\langle \mathcal{Q}_{m,n}(\mathbf{C};\mathbf{R}) \left[\left(\mathbf{R}^k - \lambda^k\right) x_i + \lambda^k x_i\right] \mid \left(\mathbf{R}^k - \lambda^k\right) x_i + \lambda^k x_i\right\rangle \\
=& |\lambda|^{2k} \langle \mathcal{Q}_{m,n}(\mathbf{C};\mathbf{R})x_i \mid x_i\rangle \quad i \to +\infty \\
=& |\lambda|^{2k} \left\langle \sum_{j=0}^{m} (-1)^j \binom{m}{j} \mathbf{R}^{*m-j}\Lambda_n(\mathbf{C},\mathbf{R})\mathbf{C}\mathbf{R}^{m-j}x_i \mid x_i\right\rangle \\
=& |\lambda|^{2k} \left\langle \sum_{j=0}^{m} (-1)^j \binom{m}{j} \Lambda_n(\mathbf{C},\mathbf{R})\mathbf{C}\mathbf{R}^{m-j}x_i \mid \mathbf{R}^{m-j}x_i\right\rangle \quad i \to +\infty \\
=& |\lambda|^{2k} \Bigg\langle \sum_{j=0}^{m} (-1)^j \binom{m}{j} \Lambda_n(\mathbf{C},\mathbf{R})\mathbf{C} \left(\mathbf{R}^{m-j} - \lambda^{m-j}x_i\right) \\
&+ \lambda^{m-j}x_i \mid \left(\mathbf{R}^{m-j} - \lambda^{m-j}\right) x_i + \lambda^{m-j}x_i\Bigg\rangle \\
=& |\lambda|^{2k} \left\langle \sum_{j=0}^{m} (-1)^j \binom{m}{j} \Lambda_n(\mathbf{C},\mathbf{R})\mathbf{C}\lambda^{m-j}x_i \mid \lambda^{m-j}x_i\right\rangle \quad i \to +\infty \\
=& |\lambda|^{2k} \left(1 - |\lambda|^2\right)^m \langle \Lambda_n(\mathbf{C},\mathbf{R})\mathbf{C}x_i \mid x_i\rangle \\
=& |\lambda|^{2k} \left(1 - |\lambda|^2\right)^m \left\langle \sum_{r=0}^{n} (-1)^r \binom{n}{r} \mathbf{R}^{*n-r}\mathbf{C}\mathbf{R}^r x_i \mid x_i\right\rangle \quad i \to +\infty \\
=& |\lambda|^{2k} \left(1 - |\lambda|^2\right)^m \left\langle \sum_{r=0}^{n} (-1)^r \binom{n}{r} \mathbf{C}\mathbf{R}^r x_i \mid \mathbf{R}^{n-r}x_i\right\rangle \quad i \to +\infty \\
=& |\lambda|^{2k} \left(1 - |\lambda|^2\right)^m \Bigg\langle \sum_{r=0}^{n} (-1)^r \binom{n}{r} \mathbf{C}\left(\mathbf{R}^r - \lambda^r\right) x_i \\
&+ \lambda^r x_i \mid \left(\mathbf{R}^{n-r} - \lambda^{n-r}\right) x_i + \lambda^{n-r}x_i\Bigg\rangle \quad i \to +\infty
\end{aligned}$$

$$
\begin{aligned}
&=|\lambda|^{2k}\left(1-|\lambda|^2\right)^m\left\langle\sum_{r=0}^{n}(-1)^r\binom{n}{r}\mathbf{C}\lambda^r x_i \mid \lambda^{n-r}x_i\right\rangle \quad i\to+\infty\\
&=|\lambda|^{2k}\left(1-|\lambda|^2\right)^m\left(\lambda-\overline{\lambda}\right)^n\langle\mathbf{C}x_i \mid x_i\rangle \quad i\to+\infty\\
&=|\lambda|^{2k}\left(1-|\lambda|^2\right)^m(2Im(\lambda))^n\langle\mathbf{C}x_i \mid x_i\rangle \quad i\to+\infty.
\end{aligned}
$$

Now, since $0\notin\sigma_a(\mathbf{C})$ it follows that $\langle\mathbf{C}x_i \mid x_i\rangle \not\longrightarrow 0$ as $i\to\infty$. Consequently, $\lambda=0$ or $|\lambda|=1$ or $\lambda\in\mathbb{R}$. This completes the proof. □

Proposition 3.2. *Let $\mathbf{R}$ be a k-quasi-$(m,n,\mathbf{C})$-isosymmetric operator for some conjugation $\mathbf{C}$ and $\lambda\in\mathbb{C}$ with $Im(\lambda)\neq 0$, then the following properties hold.*

(i) *If $\lambda\in\sigma_{ap}(\mathbf{R})$, then $\overline{\lambda}\in\sigma_{ap}(\mathbf{R}^*)$.*
(ii) *If $\lambda\in\sigma_p(\mathbf{R})$, then $\overline{\lambda}\in\sigma_p(\mathbf{R}^*)$.*

Proof. (i) Let $\lambda\in\sigma_{ap}(\mathbf{R})$ then there exists a sequence $(x_i)_{i\geq 0}$, with $\|x_i\|=1$ such that $(\mathbf{R}-\lambda I)x_i\to 0$ as $i\to+\infty$. We have $(\mathbf{R}^{\mathbf{j}}-\lambda^j I)x_i\to 0$ for all positive integers j. Under the hypothesis $\mathbf{R}$ is a k-quasi-$(m,n,\mathbf{C})$-isosymmetric operator, then

$$
\begin{aligned}
0&=\mathcal{Q}_{m,n,k}(\mathbf{C};\mathbf{R})x_i\\
&=\mathbf{R}^{*k}\mathcal{Q}_{m,n}(\mathbf{C};\mathbf{R})\mathbf{R}^k x_i\\
&=\mathbf{R}^{*k}\mathcal{Q}_{m,n}(\mathbf{C};\mathbf{R})\left[\left(\mathbf{R}^k-\lambda^k\right)x_i+\lambda^k x_i\right]\\
&=\lambda^k\mathbf{R}^{*k}\mathcal{Q}_{m,n}(\mathbf{C};\mathbf{R})x_i, \quad i\to+\infty\\
&=\lambda^k\mathbf{R}^{*k}\sum_{j=0}^{m}(-1)^j\binom{m}{j}\mathbf{R}^{*m-j}\Lambda_n(\mathbf{C};\mathbf{R})\mathbf{C}\mathbf{R}^{m-j}\mathbf{C}x_i\\
&=\left(\lambda^k\mathbf{R}^{*k}\sum_{j=0}^{m}(-1)^j\binom{m}{j}\mathbf{R}^{*m-j}\Lambda_n(\mathbf{C};\mathbf{R})\mathbf{C}\mathbf{R}^{m-j}\mathbf{C}x_i\right)\\
&=\left(\lambda^k\mathbf{R}^{*k}\sum_{j=0}^{m}(-1)^j\binom{m}{j}\mathbf{R}^{*m-j}\Lambda_n(\mathbf{C};\mathbf{R})\mathbf{C}\left[\left(\mathbf{R}^{m-j}-\lambda^{m-j}\right)\mathbf{C}x_i+\lambda^{m-j}\mathbf{C}x_i\right]\right)\\
&=\left(\lambda^k\mathbf{R}^{*k}\sum_{j=0}^{m}(-1)^j\binom{m}{j}\mathbf{R}^{*m-j}\Lambda_n(\mathbf{C};\mathbf{R})\overline{\lambda}^{m-j}x_i\right), \quad i\to+\infty\\
&=\left(\lambda^k\left(\mathbf{I}-\overline{\lambda}\mathbf{R}^*\right)^m\mathbf{R}^{*k}\Lambda_n(\mathbf{C};\mathbf{R})x_i\right)\\
&=\left(\lambda^k\left(\mathbf{I}-\overline{\lambda}\mathbf{R}^*\right)^m\mathbf{R}^{*k}\sum_{r=0}^{n}(-1)^r\binom{n}{r}\mathbf{R}^{*n-r}\mathbf{C}\mathbf{R}^r\mathbf{C}x_i\right)\\
&=\left(\lambda^k\left(\mathbf{I}-\overline{\lambda}\mathbf{R}^*\right)^m\mathbf{R}^{*k}\sum_{r=0}^{n}(-1)^r\binom{n}{r}\mathbf{R}^{*n-r}\mathbf{C}\left[\left(\mathbf{R}^r-\lambda^r\right)\mathbf{C}x_i+\lambda^r\mathbf{C}x_i\right]\right)\\
&=\left(\lambda^k\left(\mathbf{I}-\overline{\lambda}\mathbf{R}^*\right)^m\mathbf{R}^{*k}\sum_{r=0}^{n}(-1)^r\binom{n}{r}\mathbf{R}^{*n-r}\mathbf{C}\overline{\lambda}^r x_i\right), \quad i\to+\infty\\
&=\lambda^k\mathbf{R}^{*k}\left(\mathbf{I}-\overline{\lambda}\mathbf{R}^*\right)^m\left(\overline{\lambda}\mathbf{I}-\mathbf{R}^*\right)^n x_i.
\end{aligned}
$$

Therefore, since $\lambda\neq 0$, then $\lim_{i\to+\infty}\mathbf{R}^{*k}\left(I-\overline{\lambda}\mathbf{R}^*\right)^m\left(\overline{\lambda}\mathbf{I}-\mathbf{R}^*\right)^n x_i=0$. If $(\overline{\lambda}-\mathbf{R}^*)$ is bounded from below, then so is $\left(\overline{\lambda}-\mathbf{R}^*\right)^n$, then there exist $M>0$ such that

$\| \left(\overline{\lambda} - \mathbf{R}^*\right)^n x\| \geq M\|x\|$ for all $x \in \mathcal{K}$. This implies that

$$\|\mathbf{R}^{*k}\left(\mathbf{I} - \overline{\lambda}\mathbf{R}^*\right)^m \left(\overline{\lambda}\mathbf{I} - \mathbf{R}^*\right)^n x_i\| \geq M\|\mathbf{R}^{*k}\left(1 - \overline{\lambda}\mathbf{R}^*\right)^m x_i\|.$$

Consequently,

$$\begin{aligned}
\lim_{i\to+\infty} \mathbf{R}^{*k}\left(\mathbf{I} - \overline{\lambda}\mathbf{R}^*\right)^m x_i = 0 \quad &\Longrightarrow \quad \lim_{i\to+\infty} \left\langle \mathbf{R}^{*k}\left(\mathbf{I} - \overline{\lambda}\mathbf{R}^*\right)^m x_i, x_i \right\rangle = 0 \\
&\Longrightarrow \quad \lim_{i\to+\infty} \left\langle x_i \mid (\mathbf{I} - \lambda\mathbf{R})^m \mathbf{R}^k x_i \right\rangle = 0 \\
&\Longrightarrow \quad \lim_{i\to+\infty} \left\langle x_i \mid \left(\overline{\lambda}\mathbf{I} - \mathbf{R}\right)^m \left(\mathbf{R}^k - \lambda^k\right) x_i \right\rangle = 0 \\
&\qquad + \lim_{i\to+\infty} \left\langle x_i \mid \lambda^k \left(\overline{\lambda}\mathbf{I} - \mathbf{R}\right)^m x_i \right\rangle = 0 \\
&\Longrightarrow \quad \lim_{i\to+\infty} \left\langle x_i \mid \lambda^k \left(\overline{\lambda}\mathbf{I} - \mathbf{R}\right)^m x_i \right\rangle = 0 \\
&\Longrightarrow \quad \lambda^k \left(\lambda - \overline{\lambda}\right)^m = 0.
\end{aligned}$$

The hypothesis $Im(\lambda) \neq 0$ gives the contradiction. Hence $\left(\overline{\lambda} - \mathbf{R}^*\right)$ is not lower bounded and therefore $\overline{\lambda} \in \sigma_{ap}(\mathbf{R}^*)$.
(ii) The statement (ii) follows from the statement (i) so we omitted its proof. □

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