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A KIND OF ROTATIONAL SURFACES WITH A LIGHT-LIKE AXIS IN CONFORMALLY FLAT PSEUDO-SPACES OF DIMENSIONAL THREE

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Abstract. In this work, we define the rotational surface with a light-like axis in conformally flat pseudo-spaces $\left(\mathbb{E}_3^1\right)_\lambda$, where λ is a radial-type conformal factor. We relate the principal curvatures of a non-degenerate surface that belongs to conformally equivalent spaces $\left(\mathbb{E}_3^1\right)_\lambda$ and $\mathbb{R}_1^3$, based on the radial conformal factor. Thus, we establish a relationship between the radial conformal factor and the profile curve of the rotational flat surface in $\left(\mathbb{E}_3^1\right)_\lambda$, but also for that of the rotational surface with zero extrinsic curvature.

1. Introduction

The theory of surfaces is one of the significant subfields of study that belong to the field of differential geometry. This theory has a wide variety of applications. For instance, it is used in computer graphics to create 3D models of objects, in physics to describe the behavior of fluids and solids, and in engineering to design structures with optimal shapes [1, 2].

In contrast to the creation of a helicoidal surface, which has been differently characterized in a recent publication [3], the formation of a rotational surface is achieved only through the rotation of a curve around an axis. The investigation of rotational surfaces has been the subject of considerable scholarly research. To access studies done in recent years, refer to references [4–6]. The study of special surfaces, such as rotational and helicoidal surfaces, is conducted in the setting of conformally flat spaces. Conformally flat spaces possess distinctive characteristics through the utilization of their conformal factors. The determination of the proper conformal factor is important for undertaking surveys of the aforementioned surfaces in conformally flat spaces. A function f is said to be invariant under a transformation T

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of the space into itself if the condition $f(Tx) = f(x)$ is satisfied for all x. If the conformal factor λ is a function that meets this criterion, it is reasonable to consider such surfaces in conformally flat spaces. An estimation for this type of function can be derived from the Cartesian equation of geometric shapes such as the sphere and the cylinder. In contrast to the cylinder type, which exhibits invariance under both rotational and translational symmetries, the spherical type is only invariant under rotational symmetry. For more on research done in the framework of the spherical type $t := x_1^2 + x_2^2 + x_3^2$, see [7, 8]. For another type, see [9–15]. In the aforementioned studies, the authors consider the various conformal factors, such as $\sqrt{t}$, $\frac{1}{\sqrt{t}}$, and e^{-t}. It is worth noting that the first two factors contribute to the formation of the generic metric, whereas the third factor serves as a metric that is a solution to Einstein's equation.

Yerlikaya [14] introduces the conformally flat pseudo-space of dimensional three, and presents a non-degenerate surface's curvatures for an arbitrary conformal factor. But, this work is based on the utilization of the radial conformal factor as the framework. From this perspective, rotational surfaces in conformally flat pseudo-spaces are analyzed.

2. Basic Notations

Denote the Minkowski space by $\mathbb{R}_1^3$, defined by the Minkowski metric $g(x, y) = -x_1y_1 + x_2y_2 + x_3y_3$ with respect to a cononical basis $\{e_1, e_2, e_3\}$ of $\mathbb{R}_1^3$, where $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$. Observe that for a pseudo-orthonormal basis $\{\xi_1, \xi_2, \xi_3\}$ of $\mathbb{R}_1^3$, the metric becomes $g(x, y) = x_1y_3 + x_2y_2 + x_3y_1$. In a such basis, the following equalities hold

$$g(\xi_1, \xi_1) = g(\xi_1, \xi_2) = g(\xi_2, \xi_3) = g(\xi_3, \xi_3) = 0, \tag{1}$$

$$g(\xi_1, \xi_3) = g(\xi_2, \xi_2) = 1. \tag{2}$$

For some tools regarding the transition matrix given by

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix}, \tag{3}$$

see [16]. The rotational motion about the null coordinate axis $O\xi_3$ is represented by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \to A^{-1}RA \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

i.e.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \to \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} 1 + \frac{\theta^2}{2} & -\frac{\theta^2}{2} & \theta \\ \frac{\theta^2}{2} & 1 - \frac{\theta^2}{2} & \theta \\ \theta & -\theta & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

or the more useful form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ -\frac{t^2}{2} & -t & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \tag{4}$$

where $t = -\sqrt{2}\theta$.

Equipped the Minkowski space $\mathbb{R}_1^3$ with a conformally flat pseudo-metric given by the angle-bracket notation

$$\langle w_1, w_2 \rangle_{g_\lambda} = \frac{1}{\lambda^2(p)} \langle w_1, w_2 \rangle_L, \qquad \forall w_1, w_2 \in T_p\mathbb{R}_1^3,\ \forall p \in \mathbb{R}_1^3,$$

the resulting space is said to be the complete pseudo-Riemannian manifold if the conformal factor λ is bounded. From now on, unless otherwise stated, this pseudo-manifold shall be mentioned as the conformally flat pseudo-space, represented by $\left(\mathbb{E}_3^1\right)_\lambda$. Here, note that the pseudo-metric $\langle , \rangle_L$ is the Minkowski metric whose coefficients are those of Eqs. (1) and (2).

3. Surfaces in a conformally flat pseudo-space with radial conformal metrics $\left(\mathbb{E}_3^1\right)_{\lambda(r)}$

In [14], the author calculates the principal curvatures of a non-degenerate parameterized surface for an arbitrary conformal factor in the conformally flat pseudo-space. Now, we'll modify the process so that it works with the radial conformal factor

$$\lambda = \lambda(r), \quad r = 2x_1x_3 + x_2^2, \tag{5}$$

which implies the spherical type with respect to the pseudo-orthonormal basis of $\mathbb{R}_1^3$. Consider a non-degenerate parametrized surface $M = X(U)$ in the Minkowski space as

$$\begin{aligned} X &: U \subset \mathbb{R}^2 \to \mathbb{R}_1^3 \\ &(s,t) \to X(s,t) = (x_1(s,t), x_2(s,t), x_3(s,t)). \end{aligned}$$

Since this surface also belongs to a pseudo-space that is conformal to the Minkowski space, we can write $\widetilde{N}(s,t) = (\lambda N)(s,t)$ for (s,t) in some planar domain, where N and $\widetilde{N}$ denote the normal vector fields in Minkowski space and conformally flat pseudo-space, respectively. Let $\bar{\nabla}$ be the Levi-Civita connection of $\left(\mathbb{E}_3^1\right)_{\lambda(r)}$. Thus, we get

$$\bar{\nabla}_{X_{,s}} \widetilde{N} = \bar{\nabla}_{X_{,s}} (\lambda N) = X_s(\lambda) N + \lambda \bar{\nabla}_{X_{,s}} N, \tag{6}$$

where $X_{,s}$ denotes the partial derivative of X with respect to the parameter s. Using the properties of the connection $\bar{\nabla}$ and considering N as the linear combination of the pseudo-basis, we write

$$\bar{\nabla}_{X_{,s}} N = N_{,s} + \sum_{i,j,k=1}^{3} X^{i}_{,s} N^{j} \Gamma^{k}_{ij} \xi_k, \tag{7}$$

where Γ^k_{ij} denote the Christoffel symbols of the conformal pseudo metric. Note that Eq. (7) holds for the parameter t, as well.

Taking Eq. (5) into account, we have $\frac{\partial \lambda}{\partial x_i} = \frac{\partial \lambda}{\partial r} \frac{\partial r}{\partial x_i}$. From now on, we use the notation $\frac{\partial \lambda}{\partial r} = \dot{\lambda}$. Thus, we can write

$$\Gamma^k_{ij} = -\bar{g}_{jk} \frac{\epsilon_j}{\epsilon_k} \frac{\dot{\lambda}(r)}{\lambda} \frac{\partial r}{\partial x_i} - \bar{g}_{ik} \frac{\epsilon_i}{\epsilon_k} \frac{\dot{\lambda}(r)}{\lambda} \frac{\partial r}{\partial x_j} + \bar{g}_{ij} \frac{\epsilon_i}{\epsilon_k} \frac{\dot{\lambda}(r)}{\lambda} \frac{\partial r}{\partial x_k}, \tag{8}$$

where $\epsilon_i = \bar{g}_{ii}$. From Eq. (8) together with Eq. (5), we get

$$\begin{aligned} \Gamma^2_{11} = \Gamma^3_{11} = \Gamma^3_{12} = \Gamma^1_{13} = \Gamma^3_{13} = \Gamma^1_{23} = \Gamma^1_{33} = \Gamma^2_{33} = 0, \\ \Gamma^1_{11} = 2\Gamma^2_{12} = -2\Gamma^3_{22} = -\frac{4x_3 \dot{\lambda}(r)}{\lambda} \\ \Gamma^1_{12} = -\Gamma^2_{13} = \Gamma^2_{22} = \Gamma^3_{23} = -\frac{2x_2 \dot{\lambda}(r)}{\lambda} \\ -2\Gamma^1_{22} = 2\Gamma^2_{23} = \Gamma^3_{33} = -\frac{4x_1 \dot{\lambda}(r)}{\lambda} \end{aligned} \tag{9}$$

Theorem 1. *Let $X : U \to \mathbb{R}^3_1$ be a non-degenerate surface parametrized as $X(s,t) = (x_1(s,t), x_2(s,t), x_3(s,t))$ in the Minkowski space $\mathbb{R}^3_1$. Consider $X(U)$ as a non-degenerate surface in a conformally flat pseudo-space $\left(\mathbb{E}^1_3\right)_{\lambda(r)}$. Then, the eigenvalues $\widetilde{k}_l$ of X in $\left(\mathbb{E}^1_3\right)_{\lambda(r)}$ are calculated as*

$$\widetilde{k}_l = \lambda k_l - 2\dot{\lambda} \left\langle (x_1, x_2, x_3), N \right\rangle, \quad 1 \leq l \leq 2, \tag{10}$$

where N denotes the normal Gauss mapping of X in $\mathbb{R}^3_1$ and k_l are the eigenvalues of N.

Proof. Let's proceeed with the proof for the parameter s. Putting (9) into Eq. (7), we have

$$\bar{\nabla}_{X_{,s}} N = N_{,s} - \frac{2\dot{\lambda}}{\lambda} \langle X, N \rangle X_{,s} - \frac{2\dot{\lambda}}{\lambda} \langle X_{,s}, X \rangle N.$$

Substituting this into Eq. (6), we obtain

$$\bar{\nabla}_{X_{,s}} \widetilde{N} = \lambda N_{,s} - 2\dot{\lambda} \langle X, N \rangle X_{,s}. \tag{11}$$

Taking $N_{,s} = k_1 X_{,s}$ and $\bar{\nabla}_{X_{,s}} \widetilde{N} = \widetilde{k}_1 X_{,s}$ into account and using Eq. (11), we obtain

$$\widetilde{k}_1 = \lambda k_1 - 2\dot{\lambda} \langle X, N \rangle , \tag{12}$$

which concludes the proof. □

3.1. **Rotational Surfaces with a light-like axis in $\left(\mathbb{E}_3^1\right)_{\lambda(r)}$.** We now consider the Gauss and extrinsic curvatures of a non-degenerate rotational surface in conformally flat pseudo-spaces $\left(\mathbb{E}_3^1\right)_{\lambda(r)}$, as it relates to the radial conformal factor. As mentioned in the introduction, helicoidal surfaces are described as the general category to which rotational surfaces belong. For this reason, the ability to define helicoidal surfaces in conformally flat pseudo-spaces, as made possible in [14], also allows for the definition of a new type of rotational surface in these spaces.

Let $\gamma(s) = (s, 0, f(s))$, $s > 0$ be a curve $x_1 x_3$-plane defined on $I \subset$ R, which is called the profile curve. Applying this curve to the rotation in Eq. (4), in the following way:

$$\begin{pmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ -\frac{t^2}{2} & -t & 1 \end{pmatrix} \begin{pmatrix} s \\ 0 \\ f(s) \end{pmatrix},$$

we get a non-degenerate surface given by the parametric form

$$X : I \times \mathbb{R} \to \left(\mathbb{E}_3^1\right)_{\lambda(r)}$$

$$(s,t) \to X(s,t) = \left(s, st, f(s) - \frac{st^2}{2}\right), \tag{13}$$

which implies that it is a rotational surface in $\left(\mathbb{E}_3^1\right)_{\lambda(r)}$, where $f(s)$ is a function defined on an open interval I of $\mathbb{R}$.

Lemma 1. *Let $X(s,t) = \left(s, st, f(s) - \frac{st^2}{2}\right)$ be a rotational surface in $\left(\mathbb{E}_3^1\right)_{\lambda(r)}$. Thus, the Gaussian curvature of X is computed as*

$$K = \frac{-\epsilon \lambda^2}{s\sqrt{2f'}} \frac{\partial}{\partial s} \left(\frac{\lambda - 2\dot{\lambda} s \left(f + s f'\right))}{\lambda \sqrt{2f'}} \right), \tag{14}$$

where $\dot{\lambda} = \frac{d\lambda}{dr}$ and $\epsilon = \pm 1$.

Proof. To find the Gaussian curvature of X in the conformally flat pseudo-space $\left(\mathbb{E}_3^1\right)_{\lambda(r)}$, we need to calculate the coefficients of the first fundamental form of X with respect to the conformal metric. Then, it is easily seen that

$$\widetilde{E} = \frac{2f'}{\lambda^2}, \quad \widetilde{F} = 0 \quad \text{and} \quad \widetilde{G} = \frac{s^2}{\lambda^2}. \tag{15}$$

Due to $\widetilde{F}=0$, we have from [17] the knowledge that there is a formula for calculating the Gaussian curvature in the Euclidean version. Based on this knowledge, we modify, in the Minkowskian version, the formula of Gaussian curvature such that

$$K=\frac{-\epsilon}{2\sqrt{\widetilde{E}\widetilde{G}}}\left(\frac{\partial}{\partial t}\left(\frac{\widetilde{E}_t}{\sqrt{\widetilde{E}\widetilde{G}}}\right)+\frac{\partial}{\partial s}\left(\frac{\widetilde{G}_s}{\sqrt{\widetilde{E}\widetilde{G}}}\right)\right). \tag{16}$$

Hence, together with $\widetilde{E}_t=0$ and $\widetilde{G}_s=\frac{2s\lambda^2-4\lambda\dot{\lambda}\left(f+sf'\right)s^2}{\lambda^4}$, using Eq. (16), we get Eq. (14). This concludes the proof.

□

Theorem 2. *Let $X(s,t)=\left(s,st,f(s)-\frac{st^2}{2}\right)$ be a rotational surface in $\left(\mathbb{E}_3^1\right)_{\lambda(r)}$. Thus, $X(s,t)$ is flat in $\left(\mathbb{E}_3^1\right)_{\lambda(r)}$ if and only if $\lambda=\lambda(2sf)=e^{-\int\frac{c_1\sqrt{2f'}-1}{s}ds}$, $c_1\neq 0$.*

Proof. It is clear from Eq. (14) that the necessary condition for X to be flat in $\left(\mathbb{E}_3^1\right)_{\lambda(r)}$ have to satisfy the following equation

$$\frac{s\lambda-2s^2\dot{\lambda}\left(f+sf'\right)}{\lambda\sqrt{2s^2f'}}=c_1. \tag{17}$$

Hence, if $c_1=0$, we get a contradiction about the completeness of the metric. If $c_1\neq 0$, then Eq. (17) becomes $\frac{\dot{\lambda}}{\lambda}=\frac{c_1\sqrt{2f'}-1}{2s(f+sf')}$. By integrating both sides, we obtain the desired outcome. □

Lemma 2. *Let $X(s,t)=\left(s,st,f(s)-\frac{st^2}{2}\right)$ be a rotational surface in $\left(\mathbb{E}_3^1\right)_{\lambda(r)}$. Thus, the extrinsic curvature of X is computed as*

$$\widetilde{K}_E=\frac{-\epsilon}{4sf'^2}\left(\lambda f''-4\dot{\lambda}f'\left(f-sf'\right)\right)\left(\lambda+2s\dot{\lambda}\left(f-sf'\right)\right), \tag{18}$$

where $\epsilon=\pm 1$.

Proof. If we proceed through the steps of proving Lemma (1) for the Minkowskian metric, then the coefficients of the first fundamental form are as follows:

$$E=2f',\quad F=0\ \text{ and }\ G=s^2, \tag{19}$$

and the coefficients of the second fundamental form are calculated as

$$e=-\frac{sf''}{\alpha},\quad f=0\ \text{ and }\ g=\frac{s^2}{\alpha}, \tag{20}$$

where $\alpha=\sqrt{2s^2f'}$. On the other hand, taking into account the partial derivatives of X, we find

$$\widetilde{k}_i=\lambda k_i-4\dot{\lambda}\frac{sff'\left(1-sf'\right)}{\alpha}. \tag{21}$$

Ultimately, using together Eqs. (19) and (20) with Eq. (21), we get

$$\widetilde{K}_E = \widetilde{k}_1 \widetilde{k}_2 = \frac{-\epsilon}{4sf'^2}\left(\lambda f'' - 4\dot{\lambda} f'\left(f - sf'\right)\right)\left(\lambda + 2s\dot{\lambda}\left(f - sf'\right)\right). \qquad (22)$$

□

Theorem 3. *Let* $X(s,t) = \left(s, st, f(s) - \frac{st^2}{2}\right)$ *be a rotational surface in* $\left(\mathbb{E}_3^1\right)_{\lambda(r)}$. *Thus,* $X(s,t)$ *has zero extrinsic curvature in* $\left(\mathbb{E}_3^1\right)_{\lambda(r)}$ *if and only if either one of the next two equations*

$$\lambda = \lambda(2sf) = \frac{c_1\sqrt{f'}}{f - sf'} \text{ or } \lambda = \lambda(2sf) = e^{-\int \frac{f+sf'}{s(f-sf')}ds} \qquad (23)$$

are satisfied, where c_1 *is a positive real number.*

Proof. In order for X to have zero extrinsic curvature in $\left(\mathbb{E}_3^1\right)_{\lambda(r)}$, the following equations must be met:

$$\lambda f'' - 4\dot{\lambda} f'\left(f - sf'\right) = 0 \text{ or } \lambda + 2s\dot{\lambda}\left(f - sf'\right) = 0.$$

Of these, the first one becomes $\frac{\dot{\lambda}}{\lambda} = \frac{f''}{4f'(f-sf')}$. Using the integration, we get $\lambda = \frac{c_1\sqrt{f'}}{f-sf'}$. As similar to this, we find the other one. The proof concludes here. □

Remark 1. *In the first equality of Eq. (23), for* $\lambda(r) = \frac{1}{\sqrt{r}}$, *rotational surfaces* X *with zero extrinsic curvature are rational kinds. More clearly, from Eq. (18), when* $\lambda(r) = \frac{1}{\sqrt{r}}$, $\widetilde{K}_E = 0$ *if and only if* $sff'' + ff' - sf'^2 = 0$, *whose general solution is* $f(s) = ns^m$, *where* m *is a constant and* n *is a positive real number. Rotational surfaces with zero extrinsic curvature can be determined to be polynomial in character with isothermal parameters by a special solution of the differential equation mentioned above. In the second one, for* $\lambda(r) = e^{-r}$, $\widetilde{K}_E = 0$ *if and only if it satisfies the equation* $2s^2f' - 2sf + 1 = 0$, *which ensures that the general solution is* $f(s) = ms + \frac{1}{4s}$, *where* m *is a real number. By using a special solution of the differential equation, we just talked about above, we can figure out that rotational surfaces with zero extrinsic curvature are of constant Gaussian curvature. Both conformal factors are useful, but in different ways for different models, as was mentioned in the introduction.*

Example 1. *Let's use Theorem 3 to describe a rotational surface with zero extrinsic curvature in* $\left(\mathbb{E}_3^1\right)_{\frac{1}{\sqrt{r}}}$. *From Remark 1, for* $\lambda(r) = \frac{1}{\sqrt{r}}$, *we have the knowledge whose profile curve will be* $f(s) = ns^m$. *Substituting this profil curve into Eq. (13), we get the parametrization of a rotational surface with zero curvature surface as follows:*

$$X(s,t) = \left(s, st, ns^m - \frac{st^2}{2}\right).$$

We now plot it putting for $m = 3$ *and* $n = 2$. *See Fig. (1).*

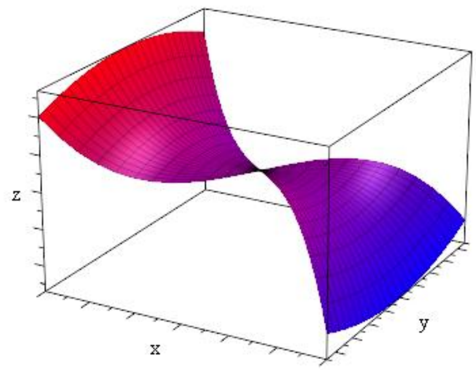

FIGURE 1. The graphic belongs to a rotational surface of rational kind with zero extrinsic curvature in $\left(\mathbb{E}_3^1\right)_{\frac{1}{\sqrt{r}}}$.

We also sketch it out with respect to the constants $m = 3$ and $n = \frac{1}{6}$ that serves as the isothermal parametrization condition. See Fig. (2).

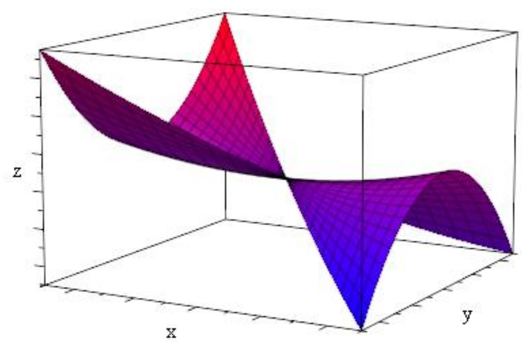

FIGURE 2. The graphic belongs to a rotational surface of rational type with zero extrinsic curvature having the isothermal parameter in $\left(\mathbb{E}_3^1\right)_{\frac{1}{\sqrt{r}}}$.

Example 2. *As similar to Example (1), the profile curve of a rotational surface with zero curvature in $\left(\mathbb{E}_3^1\right)_{e^{-r}}$ is $f(s) = ms + \frac{1}{4s}$. Applying this to Eq. (13) yields*

$$X(s,t) = \left(s, st, ms + \frac{1}{4s} - \frac{st^2}{2}\right).$$

For $m = 1$, see Fig. (3).

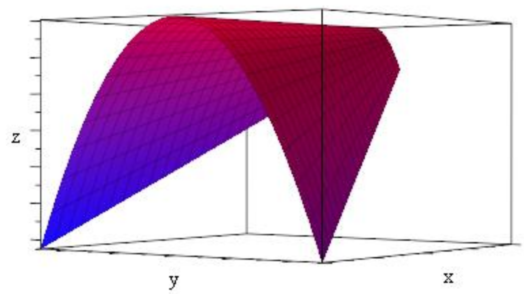

FIGURE 3. The graphic belongs to a rotational surface of with zero extrinsic curvature in $\left(\mathbb{E}_3^1\right)_{e^{-r}}$.

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