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RESEARCH PAPER

Optical solitons of the complex Ginzburg–Landau equation having dual power nonlinear form using φ^6-model expansion approach

Muhammad Abubakar Isah [1,*,‡] and **Asif Yokus** [1,2,‡]

[1]Department of Mathematics, Faculty of Science, Firat University, 23200 Elazig, Türkiye, [2]Application and Research Center Advisory Board Member, Istanbul Commerce University, Istanbul, Türkiye

*Corresponding Author
‡myphysics_09@hotmail.com (Muhammad Abubakar Isah); asfyokus@firat.edu.tr (Asif Yokus)

Abstract

This paper employs a novel φ^6-model expansion approach to get dark, bright, periodic, dark-bright, and singular soliton solutions to the complex Ginzburg-Landau equation with dual power-law nonlinearity. The dual-power law found in photovoltaic materials is used to explain nonlinearity in the refractive index. The results of this paper may assist in comprehending some of the physical effects of various nonlinear physics models. For example, the hyperbolic sine arises in the calculation of the Roche limit and the gravitational potential of a cylinder, the hyperbolic tangent arises in the calculation of the magnetic moment and the rapidity of special relativity, and the hyperbolic cotangent arises in the Langevin function for magnetic polarization. Frequency values, one of the soliton's internal dynamics, are used to examine the behavior of the traveling wave. Finally, some of the obtained solitons' three-, two-dimensional, and contour graphs are plotted.

Keywords: φ^6-model expansion method; complex Ginzburg–Landau equation; soliton solutions; dual power-law nonlinearity

AMS 2020 Classification: 35C07; 35C08; 35G20; 35C08

1 Introduction

The study of surfaces [1, 2] in geometry [3, 4] and a wide range of mechanical problems were the first implementations of partial differential equations. In the nineteenth century, eminent mathematicians from all over the world showed a significant interest in researching a variety of problems arising from partial differential equations [5]. Optical solitons have emerged as a key study issue in the physical and natural sciences. Solitons have been discovered to play an important role in several disciplines of research, including optical fibers [6, 7], plasma physics

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[8], nonlinear optics, and many others [9, 10]. Solitons may spread across trans-continental and trans-oceanic distances in fiber optics. Solitons are solutions to the nonlinear partial differential equations that describe a single moving wave. Optical solitons, which are formed as the result of a perfect balance between dispersion (or diffraction) and nonlinearity in a nonlinear medium, are frequently used in telecommunications and electromagnetics. Optical fibers are solutions to the Nonlinear Schrodinger equations. Soliton solutions contain particle-like structures, such as magnetic monopoles, as well as extended structures, such as domain walls and cosmic strings, which have implications in the cosmology of the early universe. Periodic solutions, such as $cos(x+t)$, are periodic traveling wave solutions [11, 12]. Dark soliton refers to a solitary wave with lower intensity than the background, bright soliton refers to a solitary wave with higher peak intensity than the background, and singular soliton refers to a solitary wave with discontinuous derivatives; examples of such solitary waves include compactions, which have finite (compact) support and peakons, whose peaks have a discontinuous derivative, dark solitons are modeled by the $\tan h$ functions whilst bright solitons are modeled by the $\sec h$. Understanding the dynamics of solitons can lead to a better understanding of the physics of the phenomena in which they exist. As a result, a number of sophisticated mathematical techniques have been developed to generate soliton solutions for a wide range of physical models such as the Kadomtsev–Petviashvili equation [13], the Benjamin–Ono equation [14], the disturbance effect in intracellular calcium dynamic on fibroblast cells [15], the Fisher equation [16], the nonlinear Schrödinger equation [17, 18], the Sharma–Tasso–Olver equation [19], the Murnaghan model [20], the Kaup-Kupershmidt equation [21], Navier–Stokes equation [22], the Zakharov–Kuznetsov equation [23], the B-type Kadomtsev–Petviashvili–Boussinesq equation [24] and others [25–27]. Recent analytical methods for solving PDEs, such as the eMETEM method [28], the generalized exponential rational function method [29], the extended sinh-Gordon equation expansion method [30], the q-homotopy analysis transform technique [31], the new extended direct algebraic method [32], the direct method [33], the Kudryashov's new function method [34], the split-step Fourier transform [35], the new modified unified auxiliary equation method [36], the $\left(\frac{1}{G'}\right)$-expansion method [37–39], the Jacobi elliptic functions [40].

The Ginzburg-Landau equation GLE is one of the most prominent partial differential equations in mathematics and physics, it was brought into the study of superconducting phenomenology theory by Ginzburg and Landau in the twentieth century. The GLE is commonly used to describe the propagation of optical solitons across optical fibers over extended distances. As a result, it is critical to examine the dynamic behavior of the GLE. Many researchers have recently solved the Complex Ginzburg-Landau equation CGLE with dual power law nonlinearity, among them; Arshed [41] solved this equation with the help of The exp(- $\phi(\xi)$)-expansion method and received different forms of solitons such as hyperbolic, rational and trigonometric functions. Jacobi's elliptic function expansion method is used to obtain some dark and periodic soliton solutions by Abdou et al. [42]. Al-Ghafri and Khalil [43] used The relation between the Weierstrass elliptic function and hyperbolic functions to derive optical soliton and period waves, in [44, 45], the trial solutions approach, exp(- $\phi(\xi)$)-expansion method and $\left(\frac{G'}{G^2}\right)$-expansion method are used, the $\left(\frac{G'}{G}\right)$ −method is used to secure the soliton solutions by Li Zhao, et al. [46], the other methods includes GPRE method [47].

The main objective of this study is to develop new solitons for the CGLE with dual power nonlinearity using the recently developed φ^6–model expansion method [18, 48], in which, to our knowledge, ıt has not been studied yet using the proposed technique. These new solitons include dark, bright, singular, rational, combined periodic, combined singular and periodic solitary wave

solutions.
The following is the outline for this paper; the mathematical analysis of the model will be given in Section 2. In Section 3, we will present the description of the φ^6–model expansion method. In Section 4, the φ^6–model expansion method will be applied to the CGLE model with dual power nonlinearity to get new travelling wave solutions to the model. Additionally, the physical structure of the traveling wave solution is graphically displayed in the related 2D, 3D and contour graphs. The physical dynamics of the soliton solutions are explored in Section 5, while the conclusions will be drawn in Section 6.

2 Mathematical analysis of the model

One of the most well-known partial differential equations in mathematics and physics, the Ginzburg-Landau equation was developed in the 20th century by Ginzburg and Landau and used to examine the superconducting phenomenology hypothesis. The propagation of optical solitons over optical fibers over long distances is frequently described using the GLE. The authors [41, 47] give the dimensionless shape of CGLE that will be investigated in this article as

$$iQ_t + aQ_{xx} + bF(|Q|^2)Q = \frac{1}{|Q|^2 Q^\star}\left[\alpha |Q|^2 (|Q|^2)_{xx} - \beta \left\{(|Q|^2)_x\right\}^2\right] + \gamma Q, \tag{1}$$

where x is the non-dimensional distance along the fibers and t is time in dimensionless form; a, b, α, β and γ are valued constants. The coefficients a and b are determined by the group velocity dispersion (GVD) and nonlinearity respectively. The terms with α, β and γ result from perturbation effects, specifically detuning.
The F in Eq.(1) is a real-valued algebraic function that must be smooth. $F(|Q|^2)Q$ is continuously differentiable k times,implying that

$$F(|Q|^2)Q \in \cup_{m,n=1}^{\infty} C^k\left((-n,n)\times(-m,m);R^2\right), \tag{2}$$

by setting

$$\alpha = 2\beta. \tag{3}$$

Eq. (1) turns to

$$iQ_t + aQ_{xx} + bF(|Q|^2)Q = \frac{\beta}{|Q|^2 Q^\star}\left[2|Q|^2 (|Q|^2)_{xx} - \left\{(|Q|^2)_x\right\}^2\right] + \gamma Q. \tag{4}$$

To solve Eq. (1), the standard decomposition into phase-amplitude components:

$$Q(x,t) = U(\zeta)e^{i(-kx+\omega t+\theta)}, \tag{5}$$

and the wave variable ζ is given by

$$\zeta = (x - vt), \tag{6}$$

the function U represents the pulse shape and v is the soliton's velocity. In the phase factor, k denotes the soliton frequency, ω the soliton wave number and the phase constant θ. Substituting the amplitude-phase decomposition into Eq. (4) results in the following couple of equations after breaking into real and imaginary parts:

$$-\left(ak^2+\gamma+\omega\right)U+bF\left(U^2\right)U+(a-4\beta)\,U''=0, \tag{7}$$

and

$$v=-2ka. \tag{8}$$

In the following part, Eq. (7) will be examined by using dual power law nonlinearity.

3 Description of the proposed technique

According to Zayed et al. [27, 41] the following are the key steps of a recent φ^6-model expansion method:
Step-1: Consider the following nonlinear evolution equation for $Q=Q(x,t)$.

$$G(Q,Q_x,Q_t,Q_{xx},Q_{xt},Q_{tt},...)=0, \tag{9}$$

here G is a polynomial of $Q(x,t)$ and its highest order partial derivatives, including its nonlinear terms.
Step-2: Making use of the wave transformation

$$Q(x,t)=Q(\zeta),\quad \zeta=x-vt, \tag{10}$$

where v represents wave velocity and Eq. (9) can be converted into the nonlinear ordinary differential equation shown below

$$\Omega(Q,Q',QQ',Q'',...)=0, \tag{11}$$

where the derivatives with respect to ζ are represented by the prime. **Step-3**: Suppose that the formal solution to Eq. (11) exists:

$$Q\left(\zeta\right)=\sum_{i=0}^{2N}\alpha_iU^i(\zeta), \tag{12}$$

where $\alpha_i(i=0,1,2,\ldots,N)$ are to be determined constants, N can be obtained using the balancing rule and $U(\zeta)$ satisfies the auxiliary NLODE;

$$\begin{aligned}U'^2(\zeta)&=h_0+h_2U^2(\zeta)+h_4U^4(\zeta)+h_6U^6(\zeta),\\ U''(\zeta)&=h_2U(\zeta)+2h_4U^3(\zeta)+3h_6U^5(\zeta),\end{aligned} \tag{13}$$

where $h_i(i=0,2,4,6)$ are real constants that will be discovered later.

Step-4: It is well known that the solution to Eq. (13) is as follows;

$$U(\zeta)=\frac{P(\zeta)}{\sqrt{fP^2(\zeta)+g}}, \tag{14}$$

provided that $0 < fP^2(\zeta) + g$ and $P(\zeta)$ is the Jacobi elliptic equation solution

$$P'^2(\zeta) = l_0 + l_2P^2(\zeta) + l_4P^4(\zeta), \tag{15}$$

where $l_i(i = 0, 2, 4)$ are unknown constants to be determined, f and g are given by

$$f = \frac{h_4(l_2 - h_2)}{(l_2 - h_2)^2 + 3l_0l_4 - 2l_2(l_2 - h_2)},$$
$$g = \frac{3l_0h_4}{(l_2 - h_2)^2 + 3l_0l_4 - 2l_2(l_2 - h_2)}, \tag{16}$$

under the restriction condition

$$h_4^2(l_2 - h_2)[9l_0l_4 - (l_2 - h_2)(2l_2 + h_2)] + 3h_6[-l_2^2 + h_2^2 + 3l_0l_4]^2 = 0. \tag{17}$$

Step-5: According to [18, 25–27, 41, 48] it is well known that the Jacobi elliptic solutions of Eq. (15) can be calculated when $0 < m < 1$. We can have the exact solutions of Eq. (9) by substituting Eqs. (14) and (15) into Eq. (12).

Function	$m \to 1$	$m \to 0$	Function	$m \to 1$	$m \to 0$
$sn(\zeta, m)$	$tanh(\zeta)$	$sin(\zeta)$	$ds(\zeta, m)$	$csch(\zeta)$	$csc(\zeta)$
$cn(\zeta, m)$	$sech(\zeta)$	$cos(\zeta)$	$sc(\zeta, m)$	$sinh(\zeta)$	$tan(\zeta)$
$dn(\zeta, m)$	$sech(\zeta)$	1	$sd(\zeta, m)$	$sinh(\zeta)$	$sin(\zeta)$
$ns(\zeta, m)$	$coth(\zeta)$	$csc(\zeta)$	$nc(\zeta, m)$	$cosh(\zeta)$	$sec(\zeta)$
$cs(\zeta, m)$	$csch(\zeta)$	$cot(\zeta)$	$cd(\zeta, m)$	1	$cos(\zeta)$

4 Application of the φ^6-model expansion method

The dual power law is found in photovoltaic materials and is used to explain non-linearity in the refractive index. The formula for this law is $F(U) = U^n + rU^{2n}$, where r is a constant. When $n = 1$, is the non-linearity of the parabolic law which is the subset of the dual-power law [41, 45, 49], when $r = 0$ and $n = 1$, the relationship becomes the Kerr law of non-linearity, which is formed from the fact that a light wave in an optical fiber undergoes nonlinear responses owing to non-harmonic electron mobility in the presence of an external electric field [41].
For this non-linearity, Eq. (7) is reduced to

$$-\left(ak^2 + \gamma + \omega\right)U + b\left(U^{2n+1} + rU^{4n+1}\right) + (a - 4\beta)U'' = 0, \tag{18}$$

where $N = \frac{1}{2n}$ is obtained by balancing the higher order derivative and the nonlinear term in the above equation. The following transformation is used to achieve closed-form solutions

$$U = p^{\frac{1}{2n}},$$

that will reduce Eq. (18) into the ODE

$$-4n^2\left(ak^2 + \gamma + \omega\right)p^2 + 4bn^2p^2\left(p + rp^2\right) + (a - 4\beta)\left(2npp'' + (1 - 2n)\left(p'\right)^2\right) = 0, \tag{19}$$

by balancing the highest order derivative and the highest nonlinear term in the above equation

$4N = 2N + 2$ we get $N = 1$, we obtain the following by substituting $N = 1$ in Eq. (12)

$$p(\zeta) = \alpha_0 + \alpha_1 U(\zeta) + \alpha_2 U^2(\zeta), \tag{20}$$

where α_0, α_1 and α_2 are constants to be determined.

We obtain the following algebraic equations by substituting Eq. (20) along with Eq. (13) into Eq. (19) and setting the coefficients of all powers of $U^i(\zeta)$, $i = 0, 1, \ldots, 8$ to zero

$$\begin{aligned}
U^0(\zeta) \;\; &; \;\; -4n^2\left(ak^2+\gamma+\omega\right)\alpha_0^2 + 4bn^2\alpha_0^3\left(1+r\alpha_0\right) \\
&\quad -\left(-1+2n\right)\left(a-4\beta\right)h_0\alpha_1^2 + 4n\left(a-4\beta\right)h_0\alpha_0\alpha_2 = 0, \\
U^1(\zeta) \;\; &; \;\; -2\alpha_1\left(n\alpha_0\left(4n\left(ak^2+\gamma+\omega\right) - \left(a-4\beta\right)h_2 - 2bn\alpha_0\left(3+4r\alpha_0\right)\right) + 2\left(-1+n\right)\left(a-4\beta\right)h_0\alpha_2\right) = 0, \\
U^2(\zeta) \;\; &; \;\; \left(\left(a-4\beta\right)h_2 - 4n^2\left(ak^2+\gamma+\omega-3b\alpha_0\left(1+2r\alpha_0\right)\right)\right)\alpha_1^2 \\
&\quad +4\alpha_2\left(n\alpha_0\left(-2n\left(ak^2+\gamma+\omega\right)+2\left(a-4\beta\right)h_2+bn\alpha_0\left(3+4r\alpha_0\right)\right)-\left(-1+n\right)\left(a-4\beta\right)h_0\alpha_2\right) = 0, \\
U^3(\zeta) \;\; &; \;\; 2\alpha_1^3\left(2n\left(a-4\beta\right)h_4\alpha_0 + 2bn^2\left(1+4r\alpha_0\right)\right) \\
&\quad +2\alpha_1\alpha_2\left(\left(2+n\right)\left(a-4\beta\right)h_2 - 4n^2\left(ak^2+\gamma+\omega-3b\alpha_0\left(1+2r\alpha_0\right)\right)\right) = 0, \\
U^4(\zeta) \;\; &; \;\; 4bn^2r\alpha_1^4 + 12bn^2\left(1+4r\alpha_0\right)\alpha_1^2\alpha_2 + \left(a-4\beta\right)h_4\left(\left(1+2n\right)\alpha_1^2 + 12n\alpha_0\alpha_2\right) \\
&\quad -4\left(-\left(a-4\beta\right)h_2 + n^2\left(ak^2+\gamma+\omega-3b\alpha_0\left(1+2r\alpha_0\right)\right)\right)\alpha_2^2 = 0, \\
U^5(\zeta) \;\; &; \;\; 6\alpha_1 n\left(a-4\beta\right)h_6\alpha_0 + 4\alpha_1\alpha_2\left(\left(1+2n\right)\left(a-4\beta\right)h_4 + bn^2\left(4r\alpha_1^2 + 3\left(1+4r\alpha_0\right)\alpha_2\right)\right) = 0, \\
U^6(\zeta) \;\; &; \;\; \left(a-4\beta\right)h_6\left(\left(1+4n\right)\alpha_1^2 + 16n\alpha_0\alpha_2\right) + 4\alpha_2^2\left(\left(1+n\right)\left(a-4\beta\right)h_4 + bn^2\left(6r\alpha_1^2+\alpha_2+4r\alpha_0\alpha_2\right)\right) = 0, \\
U^7(\zeta) \;\; &; \;\; 2\alpha_1\alpha_2\left(\left(2+7n\right)\left(a-4\beta\right)h_6 + 8bn^2r\alpha_2^2\right) = 0, \\
U^8(\zeta) \;\; &; \;\; 4\left(\left(1+2n\right)\left(a-4\beta\right)h_6\alpha_2^2 + bn^2r\alpha_2^4\right) = 0.
\end{aligned}$$

We get the following result after solving the resulting system:

$$\begin{aligned}
\alpha_0 &= \frac{-1-2n}{4\left(1+n\right)r}, \quad \alpha_1 = \frac{\sqrt{\left(1+2n\right)h_4}\sqrt{-a+4\beta}}{2n\sqrt{br}}, \quad \alpha_2 = 0, \\
h_0 &= \frac{b^2n^4\left(1+2n\right)^2}{16r^2h_4\left(1+n\right)^4\left(a-4\beta\right)^2}, \quad h_2 = \frac{bn^2\left(1+2n\right)}{2r\left(1+n\right)^2\left(a-4\beta\right)}, \\
h_6 &= 0, \quad \omega = -1\left(ak^2+\gamma+\frac{b\left(1+2n\right)}{4r\left(1+n\right)^2}\right).
\end{aligned} \tag{21}$$

In view of Eqs. (14), (20) and (21) along with the Jacobi elliptic functions in the table above, we obtain the following exact solutions of Eq. (18).

1. If $l_0 = 1$, $l_2 = -(1+m^2)$, $l_4 = m^2$, $0 < m < 1$, then $P(\zeta) = sn(\zeta, m)$ or $P(\zeta) = cd(\zeta, m)$, we have

$$Q_1\left(x,t\right) = \left[\frac{-1-2n}{4(1+n)r} + \frac{\sqrt{(1+2n)h_4}\sqrt{-a+4\beta}}{2n\sqrt{br}}\left(\frac{sn(\zeta,m)}{\sqrt{f(sn(\zeta,m))^2+g}}\right)\right]^{\frac{1}{2n}} e^{i(-kx+\omega t+\theta)}, \tag{22}$$

or

$$Q_{1,0}\left(x,t\right)=\left[\frac{-1-2n}{4(1+n)r}+\frac{\sqrt{(1+2n)h_4}\sqrt{-a+4\beta}}{2n\sqrt{br}}\left(\frac{cd(\zeta,m)}{\sqrt{f(cd(\zeta,m))^2+g}}\right)\right]^{\frac{1}{2n}}e^{i(-kx+\omega t+\theta)}, \quad (23)$$

such that $0<2n\sqrt{br}\left(4\left(1+n\right)r\right)$, $\zeta=x-vt$ and f and g in Eq. (16) are given by

$$f=\frac{(1+m^2+h_2)h_4}{1-m^2+m^4-h_2^2}, \qquad g=\frac{-3h_4}{1-m^2+m^4-h_2^2},$$

under the restriction condition

$$-h_4^2\left(-1-m^2-h_2\right)\left(-1+2m^2-h_2\right)\left(-2+m^2+h_2\right)=0.$$

If $m\longrightarrow 1$, then the soliton solution is obtained

$$Q_{1,1}\left(x,t\right)=\left[\frac{-1-2n}{4(1+n)r}+\frac{\sqrt{(1+2n)h_4}\sqrt{-a+4\beta}}{2n\sqrt{br}}\left(\frac{\tanh(\zeta)}{\sqrt{\frac{-h_4\left(-3+(2+h_2)\tanh^2(\zeta)\right)}{-1+h_2^2}}}\right)\right]^{\frac{1}{2n}}e^{i(-kx+\omega t+\theta)}, \quad (24)$$

such that

$$h_4^2\left(-2-h_2\right)\left[-1+h_2\right]^2=0.$$

If $m\longrightarrow 0$, then the periodic wave solution is obtained

$$Q_{1,2}\left(x,t\right)=\left[\frac{-1-2n}{4(1+n)r}+\frac{\sqrt{(1+2n)(-a+4\beta)h_4}}{2n\sqrt{br}}\left(\frac{\sin(\zeta)}{\sqrt{\frac{-\left(-3+(1+h_2)\sin^2(\zeta)\right)h_4}{-1+h_2^2}}}\right)\right]^{\frac{1}{2n}}e^{i(-kx+\omega t+\theta)}, \quad (25)$$

such that

$$h_4^2\left(-1-h_2\right)\left[\left(-2+h_2\right)\left(1+h_2\right)\right]=0.$$

2. If $l_0=1-m^2, l_2=2m^2-1, l_4=-m^2, 0<m<1$, then $P(\zeta)=cn(\zeta,m)$, therefore

$$Q_2\left(x,t\right)=\left[\frac{-1-2n}{4(1+n)r}+\frac{\sqrt{(1+2n)(-a+4\beta)h_4}}{2n\sqrt{br}}\left(\frac{cn(\zeta,m)}{\sqrt{f(cn(\zeta,m))^2+g}}\right)\right]^{\frac{1}{2n}}e^{i(-kx+\omega t+\theta)}, \quad (26)$$

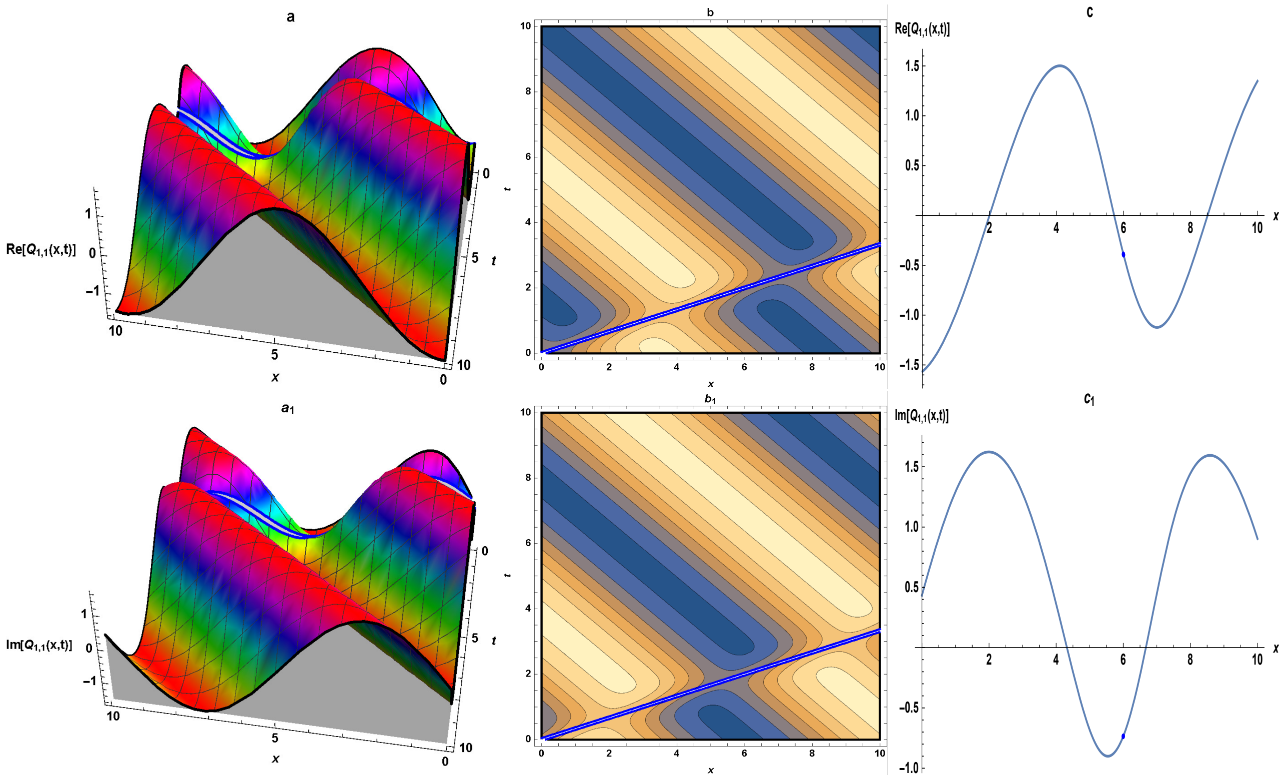

Figure 1. The 3Ds (a), (a_1), contours (b),(b_1) and 2Ds (c),(c_1) graphs of Eq. (24)

where f and g are determined by

$$f=-\frac{(-1+2m^2-h_2)h_4}{1-m^2+m^4-h_2^2},\qquad g=\frac{3\left(-1+m^2\right)h_4}{1-m^2+m^4-h_2^2},$$

under the constraint condition

$$h_4^2\left(-1+2m^2-h_2\right)\left[\left(-2+m^2+h_2\right)\left(1+m^2+h_2\right)\right]=0.$$

If $m\to 1$, then the optical soliton is retrieved

$$Q_{2,1}\left(x,t\right)=\left[\frac{-1-2n}{4\left(1+n\right)r}+\frac{\sqrt{(1+2n)\left(-a+4\beta\right)h_4}}{2n\sqrt{br}}\left(\frac{sech(\zeta)}{\sqrt{\frac{-sech^2(\zeta)h_4}{1+h_2}}}\right)\right]^{\frac{1}{2n}}e^{i(-kx+\omega t+\theta)},\qquad(27)$$

provided that

$$h_4^2\left(1-h_2\right)\left[h_2^2+h_2-2\right]=0.$$

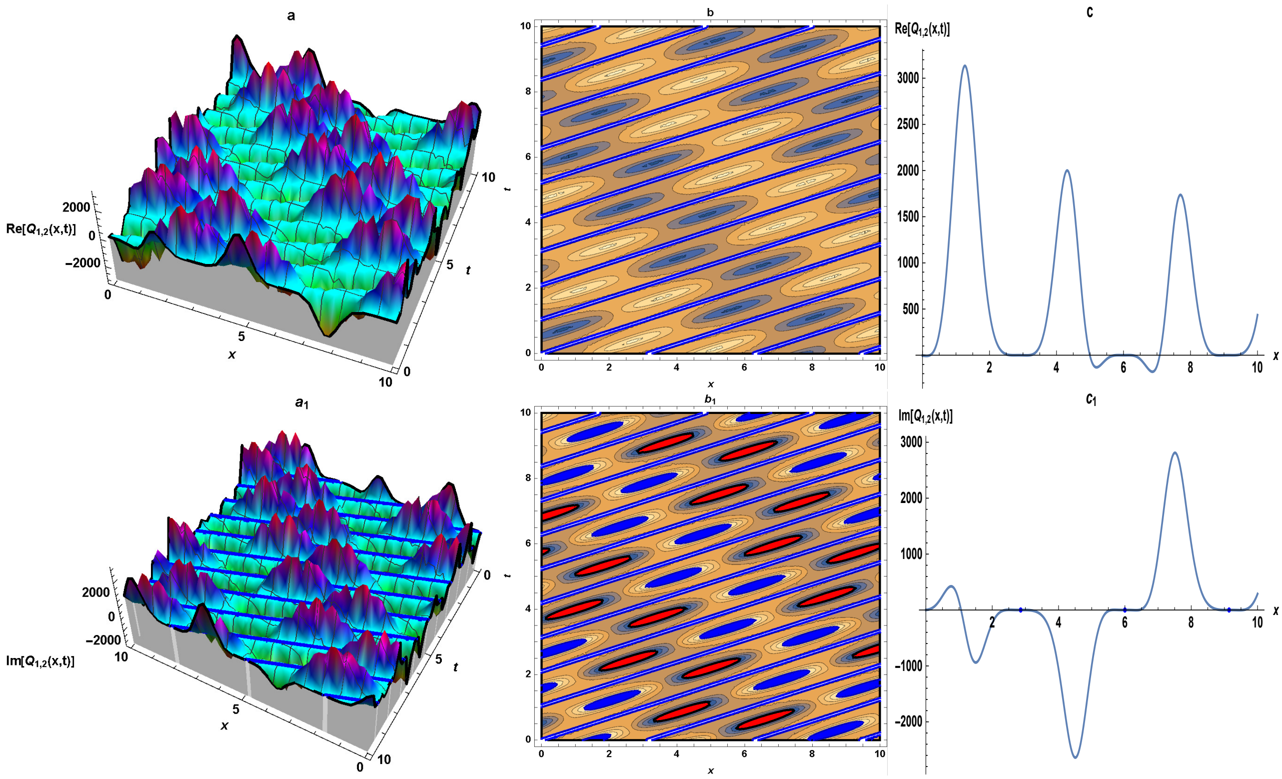

Figure 2. The 3Ds (a), (a_1), contours (b),(b_1) and 2Ds (c),(c_1) graphs of Eq. (25)

If $m \to 0$, then the periodic wave solution is obtained

$$Q_{2,2}\,(x,t) = \left[\frac{-1-2n}{4(1+n)r} + \frac{\sqrt{(1+2n)(-a+4\beta)h_4}}{2n\sqrt{br}}\left(\frac{\sin(\zeta)}{\sqrt{\frac{-\left(-3+(1+h_2)\sin^2(\zeta)\right)h_4}{-1+h_2^2}}}\right)\right]^{\frac{1}{2n}} e^{i(-kx+\omega t+\theta)}, \quad (28)$$

such that

$$h_4^2\,(-1-h_2)\,[(-2+h_2)\,(1+h_2)] = 0.$$

3. If $l_0 = m^2-1, l_2 = 2-m^2, l_4 = -1, 0 < m < 1$, then $P(\zeta) = dn(\zeta, m)$ which gives

$$Q_3\,(x,t) = \left[\frac{-1-2n}{4(1+n)r} + \frac{\sqrt{(1+2n)(-a+4\beta)h_4}}{2n\sqrt{br}}\left(\frac{dn(\zeta,m)}{\sqrt{f(dn(\zeta,m))^2+g}}\right)\right]^{\frac{1}{2n}} e^{i(-kx+\omega t+\theta)}, \quad (29)$$

where f and g are determined by

$$f = \frac{(-2+m^2+h_2)h_4}{1-m^2+m^4-h_2^2}, \qquad g = \frac{-3\left(-1+m^2\right)h_4}{1-m^2+m^4-h_2^2},$$

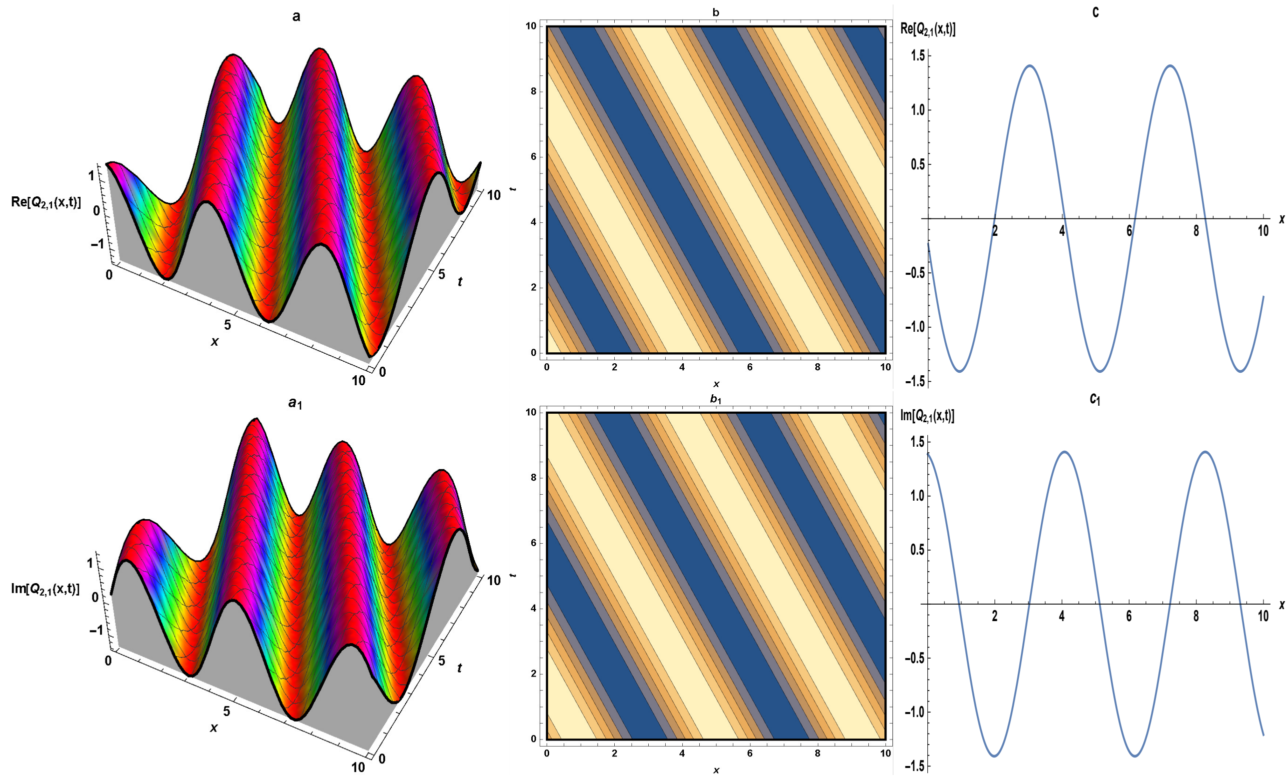

Figure 3. The 3Ds (a), (a_1), contours (b),(b_1) and 2Ds (c),(c_1) graphs of Eq. (27)

under the restriction condition

$$h_4^2\left(2-m^2-h_2\right)\left[-\left(-1+2m^2+h_2\right)\left(1+m^2+h_2\right)\right]=0.$$

If $m \to 1$, then the soliton solution is obtained

$$Q_{3,1}\left(x,t\right)=\left[\frac{-1-2n}{4\left(1+n\right)r}+\frac{\sqrt{\left(1+2n\right)\left(-a+4\beta\right)h_4}}{2n\sqrt{br}}\left(\frac{sech(\zeta)}{\sqrt{\frac{-sech^2(\zeta)h_4}{1+h_2}}}\right)\right]^{\frac{1}{2n}}e^{i(-kx+\omega t+\theta)}, \tag{30}$$

provided that

$$h_4^2\left(1-h_2\right)\left[-2+h_2+h_2^2\right]=0.$$

If $m \to 0$, then the rational solution is obtained

$$Q_{3,2}\left(x,t\right)=\left[\frac{-1-2n}{4\left(1+n\right)r}+\frac{\sqrt{\left(1+2n\right)\left(-a+4\beta\right)h_4}}{2n\sqrt{br}}\left(\frac{1}{\sqrt{\frac{-h_4}{-1+h_2}}}\right)\right]^{\frac{1}{2n}}e^{i(-kx+\omega t+\theta)}, \tag{31}$$

such that

$$h_4^2\left(2-h_2\right)\left[\left(1+h_2\right)^2\right]=0.$$

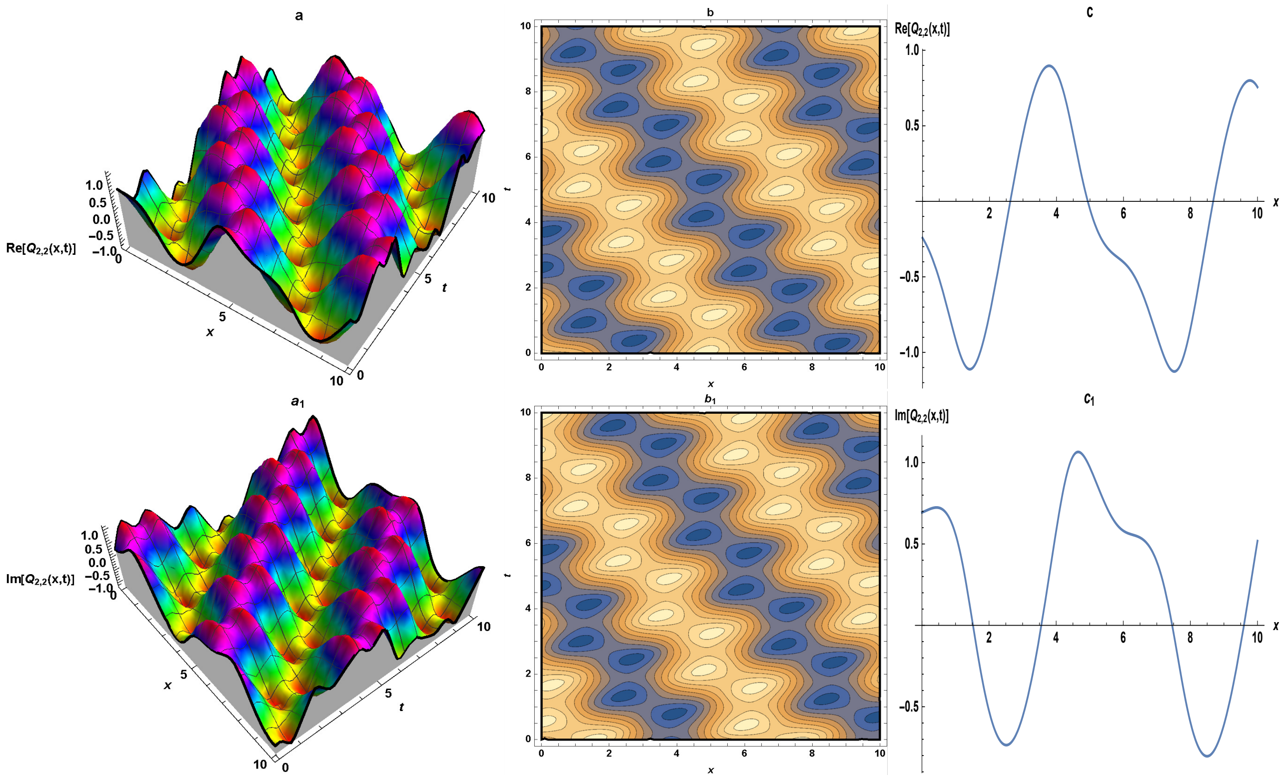

Figure 4. The 3Ds (a), (a_1), contours (b),(b_1) and 2Ds (c),(c_1) graphs of Eq. (28)

4. If $l_0 = m^2, l_2 = -\left(1+m^2\right), l_4 = 1, 0 < m < 1$, then $P(\zeta) = ns(\zeta, m)$ or $P(\zeta) = dc(\zeta, m)$, then

$$\mathrm{Q}_{4,1}\left(x,t\right) = \left[\frac{-1-2n}{4(1+n)r} + \frac{\sqrt{(1+2n)(-a+4\beta)h_4}}{2n\sqrt{br}}\left(\frac{ns(\zeta,m)}{\sqrt{f(ns(\zeta,m))^2+g}}\right)\right]^{\frac{1}{2n}} e^{i(-kx+\omega t+\theta)}, \quad (32)$$

or

$$\mathrm{Q}_{4,2}\left(x,t\right) = \left[\frac{-1-2n}{4(1+n)r} + \frac{\sqrt{(1+2n)(-a+4\beta)h_4}}{2n\sqrt{br}}\left(\frac{dc(\zeta,m)}{\sqrt{f(dc(\zeta,m))^2+g}}\right)\right]^{\frac{1}{2n}} e^{i(-kx+\omega t+\theta)}, \quad (33)$$

where f and g are given by

$$f = \frac{(1+m^2+h_2)h_4}{1-m^2+m^4-h_2^2}, \qquad g = \frac{-3m^2h_4}{1-m^2+m^4-h_2^2},$$

under the constraint condition

$$h_4^2\left(-1-m^2-h_2\right)\left[-\left(-1+2m^2-h_2\right)\left(-2+m^2+h_2\right)\right] = 0.$$

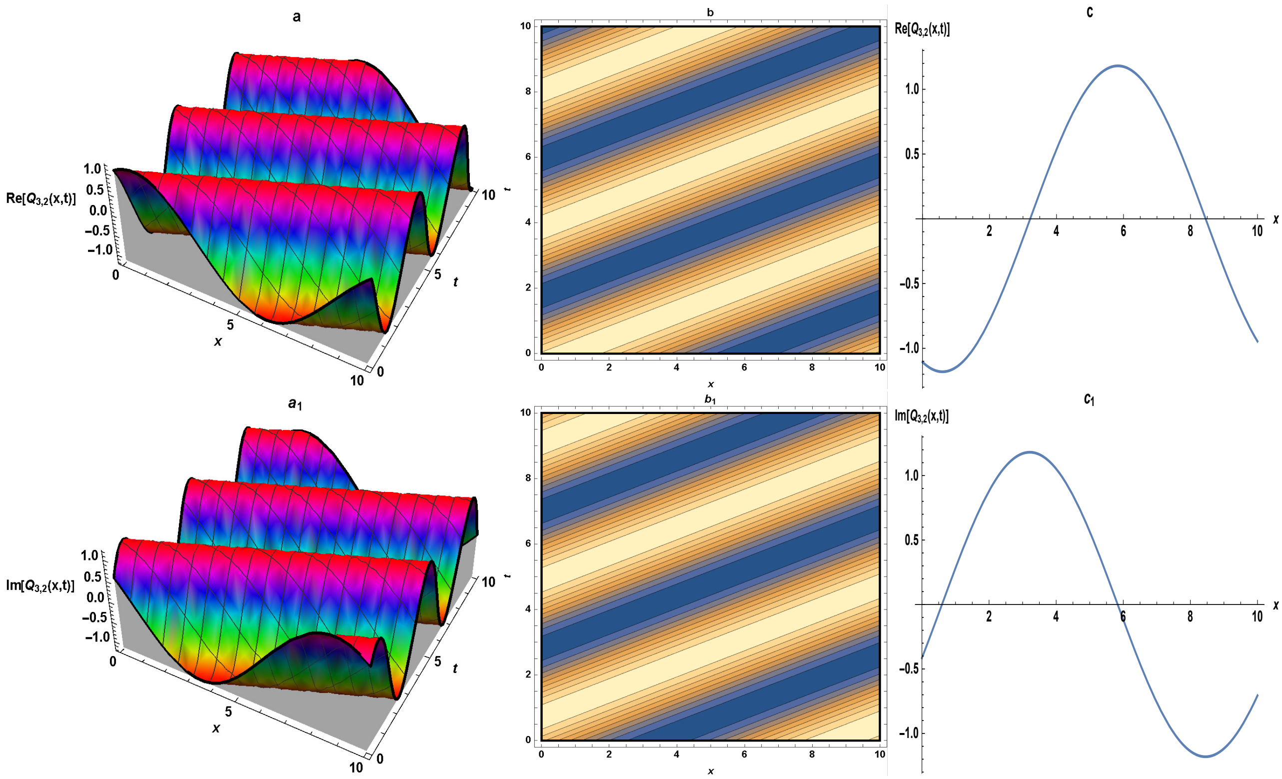

Figure 5. The 3Ds (a), (a_1), contours (b),(b_1) and 2Ds (c),(c_1) graphs of Eq. (31)

If $m \to 1$, then the dark singular soliton solution is obtained

$$Q_{4,3}\left(x,t\right) = \left[\frac{-1-2n}{4(1+n)r} + \frac{\sqrt{(1+2n)(-a+4\beta)h_4}}{2n\sqrt{br}}\left(\frac{\coth(\zeta)}{\sqrt{\frac{-(-1+h_2+(2+h_2)csch^2(\zeta))h_4}{-1+h_2^2}}}\right)\right]^{\frac{1}{2n}} e^{i(-kx+\omega t+\theta)}, \quad (34)$$

such that

$$h_4^2\left(-2-h_2\right)\left[\left(-1+h_2\right)^2\right] = 0.$$

If $m \to 0$, then the periodic wave solution is obtained

$$p_{4,4}\left(x,t\right) = \left[\frac{-1-2n}{4\left(1+n\right)r} + \frac{\sqrt{\left(1+2n\right)\left(-a+4\beta\right)h_4}}{2n\sqrt{br}}\left(\frac{\csc(\zeta)}{\sqrt{\frac{-\csc^2(\zeta)h_4}{-1+h_2}}}\right)\right]^{\frac{1}{2n}} e^{i(-kx+\omega t+\theta)}, \quad (35)$$

such that

$$h_4^2\left(-1-h_2\right)\left[\left(-2+h_2\right)\left(1+h_2\right)\right] = 0.$$

5. If $l_0 = -m^2, l_2 = 2m^2 - 1, l_4 = 1 - m^2, 0 < m < 1$, then $P(\zeta) = nc(\zeta, m)$, we have

$$Q_5(x,t) = \left[\frac{-1-2n}{4(1+n)r} + \frac{\sqrt{(1+2n)(-a+4\beta)h_4}}{2n\sqrt{br}}\left(\frac{nc(\zeta,m)}{\sqrt{f(nc(\zeta,m))^2+g}}\right)\right]^{\frac{1}{2n}} e^{i(-kx+\omega t+\theta)}, \tag{36}$$

where f and g are given by

$$f = \frac{-(-1+2m^2-h_2)h_4}{1-m^2+m^4-h_2^2}, \qquad g = \frac{3m^2h_4}{1-m^2+m^4-h_2^2},$$

under the constraint condition

$$h_4^2\left(-1+2m^2-h_2\right)\left[\left(-2+m^2+h_2\right)\left(1+m^2+h_2\right)\right] = 0.$$

If $m \to 1$, then the singular soliton solution is obtained

$$Q_{5,1}(x,t) = \left[\frac{-1-2n}{4(1+n)r} + \frac{\sqrt{(1+2n)(-a+4\beta)h_4}}{2n\sqrt{br}}\left(\frac{\cosh(\zeta)}{\sqrt{\frac{\left(-3+(1-h_2)\cosh^2(\zeta)\right)h_4}{-1+h_2^2}}}\right)\right]^{\frac{1}{2n}} e^{i(-kx+\omega t+\theta)}, \tag{37}$$

such that

$$h_4^2(1-h_2)\left[-2+h_2+h_2^2\right] = 0.$$

If $m \to 0$, then the periodic wave solution is obtained

$$Q_{5,2}(x,t) = \left[\frac{-1-2n}{4(1+n)r} + \frac{\sqrt{(1+2n)(-a+4\beta)h_4}}{2n\sqrt{br}}\left(\frac{\sec(\zeta)}{\sqrt{\frac{-\sec^2(\zeta)h_4}{-1+h_2}}}\right)\right]^{\frac{1}{2n}} e^{i(-kx+\omega t+\theta)}, \tag{38}$$

such that

$$h_4^2(-1-h_2)\left[(-2+h_2)(1+h_2)\right] = 0.$$

6. If $l_0 = -1, l_2 = 2 - m^2, l_4 = -\left(1-m^2\right), 0 < m < 1$, then $P(\zeta) = nd(\zeta, m)$ and we have

$$Q_6(x,t) = \left[\frac{-1-2n}{4(1+n)r} + \frac{\sqrt{(1+2n)(-a+4\beta)h_4}}{2n\sqrt{br}}\left(\frac{nd(\zeta,m)}{\sqrt{f(nd(\zeta,m))^2+g}}\right)\right]^{\frac{1}{2n}} e^{i(-kx+\omega t+\theta)}, \tag{39}$$

where f and g are given by

$$f = \frac{(-2+m^2+h_2)h_4}{1-m^2+m^4-h_2^2}, \qquad g = \frac{3h_4}{1-m^2+m^4-h_2^2},$$

under the constraint condition

$$h_4^2\left(2-m^2-h_2\right)\left[-\left(-1+2m^2-h_2\right)\left(1+m^2+h_2\right)\right]=0.$$

If $m\to 1$, then the singular soliton solution is obtained

$$Q_{6,1}\left(x,t\right)=\left[\frac{-1-2n}{4(1+n)r}+\frac{\sqrt{(1+2n)(-a+4\beta)h_4}}{2n\sqrt{br}}\left(\frac{\cosh(\zeta)}{\sqrt{\frac{\left(-3+(1-h_2)\cosh^2(\zeta)\right)h_4}{-1+h_2^2}}}\right)\right]^{\frac{1}{2n}}e^{i(-kx+\omega t+\theta)},\quad(40)$$

such that

$$h_4^2\left(1-h_2\right)\left[-\left(1-h_2\right)\left(2+h_2\right)\right]=0.$$

If $m\to 0$, then the periodic wave solution is obtained

$$Q_{6,2}\left(x,t\right)=\left[\frac{-1-2n}{4\left(1+n\right)r}+\frac{\sqrt{\left(1+2n\right)\left(-a+4\beta\right)h_4}}{2n\sqrt{br}}\left(\frac{1}{\sqrt{\frac{-h_4}{-1+h_2}}}\right)\right]^{\frac{1}{2n}}e^{i(-kx+\omega t+\theta)},\quad(41)$$

such that

$$h_4^2\left(2-h_2\right)\left[\left(1+h_2\right)^2\right]=0.$$

7. If $l_0=1$, $l_2=2-m^2$, $l_4=1-m^2$, $0<m<1$, then $P(\zeta)=sc(\zeta,m)$, we have

$$Q_7\left(x,t\right)=\left[\frac{-1-2n}{4\left(1+n\right)r}+\frac{\sqrt{\left(1+2n\right)\left(-a+4\beta\right)h_4}}{2n\sqrt{br}}\left(\frac{sc(\zeta,m)}{\sqrt{f\left(sc(\zeta,m)\right)^2+g}}\right)\right]^{\frac{1}{2n}}e^{i(-kx+\omega t+\theta)},\quad(42)$$

where f and g are given by

$$f=\frac{(-2+m^2+h_2)h_4}{1-m^2+m^4-h_2^2},\qquad g=\frac{-3h_4}{1-m^2+m^4-h_2^2},$$

under the constraint condition

$$h_4^2\left(2-m^2-h_2\right)\left[-\left(-1+2m^2-h_2\right)\left(1+m^2+h_2\right)\right]=0.$$

If $m\to 1$, then the singular soliton solution is obtained

$$Q_{7,1}\left(x,t\right)=\left[\frac{-1-2n}{4(1+n)r}+\frac{\sqrt{(1+2n)(-a+4\beta)h_4}}{2n\sqrt{br}}\left(\frac{\sinh(\zeta)}{\sqrt{\frac{\left(3+(1-h_2)\sinh^2(\zeta)\right)h_4}{-1+h_2^2}}}\right)\right]^{\frac{1}{2n}}e^{i(-kx+\omega t+\theta)},\quad(43)$$

such that

$$h_4^2\left(1-h_2\right)\left[-2+h_2+h_2^2\right]=0.$$

If $m \rightarrow 0$, then the periodic wave solution is obtained

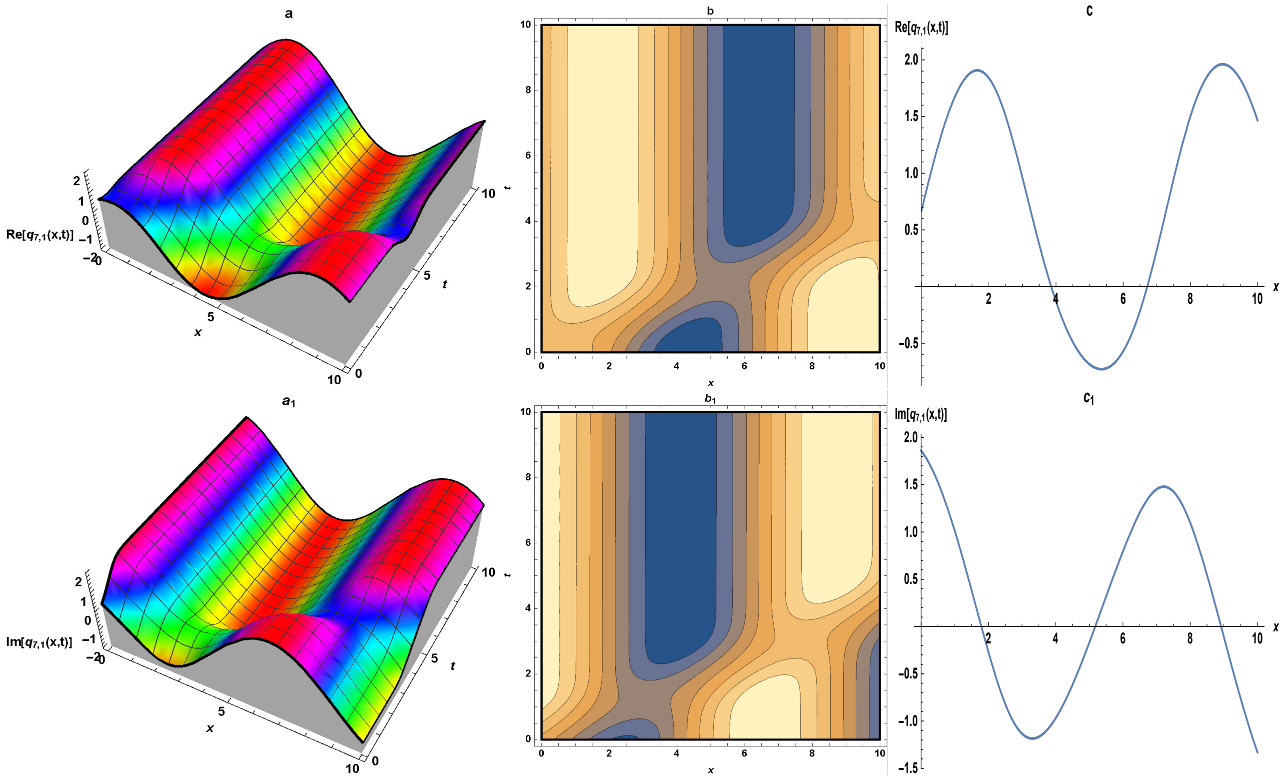

Figure 6. The 3Ds (a), (a_1), contours (b),(b_1) and 2Ds (c),(c_1) graphs of Eq. (43)

$$Q_{7,2}\left(x,t\right)=\left[\frac{-1-2n}{4(1+n)r}+\frac{\sqrt{(1+2n)(-a+4\beta)h_4}}{2n\sqrt{br}}\left(\frac{\tan(\zeta)}{\sqrt{\frac{(3-(-2+h_2)\tan^2(\zeta))h_4}{-1+h_2^2}}}\right)\right]^{\frac{1}{2n}}e^{i(-kx+\omega t+\theta)},\quad (44)$$

such that

$$h_4^2\left(2-h_2\right)\left[\left(1+h_2\right)^2\right]=0.$$

8. If $l_0=1, l_2=2m^2-1, l_4=-m^2\left(1-m^2\right), 0<m<1$, then $P(\zeta)=sd(\zeta,m)$, we have

$$Q_8\left(x,t\right)=\left[\frac{-1-2n}{4(1+n)r}+\frac{\sqrt{(1+2n)(-a+4\beta)h_4}}{2n\sqrt{br}}\left(\frac{sd(\zeta,m)}{\sqrt{f(sd(\zeta,m))^2+g}}\right)\right]^{\frac{1}{2n}}e^{i(-kx+\omega t+\theta)},\quad (45)$$

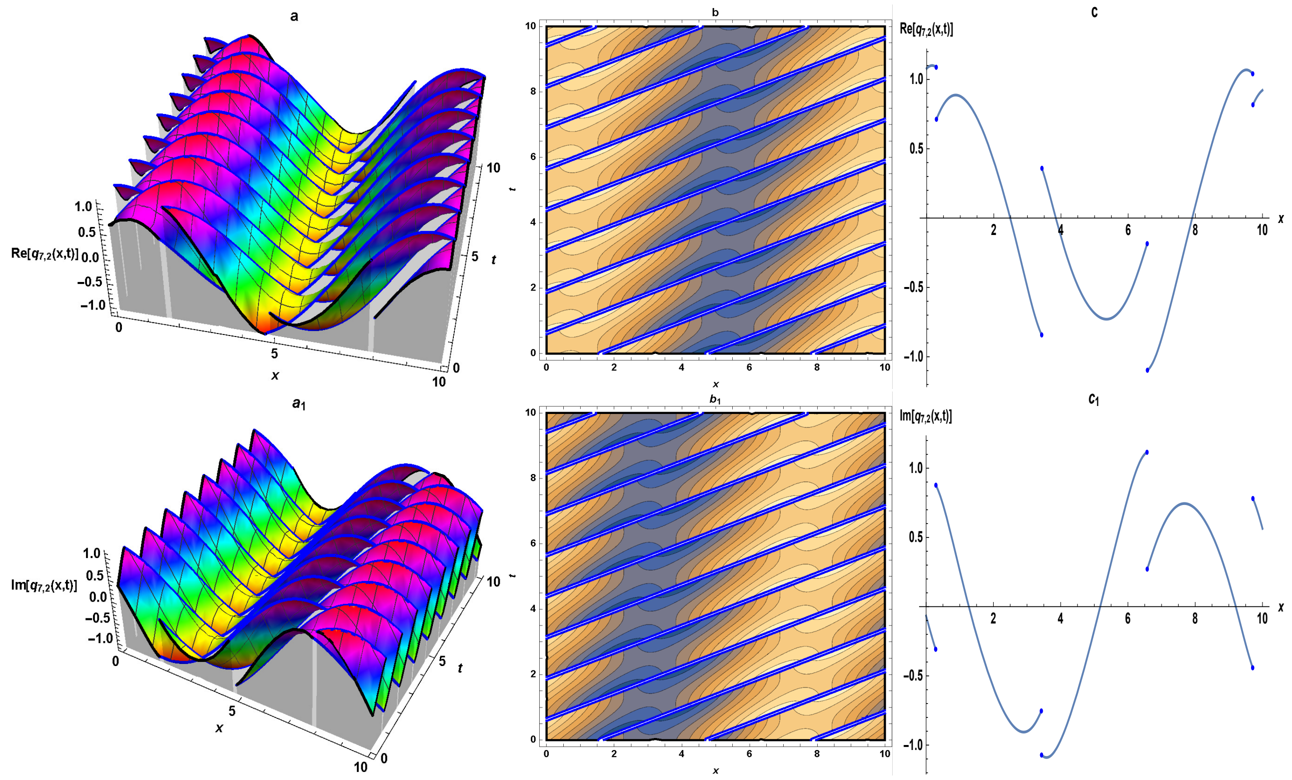

Figure 7. The 3Ds (a), (a_1), contours (b),(b_1) and 2Ds (c),(c_1) graphs of Eq. (44)

where f and g are given by

$$f=\frac{(-1+2m^2-h_2)h_4}{1-m^2+m^4-h_2^2},\qquad g=\frac{-3h_4}{1-m^2+m^4-h_2^2},$$

under the constraint condition

$$h_4^2\left(-1+2m^2-h_2\right)\left[\left(-2+m^2+h_2\right)\left(1+m^2+h_2\right)\right]=0.$$

9. If $l_0=1-m^2$, $l_2=2-m^2$, $l_4=1$, $0<m<1$, then $P(\zeta)=cs(\zeta,m)$, we have

$$Q_9\left(x,t\right)=\left[\frac{-1-2n}{4\left(1+n\right)r}+\frac{\sqrt{\left(1+2n\right)\left(-a+4\beta\right)h_4}}{2n\sqrt{br}}\left(\frac{cs(\zeta,m)}{\sqrt{f\left(cs(\zeta,m)\right)^2+g}}\right)\right]^{\frac{1}{2n}}e^{i(-kx+\omega t+\theta)},\quad(46)$$

where f and g are given by

$$f=\frac{(-2+m^2+h_2)h_4}{1-m^2+m^4-h_2^2},\qquad g=\frac{3(-1+m^2)h_4}{1-m^2+m^4-h_2^2},$$

under the constraint condition

$$h_4^2\left(2-m^2-h_2\right)\left[-\left(-1+2m^2-h_2\right)\left(1+m^2+h_2\right)\right]=0.$$

If $m \to 1$, then the singular soliton solution is obtained

$$Q_{9,1}(x,t) = \left[\frac{-1-2n}{4(1+n)r} + \frac{\sqrt{(1+2n)(-a+4\beta)h_4}}{2n\sqrt{br}} \left(\frac{csch(\zeta)}{\sqrt{\frac{-csch^2(\zeta)h_4}{1+h_2}}} \right) \right]^{\frac{1}{2n}} e^{i(-kx+\omega t+\theta)}, \quad (47)$$

such that

$$h_4^2(1-h_2)\left[-2+h_2+h_2^2\right] = 0.$$

If $m \to 0$, then the periodic wave solution is obtained

$$Q_{9,2}(x,t) = \left[\frac{-1-2n}{4(1+n)r} + \frac{\sqrt{(1+2n)(-a+4\beta)h_4}}{2n\sqrt{br}} \left(\frac{\cot(\zeta)}{\sqrt{\frac{(3+(2-h_2)\cot^2(\zeta))h_4}{-1+h_2^2}}} \right) \right]^{\frac{1}{2n}} e^{i(-kx+\omega t+\theta)}, \quad (48)$$

such that

$$h_4^2(2-h_2)\left[(1+h_2)^2\right] = 0.$$

10. If $l_0 = -m^2(1-m^2)$, $l_2 = 2m^2-1$, $l_4 = 1$, $0 < m < 1$, then $P(\zeta) = ds(\zeta,m)$, we have

$$Q_{10}(x,t) = \left[\frac{-1-2n}{4(1+n)r} + \frac{\sqrt{(1+2n)(-a+4\beta)h_4}}{2n\sqrt{br}} \left(\frac{ds(\zeta,m)}{\sqrt{f(ds(\zeta,m))^2+g}} \right) \right]^{\frac{1}{2n}} e^{i(-kx+\omega t+\theta)}, \quad (49)$$

where f and g are given by

$$f = \frac{-(-1+2m^2-h_2)h_4}{1-m^2+m^4-h_2^2}, \qquad g = \frac{-3m^2(-1+m^2)h_4}{1-m^2+m^4-h_2^2},$$

under the constraint condition

$$h_4^2\left(-1+2m^2-h_2\right)\left[\left(-2+m^2+h_2\right)\left(1+m^2+h_2\right)\right] = 0.$$

11. If $l_0 = \frac{1-m^2}{4}$, $l_2 = \frac{1+m^2}{2}$, $l_4 = \frac{1-m^2}{4}$, $0 < m < 1$, then $P(\zeta) = nc(\zeta,m) \pm sc(\zeta,m)$ or $P(\zeta) = \frac{cn(\zeta,m)}{1 \pm sn(\zeta,m)}$, we have

$$Q_{11,1}(x,t) = \left[\frac{-1-2n}{4(1+n)r} + \frac{\sqrt{(1+2n)(-a+4\beta)h_4}}{2n\sqrt{br}} \left(\frac{nc(\zeta,m) \pm sc(\zeta,m)}{\sqrt{f(nc(\zeta,m) \pm sc(\zeta,m))^2+g}} \right) \right]^{\frac{1}{2n}} e^{i(-kx+\omega t+\theta)}, \quad (50)$$

or

$$Q_{11,2}(x,t) = \left[\frac{-1-2n}{4(1+n)r} + \frac{\sqrt{(1+2n)(-a+4\beta)h_4}}{2n\sqrt{br}}\left(\frac{\frac{cn(\zeta,m)}{1\pm sn(\zeta,m)}}{\sqrt{f\left(\frac{cn(\zeta,m)}{1\pm sn(\zeta,m)}\right)^2+g}}\right)\right]^{\frac{1}{2n}} e^{i(-kx+\omega t+\theta)}, \quad (51)$$

where f and g are given by

$$f = \frac{-8(1+m^2-2h_2)h_4}{1+14m^2+m^4-16h_2^2}, \qquad g = \frac{12(-1+m^2)h_4}{1+14m^2+m^4-16h_2^2},$$

under the constraint condition

$$h_4^2\left(\frac{1}{2}\left(1+m^2-2h_2\right)\right)\left[\frac{1}{16}\left(1+(-6+m)\,m+4h_2\right)\left(1+m\,(6+m)+4h_2\right)\right] = 0.$$

If $m \to 1$, then the combined singular soliton solution

$$Q_{11,3}(x,t) = \left[\frac{-1-2n}{4(1+n)r} + \frac{\sqrt{(1+2n)(-a+4\beta)h_4}}{2n\sqrt{br}}\left(\frac{\sinh(\zeta)+\cosh(\zeta)}{\sqrt{\frac{-(\sinh(\zeta)+\cosh(\zeta))^2h_4}{1+h_2}}}\right)\right]^{\frac{1}{2n}} e^{i(-kx+\omega t+\theta)}, \quad (52)$$

or dark-bright optical soliton is obtained

$$Q_{11,4}(x,t) = \left[\frac{-1-2n}{4(1+n)\,r} + \frac{\sqrt{(1+2n)(-a+4\beta)\,h_4}}{2n\sqrt{br}}\left(\frac{\left(\frac{sech(\zeta)}{1+\tanh(\zeta)}\right)}{\sqrt{\frac{-\left(\frac{sech(\zeta)}{1+\tanh(\zeta)}\right)^2h_4}{1+h_2}}}\right)\right]^{\frac{1}{2n}} e^{i(-kx+\omega t+\theta)}, \quad (53)$$

such that

$$h_4^2\,(1-h_2)\left[-2+h_2+h_2^2\right] = 0.$$

If $m \to 0$, then the periodic wave solution is obtained

$$Q_{11,5}(x,t) = \frac{1}{2}\left[\frac{-1-2n}{4(1+n)r} + \frac{\sqrt{(1+2n)(-a+4\beta)h_4}}{2n\sqrt{br}}\left(\frac{\sec(\zeta)+\tan(\zeta)}{\sqrt{\frac{\left(3+2(1-2h_2)(\sec(\zeta)+\tan(\zeta))^2\right)h_4}{-1+16h_2^2}}}\right)\right]^{\frac{1}{2n}} e^{i(-kx+\omega t+\theta)}, \quad (54)$$

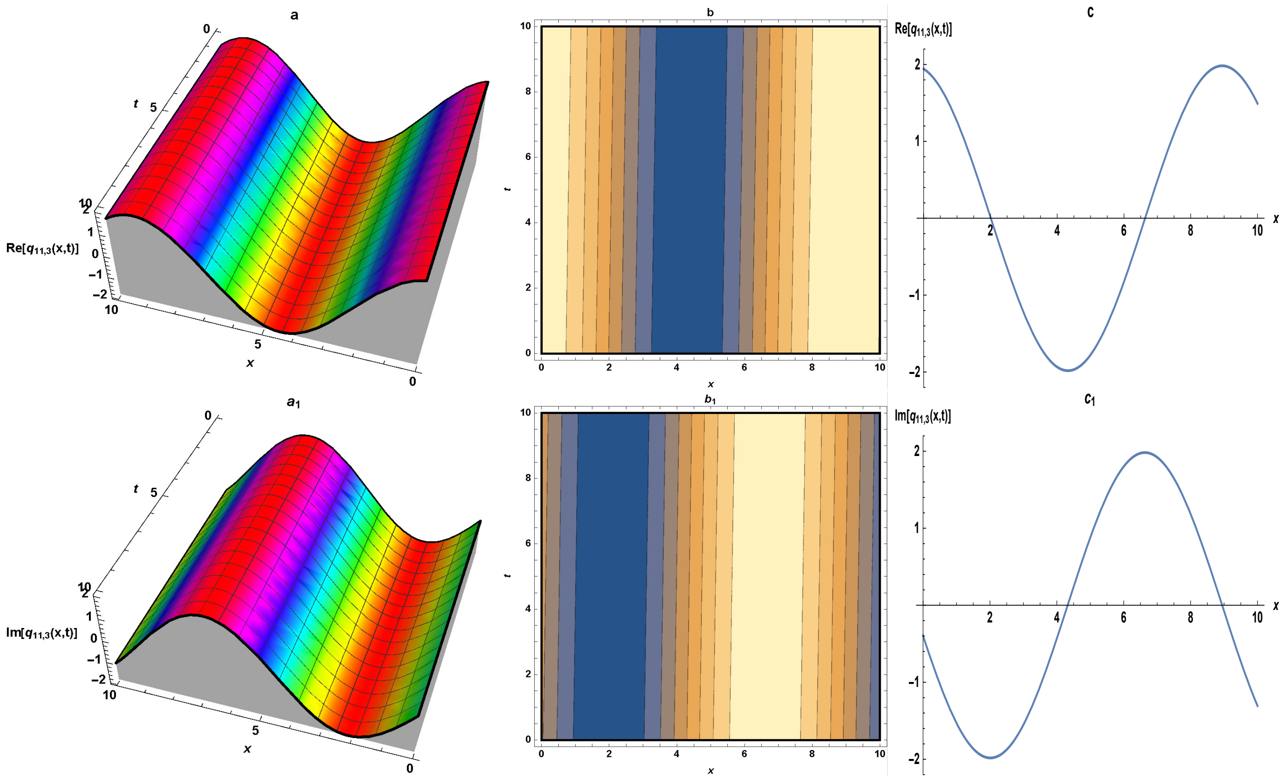

Figure 8. The 3Ds (a), (a_1), contours (b),(b_1) and 2Ds (c),(c_1) graphs of Eq. (52)

or

$$Q_{11,6}\left(x,t\right)=\frac{1}{2}\left[\frac{-1-2n}{4(1+n)r}+\frac{\sqrt{(1+2n)(-a+4\beta)h_4}}{2n\sqrt{br}}\left(\frac{\left(\frac{\cos(\zeta)}{1+\sin(\zeta)}\right)}{\sqrt{\frac{\left(3+2(1-2h_2)\left(\frac{\cos(\zeta)}{1+\sin(\zeta)}\right)^2\right)h_4}{-1+16h_2^2}}}\right)\right]^{\frac{1}{2n}}e^{i(-kx+\omega t+\theta)}, \quad (55)$$

such that

$$h_4^2\left(\frac{1}{2}-h_2\right)\left[\frac{1}{16}\left(1+4h_2\right)^2\right]=0.$$

12. If $l_0=\frac{-\left(1-m^2\right)^2}{4}$, $l_2=\frac{1+m^2}{2}$, $l_4=\frac{-1}{4}$, $0<m<1$, then $P(\zeta)=mcn(\zeta,m)\pm dn(\zeta,m)$, we have

$$Q_{12}\left(x,t\right)=\left[\frac{-1-2n}{4(1+n)r}+\frac{\sqrt{(1+2n)(-a+4\beta)h_4}}{2n\sqrt{br}}\left(\frac{mcn(\zeta,m)\pm dn(\zeta,m)}{\sqrt{f(mcn(\zeta,m)\pm dn(\zeta,m))^2+g}}\right)\right]^{\frac{1}{2n}}e^{i(-kx+\omega t+\theta)}, \quad (56)$$

where f and g are given by

$$f=\frac{-8(1+m^2-2h_2)h_4}{1+14m^2+m^4-16h_2^2}, \qquad g=\frac{12(-1+m^2)^2h_4}{1+14m^2+m^4-16h_2^2},$$

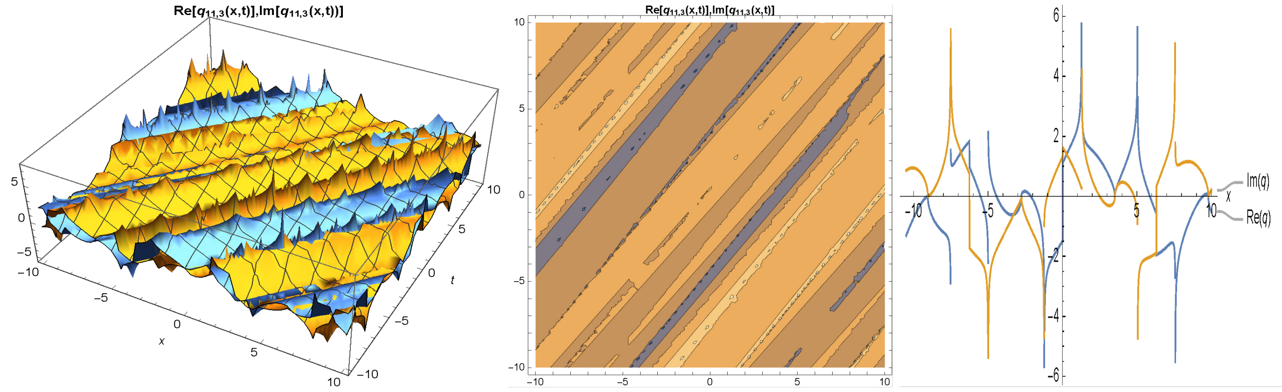

Figure 9. The 3D (a), contour (b) and 2D (c), graphs of Eq. (55)

under the constraint condition

$$h_4^2\left(\frac{1}{2}\left(1+m^2-2h_2\right)\right)\left[\frac{1}{16}\left(1+\left(-6+m\right)m+4h_2\right)\left(1+m\left(6+m\right)+4h_2\right)\right]=0.$$

13. If $l_0=\frac{1}{4}$, $l_2=\frac{1-2m^2}{2}$, $l_4=\frac{1}{4}$, $0<m<1$, then $P(\zeta)=\frac{sn(\zeta,m)}{1\pm cn(\zeta,m)}$, we have

$$\mathrm{Q}_{13}\left(x,t\right)=\left[\frac{-1-2n}{4(1+n)r}+\frac{\sqrt{(1+2n)(-a+4\beta)h_4}}{2n\sqrt{br}}\left(\frac{\frac{sn(\zeta,m)}{1\pm cn(\zeta,m)}}{\sqrt{f\left(\frac{sn(\zeta,m)}{1\pm cn(\zeta,m)}\right)^2+g}}\right)\right]^{\frac{1}{2n}}e^{i(-kx+\omega t+\theta)},\quad(57)$$

where f and g are given by

$$f=\frac{8(-1+2m^2+2h_2)h_4}{1-16m^2+16m^4-16h_2^2},\qquad g=\frac{-12h_4}{1-16m^2+16m^4-16h_2^2},$$

under the constraint condition

$$h_4^2\left(\frac{1}{2}-m^2-h_2\right)\left[\frac{1}{16}+2m^2-2m^4+\left(\frac{1}{2}-m^2\right)h_2+h_2^2\right]=0.$$

If $m\to 1$, then the dark-bright optical soliton solution is obtained

$$\mathrm{Q}_{13,1}\left(x,t\right)=\frac{1}{2}\left[\frac{-1-2n}{4(1+n)r}+\frac{\sqrt{(1+2n)(-a+4\beta)h_4}}{2n\sqrt{br}}\left(\frac{\left(\frac{\tanh(\zeta)}{1+sech(\zeta)}\right)}{\sqrt{\frac{\left(3-2(1+2h_2)\left(\frac{\tanh(\zeta)}{1+sech(\zeta)}\right)^2\right)h_4}{-1+16h_2^2}}}\right)\right]^{\frac{1}{2n}}e^{i(-kx+\omega t+\theta)},\quad(58)$$

such that

$$h_4^2\left(\frac{-1}{2}-h_2\right)\left[\frac{1}{16}\left(1-4h_2\right)^2\right]=0.$$

If $m \to 0$, then the periodic wave solution is obtained

$$Q_{13,2}(x,t) = \frac{1}{2}\left[\frac{-1-2n}{4(1+n)r} + \frac{\sqrt{(1+2n)(-a+4\beta)h_4}}{2n\sqrt{br}}\left(\frac{\left(\frac{\sin(\zeta)}{1+\cos(\zeta)}\right)}{\sqrt{\frac{\left(3+2(1-2h_2)\left(\frac{\sin(\zeta)}{1+\cos(\zeta)}\right)^2\right)h_4}{-1+16h_2^2}}}\right)\right]^{\frac{1}{2n}} e^{i(-kx+\omega t+\theta)}, \quad (59)$$

such that

$$h_4^2\left(\frac{1}{2}-h_2\right)\left[\frac{1}{16}(1+4h_2)^2\right] = 0.$$

14. If $l_0 = \frac{1}{4}$, $l_2 = \frac{1+m^2}{2}$, $l_4 = \frac{(1-m^2)^2}{4}$, $0 < m < 1$, then $P(\zeta) = \frac{sn(\zeta,m)}{cn(\zeta,m)\pm dn(\zeta,m)}$, we have

$$Q_{14}(x,t) = \left[\frac{-1-2n}{4(1+n)r} + \frac{\sqrt{(1+2n)(-a+4\beta)h_4}}{2n\sqrt{br}}\left(\frac{\frac{sn(\zeta,m)}{cn(\zeta,m)\pm dn(\zeta,m)}}{\sqrt{f\left(\frac{sn(\zeta,m)}{cn(\zeta,m)\pm dn(\zeta,m)}\right)^2+g}}\right)\right]^{\frac{1}{2n}} e^{i(-kx+\omega t+\theta)}, \quad (60)$$

where f and g are given by

$$f = \frac{-8(1+m^2-2h_2)h_4}{1+14m^2+m^4-16h_2^2}, \qquad g = \frac{-12h_4}{1+14m^2+m^4-16h_2^2},$$

under the constraint condition

$$h_4^2\left(\frac{1}{2}\left(1+m^2-2h_2\right)\right)\left[\frac{1}{16}\left(1+(-6+m)m+4h_2\right)\left(1+m(6+m)+4h_2\right)\right] = 0.$$

If $m \to 1$, then the singular soliton solution is obtained

$$Q_{14,1}(x,t) = \left[\frac{-b_1}{3b_2} + \frac{\sqrt{(1+2n)(-a+4\beta)h_4}}{2n\sqrt{br}}\left(\frac{\sinh(\zeta)}{\sqrt{\frac{\left(3+(1-h_2)\sinh^2(\zeta)\right)h_4}{-1+h_2^2}}}\right)\right]^{\frac{1}{2n}} e^{i(-kx+\omega t+\theta)}, \quad (61)$$

such that $h_4^2(1-h_2)\left[-2+h_2+h_2^2\right] = 0$. If $m \to 0$, then the combined periodic wave solution is obtained

$$Q_{14,2}(x,t) = \frac{1}{2}\left[\frac{-1-2n}{4(1+n)r} + \frac{\sqrt{(1+2n)(-a+4\beta)h_4}}{2n\sqrt{br}}\left(\frac{\left(\frac{\sin(\zeta)}{1+\cos(\zeta)}\right)}{\sqrt{\frac{\left(3+2(1-2h_2)\left(\frac{\sin(\zeta)}{1+\cos(\zeta)}\right)^2\right)h_4}{-1+16h_2^2}}}\right)\right]^{\frac{1}{2n}} e^{i(-kx+\omega t+\theta)}, \quad (62)$$

such that $h_4^2\left(\frac{1}{2}-h_2\right)\left[\frac{1}{16}(1+4h_2)^2\right] = 0$.

5 Result and physical interpretations of the findings

The φ^6-model expansion technique is used as the integration algorithm. The bright, dark, singular, dark-bright, dark singular, and combined singular soliton solutions to the complex Ginzburg-Landau equation CGLE with dual power law nonlinearity were found. The dual-power law, found in photovoltaic materials, is used to explain nonlinearity in the refractive index. The traveling wave solutions discovered in this work are both physically and mathematically helpful. The constants in the computed wave propagation solutions must be given physical meaning in order to comprehend their physical significance.

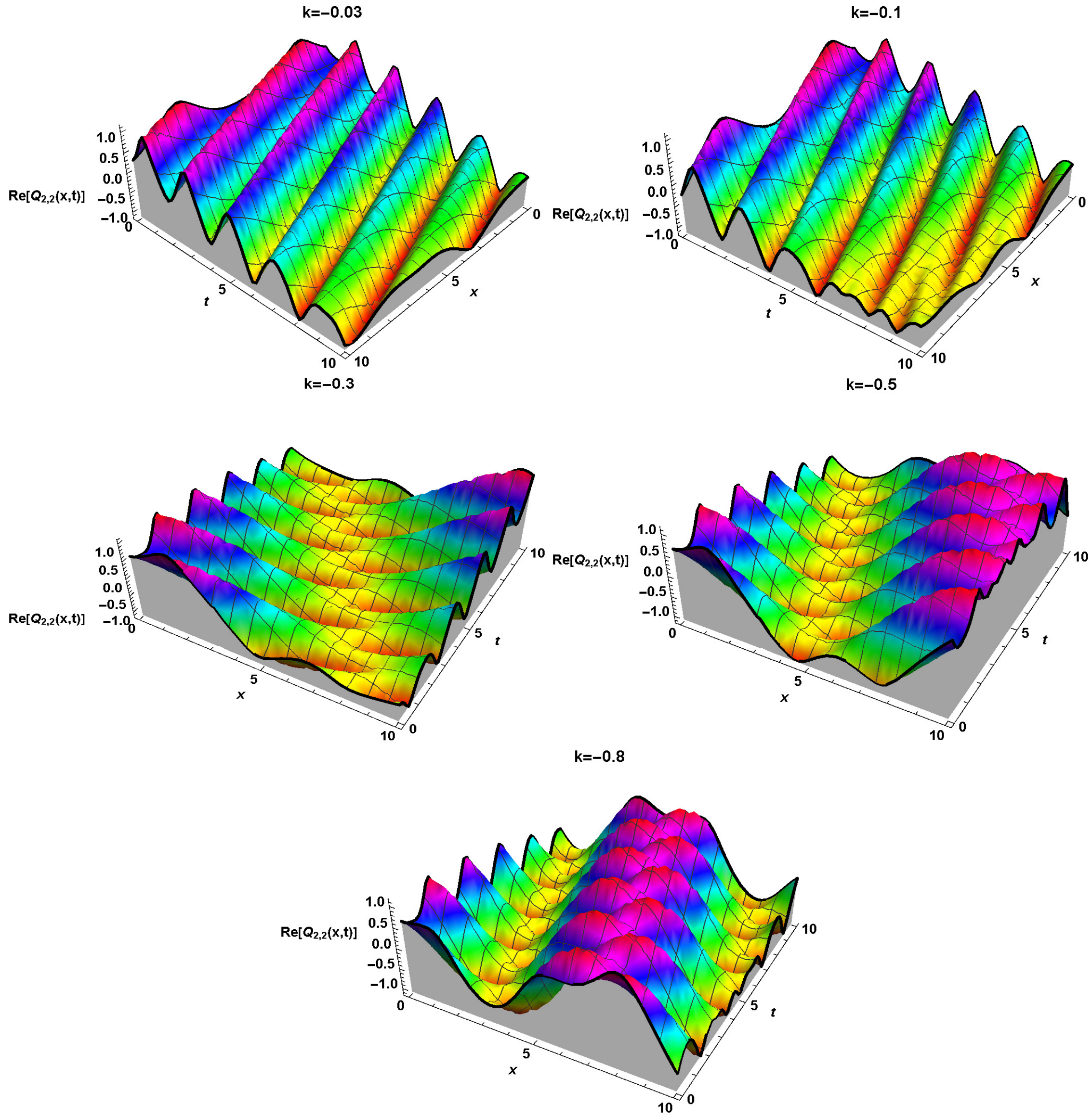

Figure 10. The 3D graphs of real part of Eq. (28) for $\alpha_0 = 0.5, \alpha_1 = 2, h_1 = 1.2, h_2 = 0.9, h_3 = 2.6, h_4 = 0.5, w = 0.2, \theta = 0.5, n = 1, v = 3, \lambda = 0.5$.

In the physics literature, Eq. (5) represents the mathematical model that assumes the envelope of a forward-moving wave pulse evolves slowly in time and space related to a period or wavelength. $Q\,(x,t) = U(x - vt)e^{i(-kx+\omega t+\theta)}$, the function U represents the pulse shape and v is the soliton's velocity. In the phase factor, ω is the soliton wave number, θ is the phase constant and k denotes the soliton frequency which is physically noteworthy in this study and whose various values will be studied for the moving wave's behavior. The parameter k is proportional to the velocity of the soliton as well as the length of the pulse. To evaluate the dynamical features and describe the evolution characteristic, we might pick appropriate values and functions for these parameters in Eq. (28).

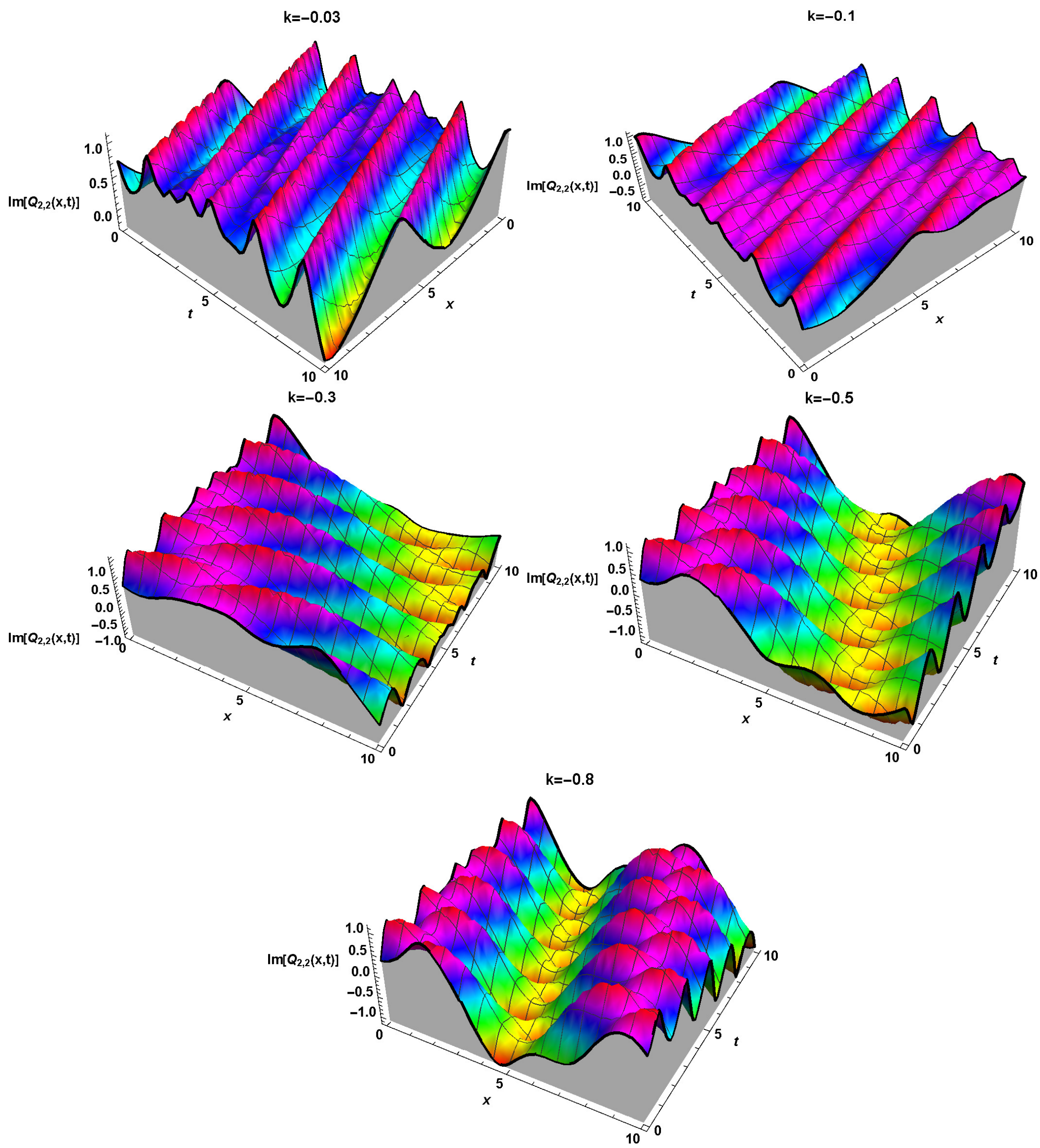

Figure 11. The 3D graphs of imaginary part of Eq. (28) for $\alpha_0 = 0.5, \alpha_1 = 2, h_1 = 1.2, h_2 = 0.9, h_3 = 2.6, h_4 = 0.5, w = 0.2, \theta = 0.5, n = 1, v = 3, \lambda = 0.5$.

Figures 10 and 11 depict the behavior of a single wave at any given time, which is crucial in the transmission of energy from one location to another. In order to provide a new perspective to the topic let us investigate the physical implications of the parameters in the transformation, known as the classical wave transformation. The velocity of the propagating wave is proportional to the variable k. The frequency of the propagating wave is directly proportional to the velocity of the wave and inversely to its wavelength. The traveling wave shows diagonal wave behavior when k is increased. The frequency of a traveling wave is proportional to the wave number. The number of waves grows as the frequency increases.

6 Conclusion

Based on the research findings, it is important to point out that the selected scheme produces a wide range of innovative solutions that are both intriguing and valuable for the governing model. The obtained results in this work are believed to describe some of the CGLE's physical impacts. This research will be crucial to the understanding of the superconducting phenomenology theory, which is frequently used to explain the long-distance propagation of optical solitons via optical fibers and it will help in studying the photovoltaic materials. The φ^6-model expansion approach is helpful and efficient for constructing optical soliton solutions for most nonlinear physical phenomena. The behavior of a traveling wave solution chosen among fourteen different solutions generated for various values of frequency is investigated. When we compare our results in this paper to the results in [41–47], we conclude that our results are unique and have not been found elsewhere, the model will also be evaluated using fractional temporal evolution to account for slow-light pulses.

Declarations

Ethical approval

The authors state that this research complies with ethical standards. This research does not involve either human participants or animals.

Consent for publication

Not applicable.

Conflicts of interest

The authors declare that they have no conflict of interest.

Data availability statement

Data availability is not applicable to this article as no new data were created or analysed in this study.

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Author's contributions

M.A.I.: Methodology, Software, Conceptualization, Validation, Investigation, Writing - Original Draft, Visualization, Data Curation. A.Y.: Methodology, Conceptualization, Validation, Investigation, Writing - Original Draft, Writing - Review & Editing, Visualization, Supervision, Project Administration. All authors discussed the results and contributed to the final manuscript.

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